LECTURE 4: OTHER RELATED MODELS

Having discussed some of the basics for simple walks, I will start talking about various different related model, to help you choose a research topic as soon as possible.

§1. MORE ON THE SELF-AVOIDING WALK

Recall the self-avoiding walk from §4 of Lecture 3. A number of interesting questions present themselves that you may wish to think about:

♠ `Open Problem 1.` One can show, by appealing to subadditivity again, that the self-avoiding walk on any regular lattice $\mathbf{L}$ satisfies

$$\lim_{n\to\infty} \frac{\log \chi_n}{n} = c(\mathbf{L}),$$

exists, and $c(\mathbf{L})$ is the *connectivity constant* of the lattice $\mathbf{L}$. In particular, $c(\mathbf{Z}^d)$ is nothing but the constant $C(d)$ of Lecture 3, §4. Two possible starting points for your research along these lines:

- ♡ Find a numerical method for computing $c(\mathbf{L})$ for some interesting lattices $\mathbf{L}$ such as $\mathbf{Z}^2$, $\mathbf{Z}^3$, or the hexagonal lattice.
- ♡ Can you numerically verify the conjecture that for the honeycomb lattice, the connectivity constant is $\sqrt{\sqrt{2}+2}$?

♠ `Open Problem 2.` It is conjectured that, quite generally, there exists a *universal constant* α such that on any lattice $\mathbf{L} \subseteq \mathbf{Z}^d$, χ_n grows like a constant times $n^\alpha (c(\mathbf{L}))^n$. The point is that α is supposed to be independent of the choice of the lattice but can (and ought to) depend on the ambient dimension d. Assuming this hypothesis, can you find a numerical approximation for α when $d = 2$? Physicists conjecture that when $d = 2$, $\alpha = \frac{43}{32}$.

- ♡ For any $x \in \mathbf{R}^d$, let $|x|^2 := x_1^2 + \cdots + x_d^2$ denote the square of the distance between x and the origin. If X_n denotes the position of the randomly selected self-avoiding path of length n in $\mathbf{Z}^d$, what is $A_n := E\{|X_n|^2\}$? It is conjectured that A_n should grow like a constant times n^β for some constant β. Can you numerically estimate β? When $d = 2$, β is conjectured to be $\frac{3}{4}$. This would suggest that self-avoiding walks grow faster than "diffusions," which is why this type of growth is called "super-diffusive." (Another phrase that refers to this property is "anamolous diffusion.")
- ♡ Can you find numerical ways to estimate $E\{|X_n|^p\}$ for some other values of $p > 2$ as well? It is conjectured that $E\{|X_n|^p\}$ should behave like the square of $E\{|X_n|^{p/2}\}$ as $n \to \infty$.

§2. DIFFUSION-LIMITED AGGREGATION (DLA)

Diffusion-limited aggregation (or DLA) is a mathematical model devised by Witten and Sanders to model crystal growth. The aim is to grow a random set in $\mathbf{Z}^d$, in successive time-steps, in order to obtain a sequence of growing random sets $A(0) \subseteq A(1) \subseteq \cdots$.

To start with, set $A(0) = \{0\}$ so this means that at "time" 0, the "crystal" is a point. Then start a random walk from infinitely (or nearly) far away from $A(0)$ and wait until this random walk hits a neighbor of $A(0)$. This defines the crystal at time 1; namely, let $A(1)$ be the set comprised of the origin together with this last value of the stopped random walk. Having created $A(n)$, create $A(n+1)$ by, once again, starting an independent random walk infinitely far awar from $A(n)$ and waiting until it hits a neighbor of $A(n)$. Add that point to $A(n)$ to create $A(n+1)$ and so on.

Although there are many predictions and conjectures, very few facts are rigorously known to be true. Here are some suggestions for interesting problems that you can try to learn about by simulation analysis. You may be able to come up with others. (Don't forget library and web research for further gaining inspiration and motivation.)

- ♠ `Open Problem 1.` One of the big open problems in this area is to decide whether or not $A(n)$ grows by growing long spindly arms. (The conjecture is that it does; this should make physical sense to you.) Can you decide if this is so? To what extent does the "shape" of $A(n)$ depend on the geometry of the lattice on which the random walks are being run?
- ♠ `Open Problem 2.` Since the notion of "shape" is not usually easy to grasp, one can ask simpler questions that are still quite interesting. For instance, how long are the arms of the DLA? (This is the title of a 1987 paper of Harry Kesten by the way.) In 1987, H. Kesten proved that for DLA on $\mathbf{Z}^d$, if $r_n := \max\{|x| : x \in A(n)\}$, then with probability one, r_n grows more slowly than $n^{2/3}$ if $d = 2$ and more slowly than $n^{2/d}$ if $d = 3$. While these results have been improved by the subsequent works of H. Kesten as well as those of G. Lawler, the known facts are very far from what is expected to be the true growth rate of $A(n)$. Can you decide what this rate is? Let me be more concrete. Suppose r_n grows like a constant times n^{β} for some exponent β. Can you find a simulation prediction for β?

§3. INTERNAL DIFFUSION-LIMITE AGGREGATION (IDLA)

In 1991, Diaconis and Fulton formulated a means by which subsets of certain commutative rings could be multiplied together. This uses a random process that is (like) a random walk on that commutative ring. When the said ring is $\mathbf{Z}^d$, their "random walk" becomes the following random process known as the internal diffusion-limit aggregation (IDLA for short):

Let $A(0) = \{0\}$; having defined $A(0), \ldots, A(n)$, we now construct $A(n+1)$ by running a random walk, independently of all else, until the random walk hits a point that is not in $A(n)$. When that happens, stop the walk and add the newly-visited point to $A(n)$ thereby creating $A(n+1)$.

This is a simpler process than the DLA, but it is far from being a simple object. Here is a fact that was shown to be true by M. Bramson, D. Griffeath, and G. Lawler (1992):

(3.1) Asymptotic Shape of the IDLA. *Let B_d denote the ball of radius 1 in $\mathbf{Z}^d$, and let ω_d denote its volume (e.g., $\omega_1 = 2$ and $\omega_2 = \pi$.) Then, as $n \to \infty$, the following happens with probability one:*

$$\left(\frac{\omega_d}{n}\right)^{\frac{1}{d}} A(n) \Rightarrow B_d, \tag{3.2}$$

where by $\Rightarrow$ I mean that for any $\varepsilon > 0$, the left-hand side is eventually contained in the ε-enlargement of B_d (i.e., ball of radius $(1+\varepsilon)$ for any ε), and eventually contains the ε-reduction of B_d (i.e., the ball of radius $(1-\varepsilon)$).

In other words, for large values of n (i.e., in large time), the IDLA set $A(n)$ looks more and more like the centered ball of radius $(n/\omega_d)^{1/d}$. For instance, when $d = 2$, this is the centered ball of radius $\sqrt{n/\pi}$.

♠ `Open Problem 1.` What happens in other lattices? For instance, what about the hexagonal or the triangular lattice? What if the lattice is inhomogeneous? (This is due to Matthew Taylor.)

♠ `Open Problem 2.` Continuing with the above, what if you have a lattice that is random? For instance, suppose you run a random walk on the infinite cluster of an independent percolation process (see §5 below). Then what behavior should you expect to see?

♠ `Open Problem 3.` One may think that $A(n)$ really looks filled in and like a ball. However, in her Ph. D. thesis, D. Eberz has proven that with probability one, there exist infinitely many n's such that $A(n)$ "has holes" in it. A good research problem would be to explore the fluctuations; i.e., to explore how different $A(n)$ is from the ball. As a concrete way to state this, consider the number of points that are (i) in $A(n)$ and not in $(n/\omega_d)^{1/d}B_d$; or are (ii) in $(n/\omega_d)^{1/d}B_d$ but not in $A(n)$. How many of them are there for large values of n? To be even more concrete, hypothesize that this number grows like a constant times n^γ. Can you estimate γ by simulation analysis?

§4. BOND PERCOLATION

For any number $0 < p < 1$, and for any lattice $\mathbf{L}$, we can define bond percolation on $\mathbf{L}$ as follows: Each edge of $\mathbf{L}$ is open with probability p and closed with probability $(1-p)$, and all edges are open/closed independently from one another. We can then say that *percolation* occurs if with positive probability, one can find some random open path that connects a given point of $\mathbf{L}$ (call it the origin) to infinity (i.e., if there is an infinite self-avoiding path emenating from the origin, all of whose edges are open.)

Let $\theta(p)$ denote the probability of percolation on a given lattice. That is, $\theta(p)$ is the probability that there is an infinite open connected path starting from the origin.

(4.1) The Critical Probability. *There exists a critical probability p_c such that whenever $p > p_c$, $\theta(p) > 0$, but when $p < p_c$, $\theta(p) = 0$.*

This follows from showing that $\theta(p)$ increases as p goes up; although it is true, this is not a trivial fact. Here is how you prove it:

Proof: On each edge e in the lattice, set down independent edge-weights X_e such that $P\{X_e \leq x\} = x$ for all $x \in [0, 1]$. In other words, X_e is uniformly distributed on $[0, 1]$. Now every time $X_e \leq p$, call that edge open, otherwise it is closed. This procedure produces the percolation process with parameter p simultaneously for all p, since $P\{e \text{ is open }\} = P\{X_e \leq p\} = p$. Moreover, if $X_e \leq p$, then for any $p' > p$, $X_e \leq p'$ also. Therefore, the percolation cluster for p is contained in the percolation cluster for p'. In particular, if there is percolation at level p, there is certainly percolation at level p'. This is another way to state that $\theta(p) \leq \theta(p')$. To finish, define p_c to be the smallest value of p such that $\theta(p) > 0$. This is well-define since θ is increasing (draw a picture!) ♣