# LECTURE 5: THE CRITICAL PERCOLATION PROBABILITY FOR BOND PERCOLATION

Recall that, in percolation, each edge in $\mathbf{Z}^d$ is open or closed with probability $p$ or $(1-p)$, and the status of all edges are independent from one another. In (4.1, Lecture 4) we showed that there exists a *critical probability* $p_c$ (sometimes written as $p_c(\mathbf{Z}^d)$ to emphasize the lattice in question), such that for all $p > p_c$, there is percolation (i.e., with positive probability, there exists an infinite connected open path from the origin), and for $p < p_c$, there is no percolation. However, this statement is completely vacuous if the numercial value of $p_c$ were trivial in the sense that $p_c$ were 0 or 1. In this lecture, we will show that this is not the case. In fact, we will show that in all dimensions $d \geq 2$,

$$\frac{1}{C(d)} \leq p_c(\mathbf{Z}^d) \leq 1 - \frac{1}{C(d)}, \tag{0.1}$$

where $C(d)$ is the connectivity constant of $\mathbf{Z}^d$; see (§4.2, lecture 3).

**(0.2) Concrete Bounds on $p_c(\mathbf{Z}^d)$.** Since that $d \leq C(d) \leq (2d)$ (§4.2, lecture 3), then it follows from (0.1) above that $\frac{1}{2d} \leq p_c(\mathbf{Z}^d) \leq 1 - \frac{1}{2d}$. This can be easily improved upon, since by §4.9 of lecture 4, $C(d) \leq (2d-1)$, so that $\frac{1}{2d-1} \leq p_c(\mathbf{Z}^d) \leq 1 - \frac{1}{2d-1}$. in particular, $p_c(\mathbf{Z}^d)$ is strictly between 0 and 1, which is the desired claim. ♣

**(0.3) The Planar Case.** The planar case deserves special mention: The previous bounds show that $p_c(\mathbf{Z}^2)$ is between $\frac{1}{3}$ and $\frac{2}{3}$. In fact, it has been shown that

- **a.** $p_c(\mathbf{Z}^2) = \frac{1}{2}$ (Harris and Kesten);
- **b.** If $p = p_c(\mathbf{Z}^2)$, then there is no percolation (Bezuidenhout and Grimmett). ♣

## §1. THE LOWER BOUND IN (0.1).

We first verify the lower bound of (0.1) on $p_c$. Note that showing $p_c \geq \frac{1}{C(d)}$ amounts to showing that whenever $p < \frac{1}{C(d)}$, then $P\{\text{percolation}\} = 0$.

First note that the chance that any self-avoiding path $\pi$ of length $n$ is open is $p^n$. Therefore,

$$\begin{aligned} E\{\#\text{ of self-avoiding paths of length } n\} &= E\left[\sum_\pi \mathbf{1}\{\pi \text{ is open}\}\right] \\ &= \sum_\pi P\{\pi \text{ is open}\} = \sum_\pi p^n, \end{aligned} \tag{1.1}$$

where $\sum_\pi$ denotes the summation over all self-avoiding paths of length $n$, and $\mathbf{1}\{\cdots\} := \mathbf{1}_{\{\cdots\}}$ is the indicator of $\{\cdots\}$. Since there are $\chi_n$ many self-avoiding paths of length $n$,

$$E\{\#\text{ of self-avoiding paths of length } n\} \leq \chi_n p^n. \tag{1.2}$$

But $\chi_n \approx \{C(d)\}^n$, where

$$\textbf{(1.3)} \qquad a_n \approx b_n \qquad \text{mean} \qquad \lim_{n\to\infty} \frac{\log a_n}{\log b_n} = 1.$$

This means that as soon as $p < \frac{1}{C(d)}$, then

$$\textbf{(1.4)} \qquad E\{\#\text{ of self-avoiding paths of length } n\} \to 0, \qquad (n \to \infty).$$

(Why? Be sure that you understand this!) But for any $n$,

$$\textbf{(1.5)} \qquad \begin{aligned} P\{\text{percolation}\} &\le P\{\#\text{ of self-avoiding paths of length } n \ge 1\} \\ &\le E\{\#\text{ of self-avoiding paths of length } n\}, \end{aligned}$$

thanks to Markov's inequality (§4.1, lecture 2). Since $P\{\text{percolation}\}$ is independent of $n$, (1.3) shows that it must be zero as long as $p < \frac{1}{C(d)}$. This shows that $p_c \ge C(d)$, which is the desired result. ♣

## §2. THE UPPER BOUND IN (0.1).

Now we want to prove the second inequality in (0.1). That is, we wish to show that if $p > 1 - \frac{1}{C(d)}$, then $P\{\text{percolation}\} > 0$. This is trickier to do, since we have to produce an open path or an algorithm for producing such a path, and this is a tall order. Instead, let us prove the (logically equivalent) converse to the bound that we are trying to prove. Namely, we show that if $P\{\text{percolation}\} = 0$, then $p \le 1 - \frac{1}{C(d)}$. For this, we need to briefly study a notion of duality for percolation, and one for graphs. From now on, we will only work with $\mathbf{Z}^2$; once you understand this case, you can extend the argument to get the upper bound in (0.1) for any $d \ge 2$.

**(2.1) The Dual Lattice.** Briefly speaking, the dual lattice $\widetilde{\mathbf{Z}^2}$ of $\mathbf{Z}^2$ is the lattice

$$\textbf{(2.2)} \qquad \widetilde{\mathbf{Z}^2} := \mathbf{Z}^2 + \left(\frac{1}{2}, \frac{1}{2}\right).$$

At this point, some of you may (and should) be asking yourselves, "*What does it mean to sum a set and a point?*" In general, $A + x$ is short-hand for the set $\{y + x;\ y \in A\}$. That is, $A + x$ is $A$ shifted by $x$. Consequently, the dual lattice $\widetilde{\mathbf{Z}^2}$ is the lattice $\mathbf{Z}^2$ shifted by $(0.5, 0.5)$. Pictorially speaking, the dual lattice $\widetilde{\mathbf{Z}^2}$ looks just like $\mathbf{Z}^2$, except that its origin is the point $(0.5, 0.5)$ instead of $(0, 0)$; i.e., its origin has been shifted by $(0.5, 0.5)$. You should plot $\widetilde{\mathbf{Z}^2}$ to see what is going on here.

**(2.3) Dual Percolation.** Each edge $\mathbf{e}$ in $\mathbf{Z}^2$ intersects a unique edge in $\widetilde{\mathbf{Z}^2}$ halfway in the middle. We can call this latter edge the *dual edge* to $\mathbf{e}$. Whenever an edge in $\mathbf{Z}^2$ is open, its dual is *declared* close, and conversely, if an edge in $\mathbf{Z}^2$ is closed, we declare its dual edge in $\widetilde{\mathbf{Z}^2}$ open. Clearly, this process creates a percolation process on the dual lattice $\widetilde{\mathbf{Z}^2}$, but the edge-probabilities are now $(1-p)$ instead of $p$. Now if there is no percolation on $\mathbf{Z}^2$, this means that on $\widetilde{\mathbf{Z}^2}$, there must exist an open "circuit" surrounding the origin. For a picture of this, see

`http://www.math.utah.edu/~davar/REU-2002/notes/lec5.html`

The probability that any given circuit, surrounding the origin, of length $n$ is dual-open is $(1-p)^n$. So,

$$E\left[\#\text{ of open circuits in }\widetilde{\mathbf{Z}^2}\text{ of length }n\right] \le C_n(1-p)^n, \tag{2.4}$$

where $C_n$ denotes the number of circuits—in $\widetilde{\mathbf{Z}^2}$—of length $n$ that surround the origin. Thus, we have shown that

$$P\left\{\text{no perocolation in }\mathbf{Z}^2\right\} \le C_n(1-p)^n. \tag{2.5}$$

We want to show that is $p$ is large enough, the above goes to zero as $n \to \infty$. To do so, we need a bound for $C_n$.

**(2.6) Bounding $C_n$.** It is easier to count the number of circuits of length $n$ in $\mathbf{Z}^2$ (not the dual) that surround the origin. This number is also $C_n$ (why?). But for a path $\pi := \pi_0, \ldots, \pi_n$ to be a circuit of length $n$ about $(0,0)$, it must be that any $(n-1)$ steps in $\pi$ form a self-avoiding path, and that $\pi$ must go through one of the points $(1,0), (1,\pm 1), (1,\pm 2), \ldots, (1, \pm\lfloor \frac{n}{2} \rfloor)$. (There are at most $(n+1)$ of these points.) Therefore, $C_n \le (n+1)\chi_{n-1}$ (why?) Recalling (1.3) above, and since $\chi_{n-1} \approx \{C(d)\}^{n-1}$, this and (2.5) show that whenever $p > 1 - \frac{1}{C(d)}$, then there can be no percolation, which is the desired result. ♣