LECTURE 7: FRACTAL PERCOLATION

§1. FRACTAL PERCOLATION

(1.1) Mandelbrot's Fractal Percolation. Consider the square $S := [0,1] \times [0,1]$. That is, S is the set of all points (x, y) such that $0 \le x \le 1$ and $0 \le y \le 1$. We will divide S into four equal-sized squares,

$$
\begin{aligned}
S_1 &:= \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right], \; S_2 := \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right], \\
S_3 &:= \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right], \; S_4 := \left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right],
\end{aligned}
\tag{1.2}
$$

For each square, you toss an independent coin; with probability $p \in (0,1)$, you keep that square, and with probability $(1-p)$ you jettison it. So now you have a random number of kept squares (some random number between 0 and 4.) Split each into four equal-sized squares, and toss an independent p-coin for each to see if you want to keep, and repeat. Fact: If p is sufficiently large, and if you continue *ad infinitum*, then with positive probability you end up with a nonempty random set that Mandelbrot calls a "random curdle," and these days is referred to as fractal percolation.

(1.3) Hard Question. Use simulation to find the critical probability p_c past which you can get fractal percolation. ♣

§2. FRACTALS AND MINKOWSKI (BOX) DIMENSION

(2.1) The Tertiary Cantor Set. Georg Cantor invented the following strange set that is nowhere dense, has length zero, and yet is uncountable. It is the archetype of what is nowadays is called a fractal.

Start with the interval $I = [0,1]$; split it into three equal parts, and jettison the middle-third to get two intervals $I_1 := [0, \frac{1}{3}]$, and $I_2 := [\frac{2}{3}, 1]$. Take the remaining two intervals, split them in threes, and jettison the middle-third interval, and repeat. After the nth stage of this construction, you will get a set C_n that is made up of 2^n intervals of length 3^{-n}. In particular, the length of C_n is $(2/3)^n$, which goes to zero. It is not hard to see that $C := \cap_n C_n \neq \emptyset$, although it has length zero. A little more work shows that it is nowhere dense.

(2.2) The Minkowski Dimension. Note that in the nth stage of the construction of the tertiary Cantor set of (2.1), we have in pricniple 3^n intervals of length 3^{-n}, but we only keep 2^n of them. Therefore, the total number of intervals of length 3^{-n} that cover the tertiary Cantor set should be 2^n. In general, let N_k denote the total number of the intervals (in higher dimensions, cubes) of length k^{-1} that cover the portion of your fractal in $[0,1]$, and define the *Minkowski* or *box dimension* of your fractal to be the number α such that $N_k \approx k^\alpha$, if such a number exists. (Recall that $a_k \approx b_k$ means that as $k \to \infty$, $\log(a_k) \div \log(b_k) \to 1$.)

(2.3) Example. Consider the tertiary Cantor set of (2.1), and check that $N_{3^{-n}} = 2^n$. Formally let $k = 3^n$ and convince yourself that as $k \to \infty$, $N_k \approx k^\alpha$ where $\alpha = \log(2)/\log(3)$. That is, the tertiary Cantor set is a "**fract**al" of "**fract**ional dimension" $\log(2)/\log(3)$ which is about equal to 0.63. ♣

(2.4) Projects for Extensions. You can try constructing other Cantor-type fractals by either (i) splitting into intervals of other sizes than $\frac{1}{3}$; (ii) retaining/jettisoning intervals by a different algorithm; or (iii) constructing higher-dimensional fractals. For instance, try starting with the square $[0,1] \times [0,1]$; split it into 9 equal-sized squares; retain all but the middle one, and repeat. ♣

(2.5) Projects for Fractal Percolation. Now go back to fractal percolation, and ask:

- ♢ What is the critical probability p_c, such that whenever $p > p_c$, you can end up with a nonempty random fractal, and when $p < p_c$, the entire construction ends at some random stage since everything has been jettisoned? The answer to this is known by theoretical considerations.
- ♢ When $p > p_c$, can you find the box dimension of the resulting random fractal? The answer to this is known by theoretical considerations.
- ♢ When $p > p_c$, can you estimate the probability that there exists a left-to-right path on the resulting random fractal? The answer to this is unknown.

(2.6) Relation to Percolation on Trees. The act of splitting each square into four equal-sized ones can be represented by a rooted tree in which each vertex splits into four vertices in the next level of the tree. Now go through the edges of this tree, and with probability p keep an edge, and with probability $(1-p)$ discard it. Question: Is there an infinite kept path starting from the root? You should make sure that you understand the following assertion: This is *exactly* the same mathematical question as, "Is there fractal percolation?"

(2.7) Relation to Branching Processes. Consider the following model for geneology of a gene: You start with one "grandmother gene." Upon death (or mutation or whatever else is the case), this gene splits into a random number of "offspring," where the offspring distribution is: With probability p^4 there are 4 offpsring; with probability $\binom{4}{1}p^3(1-p)$ there are 3 offspring; with probability $\binom{4}{2}p^2(1-p)^2$ there are 2 offspring; and with probability $(1-p)^4$ there are no offpsring. How large should p be in order for this gene population to survive forever? Make sure that you understand that this is the same mathematical problem as the one in (2.6), which is itself the same as asking whether or not one has fractal percolation.