LECTURES ON RELATIVE MOTIVIC INTEGRATION, MACPHERSON'S TRANSFORMATION, AND STRINGY CHERN CLASSES

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Preliminary draft (work in progress) - Please do not circulate.

Contents

Introduction

There notes were written to accompany my lectures at the VIGRE minicourse on "Arc Spaces and Motivic Integration", held at the University of Utah in May 2005. The main goal of these lectures is to discuss how motivic integration, if performed in the relative setting, can actually capture invariants such as Chern classes.

The first two lectures are devoted to a discussion of MacPherson's construction of Chern classes of singular varieties. We will start with reviewing the definition of Chern classes of complex manifolds and their extensions to singular varieties proposed by Mather and Schwartz-MacPherson. Then we will focus on the work of MacPherson in which he constructs a natural transformation from the functor of constructible functions to homology (or Chow groups), proving a conjecture of Deligne and Grothendieck. It is using this transformation that MacPherson proposes his theory of Chern classes for singular varieties.

The use of motivic integration to construct a new generalization of Chern classes to singular varieties is motivated by a recent result of Aluffi, where it is proven that Chern classes of smooth varieties behave well under certain birational modifications. Indeed one can show with explicit examples that this particular birational property is lost in the singular case if one considers Schwartz-MacPherson classes. This will be our motivation for what comes next.

Aiming for a theory of Chern classes that is birationally well-behaved (i.e., with a "stringy" flavor), we will apply the theory of motivic integration over a base and explain how one can extract a constructible function (defined over the base) from any given relative motivic integral. This is the content of the third lecture. Most of the technical ingredients of the general theory of motivic integration will be only sketched or quoted. Instead, the attention will be focused on the use of the theory in the relative setting, and the geometric process leading to extract constructible functions from motivic integrals will be explained in detail.

The various material covered so far is then combined in the forth lecture, where stringy Chern classes are introduced and studied. The basic idea of the construction is to apply MacPherson's transformation to certain constructible functions naturally arising (via motivi integration) from resolution of singularities. Explicit examples will be presented to compare stringy Chern classes with Schwartz-MacPherson class. Using formal properties of motivic integration, we will investigate the main properties of these classes.

The last lecture is devoted to a discussion of stringy invariants for quotient varieties: this is a beautiful part of the story, related to a classic problem known as the "McKay correspondence". The general principle is that the stringy invariants of the quotient variety, which are defined through resolution of singularities, are already encoded (in some way) in the equivariant geometry of the manifold of which we are taking the quotient. This not only is an amazing phenomenon per se, but also provides explicit formulas to compute these invariants. It can arguably be said that stringy invariants made their first appearance in this context.

Several exercises are proposed throughout, the main purpose being in most cases that of giving some concrete feeling to the reader new to the subject of what the various constructions and definitions are all about.

0.1. **Acknoledgements.** It is a pleasure to tank the organizers of the minicourse, Aaron Bertram and Christopher Hacon, for offering me the opportunity of delivering these lectures, and for providing such a nice and enjoyable stay in Salt Lake City for the period of the course.

I wish to thank Bertram and Hacon, the other lectures of the course, Manuel Blickle, Giulia Jordon and Wim Veys, and all the participants of the course, for many friutful conversations and several comments and corrections that helped me improving the exposition of these notes.

This material is based upon work supported by the National Science Foundation under agreements No. DMS-0548325 and DMS-0111298, by the University of Michigan Rackham Research Grant, and by VIGRE (?). Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

0.2. **Notation and conventions.** We work over the field of complex numbers. By *scheme* we will mean an algebraic scheme of finite type over $\operatorname{Spec}\mathbb{C}$. A *variety* is an integral scheme, and a *manifold* is a smooth variety. Subschemes and subvarieties are always assumed to be closed. A $\mathbb{Q}$-*divisor* on a normal variety is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, namely a rational combination of Weil divisors, a multiple of which is Cartier.

Finally, faithfully to the European origins of the author, the set of natural numbers $\mathbb{N}$ includes zero.

0.3. **Temporary disclaimer (!)** This is still a preliminary version, only distributed to the participants of the course. Some–hopefully most–of the errors and typos in the previous version have been corrected. Many things are still lacking, among which some of the attributions of the credits for the various results and remarks presented here, more comparisons with other works, and a complete list of references. Brief introductions to each lecture will also be included.

Comments, corrections and suggestions are of course very welcome.

Lecture 1. MATHER'S AND SCHWARTZ-MACPHERSON'S CHERN CLASSES OF SINGULAR VARIETIES

1.1. **The Chern class of a manifold.** We start by recalling a few definition from intersection theory, referring the reader to [Ful] for a full treatment and several of the properties that will be used here.

Let X be a scheme. We denote by $Z_*(X)$ the free abelian group generated by (closed) subvarieties of X. An element α of $Z_*(X)$ is called a *cycle on* X, and can be written in a unique way as a finite linear combination

$$\alpha = \sum_{i=1}^{k} n_i [V_i]$$

where $n_i \in \mathbb{Z}$ and V_i are subvarieties of X. Then the *Chow group* of X is the quotient of $Z_*(X)$ by *rational equivalence*:

$$A_*(X) := \mathbb{Z}_*(X)/\sim_{\text{rat}} .$$

We recall that a cycle $\alpha \in Z_*(X)$ is rational equivalent to zero if and only if there are subvarieties $W_1, \dots, W_t \subseteq X \times \mathbb{P}^1$ such that, denoting by $\mathrm{pr}_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$ the projection onto the second factor and fixing two distinct points 0 and ∞ in $\mathbb{P}^1$, we have

$$\alpha = \sum_{j=1}^{t} [(\mathrm{pr}_2|_{W_j})^{-1}(0)] - [(\mathrm{pr}_2|_{W_j})^{-1}(\infty)]$$

in $Z_*(X)$.

When X is proper, taking the degree of a zero-dimensional cycle gives a map $\deg : A_0(X) \to \mathbb{Z}$. The *degree* of any cycle $\alpha \in A_*(X)$ is then defined to be the degree of it zero-dimensional part, and is denoted by any of the symbols $\deg \alpha$, $\int_X \alpha$, or just $\int \alpha$.

Consider now a vector bundle E over X, and let r be the rank of E. Associated to E, there is the *total Chern class* of E: this is the sum

$$c(E) = 1 + c_1(E) + \cdots + c_r(E),$$

where each $c_i(E)$ is defined as an operator on the Chow group $A_*(X)$, the operation being given by cap products

$$c_i(E) \cap \_ : A_k(X) \to A_{k-i}(X).$$

We refer to [Ful, Section 3.2] for the definition of these operations.

Product of Chern classes of vector bundles (over the same scheme X) is defined by composition, by setting

$$c_i(E) \cdot c_j(E') \cap \_ : \ A_k(X) \xrightarrow{c_i(E)\cap\_} A_{k-i}(X) \xrightarrow{c_j(E')\cap\_} A_{k-i-j}(X).$$

We recall that if L is a line bundle and $i : V \hookrightarrow X$ is a subvariety, then $c_1(L) \cap [V] = i_*[\mathrm{div}(s_V)]$, where $s_V : V \dashrightarrow L|_V$ is any non-trivial rational section of the restriction $L|_V$ of L to V. Then, using the *splitting principle* [Ful, Section 3.2], Chern classes of vector bundles are characterized by the condition

$$c(E) = c(E') \cdot c(E'')$$

for any short exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0.$$

Moreover, if E is a globally generated vector bundle on a quasi-projective n-dimensional variety X and $s_j \in \Gamma(X, E)$ are general sections, then

$$c_i(E) \cap [X] = [\{x \in X \mid s_0(x), \dots, s_{r-i}(x) \text{ are linearly dependent}\}] \in A_{n-i}(X).$$

If X is a smooth variety and TX is the tangent bundle of X, then we obtain the class

$$c(X) := c(TX) \cap [X] \in A_*(X).$$

We call this class the *Chern class* of X. We recall that, if X is a proper smooth variety, then

$$\int_X c(X) = \chi(X) := \sum_i (-1)^i \dim H^i(X, \mathbb{Z}).$$

the topological Euler characteristic of X. Moreover, assuming that X is a smooth projective variety with globally generated tangent bundle TX, so that X admits sufficiently general (holomorphic) vector fields, we obtain a geometric interpretation of the Chern class $c(X)$ in terms of the degeneracy loci of the vector fields. In particular, we recover the following well known fact:

$$\chi(X) = \#\{\text{zeroes of a general vector field on } X\}.$$

We close this section with a comment of the notation. The choice of the symbol $c(X)$ to denote this class is not standard, and the term "Chern class" (or "total Chern class") of X is often used in literature for the class $c(TX)$ rather than its value on $[X]$. One should think at $c(X)$, as defined above, as the Poincaré dual of the cohomological Chern class of X. The reason of our choice relies on the fact that we will discuss several extensions to singular varieties of the notion of Chern class of a manifold, and all of them can only be given in some homological theory of the given variety. More precisely, all these extensions will be defined as elements in the Chow group of the variety (in some cases allowing rational coefficients).

1.2. **The Nash blowup and Mather's Chern class.** Let X be an n-dimensional variety, and assume that X is embedded in a manifold M. If $G_n(TM)$ is the Grassmann bundle over M of rank n subbundles of TM, the map

$$x \mapsto (T_xX \subseteq T_xM) \in G_n(TM)_x,$$

defined for every $x \in X_{\mathrm{reg}}$, gives a section from the smooth locus of X to $G_n(TM)$. The *Nash blowup* $\widehat{X}$ of X is the closure in $G_n(TM)$ of the image of this section. The projection to the base induces a morphism

$$\nu : \widehat{X} \to X.$$

It is a fact that (in characteristic zero, hence in our setting) this morphism is locally an isomorphism precisely over the smooth locus of X [Nob]. Moreover, if X is locally complete intersection, then the Nash blowup agrees with the blowup of the Jacobian ideal of X [Nob]; in particular the previous property also follows, in this case, from [Lip]. In general, the Nash blowup does not depend on the particular embedding chosen; in fact, it is even possible to define the Nash blowup without the use of any embedding.

Remark 1.2.1. One can construct the Nash blowup of X by thaking the closure of the natural section $X_{\mathrm{reg}} \to \mathbb{P}_X(\wedge^n \Omega^1_X)$ of the projection $\mathbb{P}_X(\wedge^n \Omega^1_X) \to X$ (see for instance [dFEI]).

Exercise 1.2.2. By considering the case of cones over smooth plane curves, argue that the Nash blowup needs no be smooth nor normal.

Throughout the above construction, $\widehat{X}$ comes naturally equipped with a rank n vector bundle, that we denote by $\widehat{T}$. This is given by the restriction to $\widehat{X}$ of the tautological vector bundle ξ over $G_n(TM)$ (we recall that the latter is the subbundle of the pullback of TM on $G_n(TM)$ whose fiber over a point $(V \subseteq T_pM) \in G_n(TM)$ is given by V). By construction, $\widehat{T}$ agrees with (the pullback of) the tangent bundle of the smooth locus of X.

The *Chern-Mather class* (or simply *Mather class*) of X is then defined to be the class

$$c_{Ma}(X) := \nu_*(c(\widehat{T}) \cap [\widehat{X}]) \in A_*(X).$$

If X is smooth, then $\widehat{X} = X$, hence $c_{Ma}(X) := c(X)$.

Exercise 1.2.3. Describe the Nash blowup of a nodal (resp. cuspidal) plane cubic $C \subset \mathbb{P}^2$, and compute its Mather class.

Observe that the dual of $\widehat{T}$ is a locally free quotient subsheaf of $\widehat{\nu}^*\Omega^1_X$. In concrete examples, it may be useful to use the following definition. A *generalized Nash blowup* of a variety X is any variety $\widetilde{X}$ that comes equipped with a proper birational morphism $\widetilde{\nu} : \widetilde{X} \to X$ and a locally free quotient

$$\widetilde{\nu}^*\Omega^1_X \to \widetilde{\Omega} \to 0$$

of rank $n = \dim X$. Observe that, if $\widetilde{\nu} : \widetilde{X} \to X$ is a generalized Nash blowup and $\widetilde{T}$ is the dual bundle of $\widetilde{\Omega}$, then for any embedding of X in a manifold M the bundle $\widetilde{T}$ is a subbundle of $\widetilde{\nu}^*TM$ extending the pullback of $TX_{\mathrm{reg}}|_U$, where TX_{reg} is the tangent bundle of the smooth locus X_{reg} of X and $U \subseteq X_{\mathrm{reg}}$ is the open subset onto which $\widetilde{\nu}$ is an isomorphism.

The Nash blowup $\widehat{X}$ can be equivalently described as the closure, in the Grassmann bundle over M of rank-n locally free quotients of Ω^1_M, of the natural section defined over the smooth locus of X. In particular $\widehat{\nu} : \widehat{X} \to X$ is a generalized Nash blowup. In fact, we have the following property: a proper birational map $\widetilde{\nu} : \widetilde{X} \to X$ is a generalized Nash blowup if and only if it factors through the Nash blowup $\widehat{\nu} : \widehat{X} \to X$:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\eta"] \arrow[dr, "\widetilde{\nu}"'] & \widehat{X} \arrow[d, "\widehat{\nu}"] \\ & X. \end{tikzcd}$$

Note that $\widetilde{T} = \eta^*\widehat{T}$. Then the projection formula implies that

$$\widetilde{\nu}_*(c(\widetilde{T}) \cap [\widetilde{X}]) = c_{Ma}(X)$$

for any generalized Nash blowup $\widetilde{\nu} : \widetilde{X} \to X$.

Exercise 1.2.4. Argue that the normalization $f : X' \to X$ of the Whitney umbrella $X = \{x^2 = y^2 z\} \subset \mathbb{A}^3$ is not a generalized Nash blowup. Conclude that a resolution of singularities of a variety needs not satisfy the requirements in the definition of generalized Nash blowup.

1.3. **MacPherson's transformation and Schwartz-MacPherson's Chern class.** Let X be a variety. For any subvariety $V \subseteq X$, we define the *characteristic function* of V to be the function

$$\mathbf{1}_V : X \to \mathbb{Z}, \quad \mathbf{1}_V(x) = \begin{cases} 1 & \text{if } x \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Then the *group of constructible functions* of X is the subgroup $F_*(X)$ of the abelian group of $\mathbb{Z}$-values functions on X freely generated by the characteristic functions of the subvarieties of X. An element $\phi \in F_*(X)$ is called a *constructible function* of X, and can be (uniquely) written in the form

$$\phi = \sum_{i=1}^{k} n_i \mathbf{1}_{V_i}, \tag{1.3.1}$$

where V_i are subvarieties of X and $n_i \in \mathbb{Z}$. Clearly, $F_*(X) \cong Z_*(X)$ as $\mathbb{Z}$-modules.

Exercise 1.3.1. Using descending induction on the dimension of the V_i, show that the expression of ϕ in (1.3.1) is unique (this amounts to say that $F_*(X)$ is freely generated by the characteristic functions $\mathbf{1}_V$ as V ranges among the subvarieties of X). Observe that unicity fails if we drop the requirement that the V_i be closed in X.

Exercise 1.3.2. The support of any constructible function on X is a *constructible subset*, namely, a finite union of locally closed subsets of X. Show that, conversely, the characteristic function of any constructible subset of X is a constructible function.

It is sometimes convenient to consider $F_*(X)$ as a ring, with the product defined pointwise. For instance, for two subvarieties V and W of X, we have $\mathbf{1}_V \cdot \mathbf{1}_W = \mathbf{1}_{V \cap W}$. Then $F_*(X)$ is a commutative ring with zero element $\mathbf{1}_\emptyset$ (the constant function $\mathbf{0}$) and identity $\mathbf{1}_X$ (the constant function $\mathbf{1}$).

Consider now a morphism $f : X \to Y$. For every subvariety $V \subseteq X$, we define the function

$$f_*\mathbf{1}_V : Y \to \mathbb{Z}, \quad f_*\mathbf{1}_V(y) := \chi_c(V \cap f^{-1}(y)),$$

where χ_c denotes the Euler characteristic with compact support (i.e., computed using cohomology with compact support). It follows by a theorem of Verdier [Ver, Corollaire (5.1)] that $f|_V$ is piecewise topologically locally trivial over a stratification of X in Zariski-locally closed subsets. This implies that

$$f_*\mathbf{1}_V \in F_*(Y).$$

Extending by linearity, we obtain a group homomorphism $f_* : F_*(X) \to F_*(Y)$.

The Euler characteristic satisfies the following two properties:

$$\chi_c(X) = \sum \chi_c(X_i)$$

for any decomposition of a variety X into a disjoint union $X = \bigsqcup X_i$ of locally closed subsets, and

$$\chi_c(Z \times Z') = \chi_c(Z)\,\chi_c(Z')$$

for any two varieties Z and Z'. Therefore, by the above mentioned result of Verdier, we have $(g \circ f)_* = g_* \circ f_*$ for any morphism g from Y to a third variety Z. Therefore we obtain a functor

$$F_* : \quad X \mapsto F_*(X), \quad (X \xrightarrow{f} Y) \mapsto (F_*(X) \xrightarrow{f_*} F_*(Y)),$$

from the category of varieties and morphisms to the category of abelian groups and homomorphisms.

If $f : X \to Y$ is a proper morphism, then we also have a push-forward of Chow groups $f_* : A_*(X) \to A_*(Y)$. The theorem of MacPherson is the existence of a natural transformation between the functors F_* and A_* over the category of varieties and proper morphisms with the additional property that, on a smooth variety X, it assigns to $\mathbf{1}_X$ the Chern class of X.

Theorem 1.3.3 ([Mac])**.** *For any variety X, there exists a group homomorphism $c_* : F_*(X) \to A_*(X)$ such that the diagram*

$$\begin{tikzcd} F_*(X) \arrow[r, "c_*"] \arrow[d, "f_*"'] & A_*(X) \arrow[d, "f_*"] \\ F_*(Y) \arrow[r, "c_*"] & A_*(Y) \end{tikzcd}$$

commutes for every proper morphism $f : X \to Y$ and

$$c_* \mathbf{1}_X = c(X)$$

whenever X is smooth.

We will refer to c_* as the *MacPherson's transformation.* The existence of such transformation was conjectured by Deligne and Grothendieck.

Granting its existence, c_* is uniquely determined by the conditions stated in the theorem. To see this, let $\phi \in F_*(X)$ be an arbitrary element. We can write

$$\phi = \sum_{i=1}^{k} b_i\, g_{i*} \mathbf{1}_{W_i},$$

where $g_i : W_i \to X$ is the resolution of the closure of an irreducible component of the support of $\phi - \sum_{j=1}^{i-1} b_j\, g_{j*} \mathbf{1}_{W_j}$ and b_i is a suitable integer. Then, using the conditions that c_* is supposed to satisfy, we compute

$$c_*\phi = \sum_{i=1}^{k} b_i\, c_* g_{i*} \mathbf{1}_{W_i} = \sum_{i=1}^{k} b_i\, g_{i*} c_* \mathbf{1}_{W_i} = \sum_{i=1}^{k} b_i\, g_{i*} c(W_i).$$

The *Chern-Schwartz-MacPherson class* (or simply *Schwartz-MacPherson class*) of an arbitrary variety X is then defined by

$$c_{SM}(X) := c_* \mathbf{1}_X \in A_*(X).$$

Exercise 1.3.4. Let X be a proper variety. Show that

$$\int_X c_{SM}(X) = \chi(X).$$

Clearly $c_{SM}(X) = c(X)$ if X is smooth. However, in general this class differs from the Mather class when X is singular.

Exercise 1.3.5. Compute the Schwartz-MacPherson class of a nodal plane cubic $C \subset \mathbb{P}^2$, and show that it does not agree with the Mather class. Then determine the difference between $c_{Ma}(X)$ and $c_{SM}(X)$ when X is the projective closure of the cone over a smooth plane curve of degree d.

Using Mather classes, we obtain a homomorphism

$$c_{Ma} : Z_*(X) \to A_*(X), \quad [V] \mapsto i_* c_{Ma}(V),$$

were V ranges among the subvarieties of X and i is its inclusion in X (note that $i_* : A_*(V) \to A_*(X)$ is well defined because i is proper). On the other hand, we have observed that there is a natural isomorphism $R : Z_*(X) \to F_*(X)$ that to each cycle $[V]$ represented by a subvariety $V \subseteq X$ associates the characteristic function $\mathbf{1}_V$. Combining these two, we obtain a homomorphism $c_{Ma} \circ R^{-1} : F_*(X) \to A_*(X)$ which clearly maps $\mathbf{1}_X$ to $c(X)$ if X is smooth. However this cannot be equal to c_* because it is not natural, in the sense that it does not commute with direct images under proper morphisms. The substance in the construction of MacPherson's transformation is precisely to correct this lack of naturality; this will be done by a choice of a different isomorphism $Z_*(X) \to F_*(X)$, which may be thought as a "change of basis". This will be the content of the next lecture.

Exercise 1.3.6. Give an example of a proper map $f : X \to Y$ such that

$$(c_{Ma} \circ R^{-1}) \circ f_* \neq f_* \circ (c_{Ma} \circ R^{-1})$$

as homomorphisms from $F_*(X)$ to $A_*(Y)$.

Lecture 2. Construction of MacPherson's transformation

2.1. **The local Euler obstruction.** The crucial definition in the construction of MacPherson's transformation is that of local Euler obstruction. Although different from the original one, we give here a intersection theoretic definition of this invariant, and refer to [Mac] for the original definition and to [G-S] for the equivalence of the two.

Recall that the *Segre class* $s(Z, Y)$ of a proper subscheme Z of a scheme Y is computed as follows. Take the blowup $g : Y' \to Y$ along Z and let E be the exceptional divisor of g. Then

$$s(Z, Y) = (g|_E)_* \sum_{j \geq 1} (-1)^{j-1} [E^j] \in A_*(Z),$$

where for short we have denoted $[E^j] := c_1(\mathcal{O}_E(E))^{j-1} \cap [E]$. If Z is regularly embedded in Y and $N_{Z/Y}$ is the normal bundle, then we also have

$$s(Z, Y) = c(N_{Z/Y})^{-1} \cap [Z].$$

Exercise 2.1.1. Check the equivalence of the two formulas for $s(Z, Y)$ when Z is a Cartier divisor.

Let X be a variety, let $\nu : \widehat{X} \to X$ be its Nash blowup, and fix a point p. Then the *local Euler obstruction* of X at p is the number

$$\mathrm{Eu}_p(X) := \int c(\widehat{T}|_{\nu^{-1}(p)}) \cap s(\nu^{-1}(p), \widehat{X}),$$

where $s(\nu^{-1}(p), \widehat{X})$ is the Segre class of $\nu^{-1}(p)$ in $\widehat{X}$. If $\sigma : \mathrm{Bl}_p \widehat{X} \to \widehat{X}$ is the blowup along $\nu^{-1}(p)$ and $D \subset \mathrm{Bl}_p \widehat{X}$ is the exceptional divisor of the blowup, then the projection formula gives

$$\mathrm{Eu}_p(X) = \int c(\sigma^* \widehat{T}|_D) \cap (\sum_{j \geq 1} (-1)^{j-1} [D^j])$$

Observe that the definition is local in nature, in the sense that if X is locally irreducible at p and $U \subset X$ is any open neighborhood of p, then $\mathrm{Eu}_p(X) = \mathrm{Eu}_p(U)$. Moreover, if X is locally reducible at p and U_i are the irreducible components of an open neighborhood $U \subset X$ of p, then we have $\mathrm{Eu}_p(X) = \sum \mathrm{Eu}_p(U_i)$.

The local Euler obstruction can be computed using any generalized Nash blowup in place of the Nash blowup.

Proposition 2.1.2. *Let $\widetilde{\nu} : \widetilde{X} \to X$ be a generalized Nash blowup of X. Then*

$$\mathrm{Eu}_p(X) = \int c(\widetilde{T}|_{\widetilde{\nu}^{-1}(p)}) \cap s(\widetilde{\nu}^{-1}(p), \widetilde{X}).$$

Proof. Let $\eta : \widetilde{X} \to \widehat{X}$ be the induced morphism. We have $\eta^* \widehat{T} = \widetilde{T}$ and $\eta_* s(\widetilde{\nu}^{-1}(p), \widetilde{X}) = s(\widehat{\nu}^{-1}(p), \widehat{X})$, hence

$$\eta_* c(\widetilde{T}|_{\widetilde{\nu}^{-1}(p)}) \cap s(\widetilde{\nu}^{-1}(p), \widetilde{X}) = c(\widehat{T}|_{\widehat{\nu}^{-1}(p)}) \cap s(\widehat{\nu}^{-1}(p), \widehat{X})$$

by projection formula. Then the assertion follows by taking degrees. $\square$

Exercise 2.1.3. Show that $\mathrm{Eu}_p(X) = 1$ if X is smooth at p, and that if q is a m-ple point of a curve C, then $\mathrm{Eu}_q(C) = m$.

A proof of the following proposition can be found in [Ken, Lemma 4].

Proposition 2.1.4. *The function*

$$\mathrm{Eu}_{-}(X) : X \to \mathbb{Z}$$

that assigns to each point $p \in X$ the local Euler obstruction $\mathrm{Eu}_p(X)$ *is constructible (that it, is an element in $F_*(X)$).*

Exercise 2.1.5. Let X be the cone over a smooth plane curve of degree $d \geq 2$, and let $p \in X$ be the vertex. Show that $\mathrm{Eu}_p(X) = 2d - d^2$. Conclude that the function $\mathrm{Eu}_{-}(X) : X \to \mathbb{Z}$ needs no be positive nor upper semi-continuous.

Using the local Euler obstruction, we define a group homomorphism

$$T : Z_*(X) \to F_*(X)$$

by assigning to each class $[V] \in Z_*(X)$ represented by a subvariety $V \subseteq X$ the function

$$\mathrm{Eu}_-(V) : X \to \mathbb{Z}, \quad p \mapsto \mathrm{Eu}_p(V),$$

where we set $\mathrm{Eu}_p(V) = 0$ if $p \notin V$. In other words, T is defined by

$$T\left(\sum m_i[V_i]\right) = \sum m_i \,\mathrm{Eu}_-(V_i)$$

for $m_i \in \mathbb{Z}$ and $V_i \subseteq X$ subvarieties. Proposition 2.1.4 implies that this is a constructible function.

Lemma 2.1.6. *T is an isomorphism.*

Proof. It suffices to check that the functions $\mathrm{Eu}_-(V)$ form a basis for $F_*(X)$ as V ranges among the subvarieties of X. □

By composing T^{-1} with the homomorphism c_{Ma} introduced in Section 1.3, we obtain the homomorphism

$$c_* := c_{Ma} \circ T^{-1} : \; F_*(X) \xrightarrow{T^{-1}} Z_*(X) \xrightarrow{c_{Ma}} A_*(X). \tag{2.1.1}$$

We need to prove that c_* satisfies the requirements listed in Theorem 1.3.3. It is clearly an homomorphism mapping $\mathbf{1}_X$ to $c(X)$ whenever X is smooth, so what is left to show is that

$$f_* c_* = c_* f_* \tag{2.1.2}$$

for every proper morphism f. The proof of this property is outlined in the following two sections.

2.2. **The graph construction.** Given a proper morphism $f : X \to Y$, where X is a smooth variety, we will construct a cycle

$$\alpha = \sum n_i[V_i] \in Z_*(Y),$$

where $V_i \subseteq Y$ are subvarieties, such that

$$c_{Ma}(\alpha) = f_* c(X) \text{ in } A_*(Y), \quad \text{and} \quad T(\alpha) = f_* \mathbf{1}_X \text{ in } F_*(Y). \tag{2.2.1}$$

Before we proceed with the construction of this cycle, let us explain why the above formulas imply the identity (2.1.2) for any proper morphism of varieties $f : X \to Y$. Given this general situation, for any $\phi \in F_*(X)$ we can find proper morphisms $g_i : W_i \to X$ with W_i smooth varieties, and integers b_i, such that

$$\phi = \sum b_i \, g_{i*} \mathbf{1}_{W_i}.$$

Note that, if c_* is defined as in (2.1.1) and we assume that (2.2.1) holds for morphisms from smooth varieties, then we have

$$g_{i*} c_* \mathbf{1}_{W_i} = c_* g_{i*} \mathbf{1}_{W_i} \quad \text{and} \quad f_* g_{i*} c_* \mathbf{1}_{W_i} = c_* f_* g_{i*} \mathbf{1}_{W_i}. \tag{2.2.2}$$

Therefore, applying $f_* c_*$ to the expression defining α and using these identities, we get

$$f_* c_* \phi = \sum b_i \, f_* c_* g_{i*} \mathbf{1}_{W_i} = \sum b_i \, f_* g_{i*} c_* \mathbf{1}_{W_i} = \sum b_i \, c_* f_* g_{i*} \mathbf{1}_{W_i} = c_* f_* \phi,$$

as desired.

Exercise 2.2.1. Fill up the details to check that (2.2.2) follows from (2.2.1).

Let us go back to our setup, so that $f : X \to Y$ is a proper morphism from a smooth variety X. We assume that Y is embedded in a smooth variety M (the general case follows by a suitable patching using local embeddings). Then the cycle α is defined using the following construction. We denote by

$$G_n := G_n(TX \oplus TM)$$

the Grassmann bundle over X: the fiber over a point $x \in X$ is given by $G_n(T_xX \oplus T_yM)$, where $y = f(x)$. For any $\lambda \in \mathbb{C}$, the graph of the vector bundle map

$$\lambda\, df : TX \to TM$$

given by λ times the differential df determines a section

$$\begin{tikzcd} G_n \arrow[d, "\pi"] \\ X \arrow[u, bend left, "\sigma_\lambda"] \end{tikzcd}$$

of the Grassmann bundle. Let $Z_\lambda \subseteq G_n$ be the image of σ_λ and $\beta_\lambda := [Z_\lambda] \in A_*(G_n)$ be the associated cycle. Identify $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, let Z be the closure in $G_n \times \mathbb{P}^1$ of the image of the map

$$X \times \mathbb{C} \to G_n \times \mathbb{C} \hookrightarrow G_n \times \mathbb{P}^1, \quad (x, \lambda) \mapsto (\sigma_\lambda(x), \lambda),$$

and let

$$Z_\infty := \mathrm{pr}_1(\mathrm{pr}_2^{-1}(\infty)) \subseteq G_n,$$

where pr_i are the restrictions to Z of the projections of $G \times \mathbb{P}^1$ onto the two factors. Then let

$$\beta_\infty := [Z_\infty] = \sum m_i[W_i] \in A_*(G_n),$$

where the W_i are the irreducible components of Z_∞. By construction, $\beta_\lambda \sim_{\mathrm{rat}} \beta_\infty$ in $A_*(G_n)$ (and in fact in $A_*(Z)$) for every $\lambda \in \mathbb{C}$.

Note that $\pi|_{Z_\lambda} : Z_\lambda \to X$ is an isomorphism for every $\lambda \in \mathbb{C}$, but in general we do not have much control on Z_∞. Intuitively, the way Z_∞ splits into different components should reflect the singularities of f.

Exercise 2.2.2. Show that if $f : X \to M$ is an immersion then $\pi|_{Z_\infty} : Z_\infty \to X$ is an isomorphism.

Let V_i be the image of W_i in Y, and let $\widehat{V_i} \to V_i$ be its Nash blowup. Then let P_i be the closure in $\widehat{V_i} \times_{V_i} W_i$ of the of the inverse image of the open set of V_i projecting isomorphically to V_i. The obtain the following commutative diagram, in which we denote the pull-back of a vector bundle by the same symbol as the bundle itself.

$$\begin{tikzcd}
\widehat{T}_i \subseteq TX \oplus TM \supseteq \xi \arrow[dr] & & & TX \oplus TM \supseteq \xi \arrow[d] & & \\
& P_i \arrow[r, "\mu_i"] \arrow[dddd, "\rho_i"] & W_i \arrow[r, hook] \arrow[dddd, "\eta_i"] & Z \arrow[r] \arrow[d, "\pi"] & \mathbb{P}^1 & \\
& & & X \arrow[u, bend left, "\sigma_\lambda"] \arrow[ddd, "f"] & TX \arrow[l] & \\
& & & & & \\
\widehat{T}_i \arrow[dr] & & & & & TM \arrow[dl] \\
& \widehat{V}_i \arrow[r, "\nu_i"] & V_i \arrow[r, hook] & Y \arrow[r, hook] & M. &
\end{tikzcd}$$

Here $\widehat{T}_i$ is the bundle on $\widehat{V}_i$ naturally associated with the Nash blowup, and ξ is the tautological bundle on G_n. Note that $\operatorname{rk} \widehat{T}_i = \dim V_i$ and $\operatorname{rk} \xi = n$ $(= \dim X)$. Note also that $\dim P_i = n$.

This construction determines the subvarieties $V_i \subseteq Y$ that we will use to define α. The integers n_i are determined as follows. We observe that the bundle $\widehat{T}_i$ over P_i is a subbundle of the bundle ξ over P_i (since $\widehat{T}_i$ is a vector bundle, it enough to check the inclusion $\widehat{T}_i \subseteq \xi$ at the generic point of P_i, and this follows by the construction and generic smoothness of V_i). Then we take the quotient $\xi/\widehat{T}_i$, which is a vector bundle over P_i of rank $n - \dim V_i$. Since this is also the relative dimension of the map ρ_i, we immediately deduce that

$$\rho_{i*}(c(\xi/\widehat{T}_i) \cap [P_i]) = k_i[\widehat{V}_i] \quad \text{in } A_*(\widehat{V}_i) \tag{2.2.3}$$

for some integer k_i. Then we define

$$n_i := m_i k_i.$$

Exercise 2.2.3. Explain why $\rho_{i*}(c(\xi/\widehat{T}_i) \cap [P_i])$ does not pick up any higher codimensional cycle on $\widehat{V}_i$.

2.3. **On the naturality of the transformation.** Recall that $f : X \to Y$ is a proper map, and X is smooth. In the previous section, assuming that Y is embedded in a smooth variety M, we geometrically determined a cycle

$$\alpha = \sum m_i k_i [V_i] \in Z_*(Y)$$

using the graph construction associated to f. Here we give a partial proof of the identities stated in (2.2.1).

We start observing that

$$\sigma_{0*}[X] = \beta_0 \sim_{\text{rat}} \beta_\infty = \sum m_i [W_i] = \sum m_i \, \mu_{i*}[P_i] \quad \text{in } A_*(Z).$$

Note also that $\sigma_0^*\xi = TX$. Then the first one of the identities in (2.2.1) comes straights out of the computation

$$\begin{aligned} f_* c(X) &= f_*\, c(TX) \cap [X] \\ &= f_*\, c(\sigma_0^*\xi) \cap [X] \\ &= f_*\pi_*\sigma_{0*}(\sigma_0^* c(\xi) \cap [X]) \\ &= f_*\pi_*\, c(\xi) \cap \sigma_{0*}[X] \\ &= \sum m_i\, f_*\pi_*\, c(\xi) \cap \mu_{i*}[P_i] \\ &= \sum m_i\, f_*\pi_*\mu_{i*}\, c(\xi) \cap [P_i] \\ &= \sum m_i\, \nu_{i*}\rho_{i*}\, c(\widehat{T}_i) \cdot c(\xi/\widehat{T}_i) \cap [P_i] \\ &= \sum m_i\, \nu_{i*}\, c(\widehat{T}_i) \cap \rho_{i*}(c(\xi/\widehat{T}_i) \cap [P_i]) \\ &= \sum m_i k_i\, \nu_{i*}\, c(\widehat{T}_i) \cap [\widehat{V}_i] \\ &= c_{Ma}(\alpha). \end{aligned}$$

The second identity in (2.2.1) is less immediate. For it, we need to prove that

$$\chi(f^{-1}(p)) = \sum n_i\, \mathrm{Eu}_p(V_i) \quad \text{for every } p \in Y. \tag{2.3.1}$$

The full proof of this equation is given in [Mac] using the original transcendental definition of local Euler obstruction as the obstruction to extend certain local real vector fields. A different proof of MacPherson theorem was given using the language of Lagrangian submanifolds, which replace altogether the use of constructible functions [Sab, Ken]. The difficulty to prove directly (2.3.1) using the definition of Euler obstructions given here is due to the lack of a general intersection theoretic formula computing $\chi(f^{-1}(p))$.

Here we give a intersection theoretic argument to verify (2.3.1) in the special case when the fiber $f^{-1}(p)$ is smooth, and refer the reader to [Mac] for the general proof. For every morphism $W \to Y$, we denote by W_p the fiber over p, by $\widetilde{W}$ the blowup of W along W_p, and by E_W the corresponding exceptional divisor.

Lemma 2.3.1. *Assuming that X_p is smooth, we have*

$$\deg c(\xi) \cap s(Z_{0,p}, Z_0) = \deg c(\xi) \cap s(Z_{\infty,p}, Z_\infty).$$

Proof. For simplicity, let us assume that X_p is connected. Let $e = \operatorname{codim}(X_p, X)$. Since smooth, X_p is locally defined by a regular sequence of length e.

We claim that Z_p is locally defined by a regular sequence of length e in Z. We can replace Y to a sufficiently small neighborhood around p (hence replace X appropriately) so that for every $y \in Y$ the fiber X_y is smooth and has dimension $\dim X_y \le \dim X_p$.

We fix an arbitrary $q \in X_p$; then the claim is that the e local equations cutting X_p near q lift to a regular sequence cutting Z_p in Z locally over q. We can assume that there is a smooth subvariety $S \subseteq X$ of dimension e passing through q and such that

$$g := f|_S : S \to M$$

is an immersion. Note that $S \cap X_p = \{q\}$ and that, since $\dim S = e$, the e equations defining X_p near q restrict to a regular sequence cutting q on S. We decompose

$$T_qX = T_qS \oplus N_q, \quad \text{where } N_q := T_qX_p,$$

and extend N_q to a vector bundle N over S such that for every $x \in S$ and $y = f(x)$ we have $N_x \supseteq T_xX_y$. Let

$$G_e = G_e(TS \oplus TM).$$

For every $\lambda \in \mathbb{C}$, the homomorphism $\lambda\, dg : TS \to TM$ determines a section $\tau_\lambda : S \to G_e$; let $\Gamma_\lambda \subseteq G_e$ be the image of τ_λ, and let $\Gamma \subseteq G_e \times \mathbb{P}^1$ be the closure determined by sending $\lambda \to \infty$. Since g is an immersion, every Γ_λ, for $\lambda \in \mathbb{P}^1$, is mapped isomorphically to S, therefore the the pullback of the e equations defining X_p in X are a regular sequence cutting Γ_p in Γ.

For any $x \in S$ and $y = g(x)$, we define

$$\psi_x :\ G_e(T_xS \oplus T_yM) \to G_n(T_xX \oplus T_yM), \quad \Lambda \mapsto \Lambda + (N_x \oplus \{0\}).$$

This determines an embedding $\psi : G_e \hookrightarrow G_n$. The condition

$$N_x \supseteq T_xX_y = \ker df|_x,$$

implies that $(df|_x)|_{T_xS} = dg|_x$, hence the constructions of σ_λ and τ_λ are compatible under ψ. In other words, we have a commutative diagram

$$\begin{tikzcd} G_e \arrow[r, hook, "\psi"] \arrow[d] & G_n \arrow[d] \\ S \arrow[r, hook] \arrow[u, bend left, "\tau_\lambda"] & X. \arrow[u, bend right, "\sigma_\lambda"'] \end{tikzcd}$$

Therefore $\Gamma = Z \cap G_e$. Still restricting to the local picture over q, both Z_p and Γ_p are cut, respectively in Z and G, by the pullback of the equations in the regular sequence cutting X_p in X near q. Since these equations form a regular sequence in Γ, they must form a regular sequence in Z, as claimed.

Going back to the original setup, we conclude that Z_p is flat over $\mathbb{P}^1$ and that the normal cone of Z_p in Z is a vector bundle. This implies that

$$\deg c(\xi) \cap s(Z_{0,p}, Z_0) = \deg c(\xi) \cap s(Z_{\infty,p}, Z_\infty)$$

by [Ful, Example 10.1.10 and Proposition 10.2]. □

Lemma 2.3.2. *We have*

$$\deg c(\xi) \cap s(P_{i,p}, P_i) = k_i \deg c(\widehat{T}_i) \cap s(\widehat{V}_{i,p}, \widehat{V}_i).$$

Proof. Let $d_i := \dim V_i$, and fix an arbitrary $j \geq 1$. Let

$$\widetilde{\rho}_i : \widetilde{P}_i \to \widetilde{\widetilde{V}}_i$$

be the induced map. Observing that $\mathcal{O}_{P_i}(E_{P_i}) = \widetilde{\rho}_i^* \mathcal{O}_{\widehat{V}_i}(E_{\widehat{V}_i})$ and applying the projection formula, we compute

$$\widetilde{\rho}_{i*}\, c_{n-d_i+h}(\xi/\widehat{T}_i) \cap [E^j_{P_i}] = \begin{cases} k_i[E^j_{\widehat{V}_i}] & \text{if } h = 0 \\ 0 & \text{if } h \geq 1, \end{cases}$$

Then, using again the projection formula, we obtain

$$\rho_{i*}\, c_{d_i-j}(\widehat{T}_i)\cdot c_{n-d_i}(\xi/\widehat{T}_i)\cap [E^j_{P_i}] = k_i \deg c_{d_i-j}(\widehat{T}_i)\cap [E^j_{\widehat{V}_i}]$$

and

$$c_{d_i-j+h}(\widehat{T}_i)\cdot c_{n-d_i-h}(\xi/\widehat{T}_i)\cap [E^j_{P_i}] = 0 \quad \text{for every } h\geq 1.$$

Note that the second formula implies that

$$c_{d_i-j}(\widehat{T}_i)\cdot c_{n-d_i}(\xi/\widehat{T}_i)\cap [E^j_{P_i}] = c_{n-j}(\xi)\cap [E^j_{P_i}].$$

Putting all together, we obtain

$$\deg c_{n_i-j}(\xi)\cap [E^j_{P_i}] = k_i \deg c_{d_i-j}(\widehat{T}_i)\cap [E^j_{\widehat{V}_i}]$$

We conclude by the definition of Segre class. □

We are now ready to verify (2.3.1) when the fiber X_p is smooth. Let $N_{X_p/X}$ be the normal bundle of X_p in X. Then, using the two lemmas just proven and basic properties of Segre classes [Ful, Section 4.2], we compute

$$\begin{aligned}
\chi(X_p) &= \deg c(TX_p)\cap [X_p] \\
&= \deg c(TX)\cdot c(N_{X_p/X})^{-1}\cap [X_p] \\
&= \deg c(TX)\cap s(X_p,X) \\
&= \deg c(\xi)\cap s(Z_{0,p},Z_0) \\
&= \deg c(\xi)\cap s(Z_{\infty,p},Z_\infty) \\
&= \sum m_i \deg c(\xi)\cap s(W_{i,p},W_i) \\
&= \sum m_i \deg c(\xi)\cap s(P_{i,p},P_i) \\
&= \sum m_i k_i \deg c(\widehat{T}_i)\cap s(\widehat{V}_{i,p},\widehat{V}_i) \\
&= \sum n_i \operatorname{Eu}_p V_i.
\end{aligned}$$

Lecture 3. Relative motivic integration and constructible functions

3.1. Motivic integration over a base. Fix a complex algebraic variety X, and let $\mathfrak{Var}_X$ be the category of X-varieties. Given an X-variety $V \xrightarrow{g} X$, we denote by $[V \xrightarrow{g} X]$, or simply by $[V]$ when X and g are clear from the context, the corresponding class modulo isomorphism over X. Moreover, we set $\mathbb{L}_X := [\mathbb{A}^1_X]$. Same notation will be used to denote the elements that these classes induce in the various rings that we are going to introduce.

Let $K_0(\mathfrak{Var}_X)$ denote the free $\mathbb{Z}$-module generated by the isomorphism classes of X-varieties, modulo the relations

$$[V] = [V\setminus W] + [W]$$

whenever W is a (closed) subvariety of an X-variety V, and both W and $V\setminus W$ are viewed as X-varieties under the restriction of the morphism $V\to X$. $K_0(\mathfrak{Var}_X)$ becomes a ring when the product is defined by setting

$$[V]\cdot[W] := [V\times_X W]$$

and extending it associatively. This ring has zero $[\emptyset]$ and for identity $[X]$.

Exercise 3.1.1. Check that the product in $K_0(\mathfrak{Var}_X)$ is well defined, and that for every X-variety $V \xrightarrow{f} X$ one has

$$[V]\cdot \mathbb{L}_X = [V\times \mathbb{A}^1],$$

where $V\times\mathbb{A}^1$ is viewed as a X-variety under the composition

$$V\times \mathbb{A}^1 \xrightarrow{\mathrm{pr}_1} V \xrightarrow{f} X.$$

Let

$$\mathcal{M}_X := K_0(\mathfrak{Var}_X)[\mathbb{L}_X^{-1}].$$

The dimension $\dim\alpha$ of an element $\alpha\in\mathcal{M}_X$ is by definition the infimum of the set of integers d for which α can be written as a finite sum

$$\alpha = \sum m_i[V_i]\mathbb{L}_X^{-b_i}$$

with $m_i\in\mathbb{Z}$ and $\dim V_i - b_i \le d$ (here the dimension of V_i is the one over $\operatorname{Spec}\mathbb{C}$). Note that $\dim[\emptyset] = -\infty$. The dimension function satisfies

$$\dim(\alpha+\beta)\le \max\{\dim(\alpha),\dim(\beta)\} \quad\text{and}\quad \dim(\alpha\cdot\beta)\le\dim(\alpha)+\dim(\beta),$$

so we obtain a structure of filtered ring on $\mathcal{M}_X$ with the filtration of $\mathcal{M}_X$ given by dimension. Completing with respect to the dimensional filtration (for $d\to-\infty$), we obtain the *relative motivic ring* $\widehat{\mathcal{M}}_X$. We will use $[V]$ also to denote the image of V in $\widehat{\mathcal{M}}_X$ under the natural map $\mathcal{M}_X\to\widehat{\mathcal{M}}_X$. We will consider the composition of maps

$$\tau=\tau_X:\ K_0(\mathfrak{Var}_X)\to\mathcal{M}_X\to\widehat{\mathcal{M}}_X.$$

All main definitions and properties valid for the motivic integration over $\operatorname{Spec}\mathbb{C}$ translate to the relative setting by simply remembering the maps over X. For instance, consider a smooth X-variety Y. Let Y_∞ be the space of arcs of Y, and denote by $\mathrm{Cyl}(Y_\infty)$ the set of cylinders on Y_∞. Then the motivic pre-measure

$$\mu^X : \mathrm{Cyl}(Y_\infty)\to\widehat{\mathcal{M}}_X$$

is defined as follows. For any cylinder $C\in\mathrm{Cyl}(Y_\infty)$, we choose an integer m such that $\pi_m^{-1}(\pi_m(C)) = C$ (here $\pi_m : Y_\infty\to Y_m$ is the truncation map to the space of mth jets). Then we put

$$\mu^X(C) := [\pi_m(C)]\mathbb{L}_X^{-m\dim Y},$$

where $\pi_m(C)$ is viewed as a constructible set over X under the composite morphism

$$Y_m\to Y\to X.$$

A standard computation shows that the definition does not depend on the choice of m.

For any effective divisor D on Y, we denote by

$$\mathrm{ord}(D) : Y_\infty\to\mathbb{N}\cup\{\infty\}$$

order function along D, and set

$$\mathrm{Cont}^p(D) := \{\gamma\in Y_\infty \mid \mathrm{ord}_\gamma(D) = p\}.$$

This is a cylinder in Y_∞. Then the relative motivic integral is defined by

$$\int_{Y_\infty}\mathbb{L}_X^{-\mathrm{ord}(D)}d\mu^X := \sum_{p\ge 0}\mu^X(\mathrm{Cont}^p(D))\mathbb{L}_X^{-p}. \tag{3.1.1}$$

This gives an element in $\widehat{\mathcal{M}}_X$.

The following *change of variables formula* is a basic (but extremely useful) property of motivic integration. A proof for integration over $\operatorname{Spec}\mathbb{C}$ can be found in [DL1], and the same proof translates in the relative setting by keeping track of the morphisms to X and keeping in mind that $Y \times \mathbb{A}^1 \cong Y \times_X \mathbb{A}^1_X$ for any X-variety Y.

Theorem 3.1.2 ([Kon])**.** *Let $g : Y' \to Y$ be a proper birational map between smooth varieties over X, and let $K_{Y'/Y}$ be the relative canonical divisor of g. Let D be an effective divisor on Y. Then*

$$\int_{Y_\infty} \mathbb{L}_X^{-\operatorname{ord}(D)} d\mu^X = \int_{Y'_\infty} \mathbb{L}_X^{-\operatorname{ord}(K_{Y'/Y}+g^*D)} d\mu^X.$$

Thanks to this formula and Hironaka's resolution of singularities, one can reduce all computations to the case in which D is a simple normal crossing divisor [KoM, Theorem 0.2 and Notation 0.4]. Throughout this paper, we will use the following notation: if E_i, with $i \in J$, are the irreducible components of a simple normal crossing $\mathbb{Q}$-divisor on a smooth variety Y, then for every subset $I \subseteq J$ we write

$$E_I^0 := \begin{cases} E_I^0 = Y \setminus E & \text{if } I = \emptyset, \\ (\cap_{i\in I} E_i) \setminus (\cup_{j \in J\setminus I} E_j) & \text{otherwise.} \end{cases}$$

Now, consider a simple normal crossing effective divisor $D = \sum_{i\in J} a_i E_i$ on a smooth X-variety Y (here E_i are the irreducible components of D). Then a standard computation (identical to the analogous one over $\operatorname{Spec}\mathbb{C}$) shows that

$$\int_{Y_\infty} \mathbb{L}_X^{-\operatorname{ord}(D)} d\mu^X = \sum_{I \subseteq J} \frac{[E_I^0]}{\prod_{i\in I}[\mathbb{P}_X^{a_i}]}. \tag{3.1.2}$$

The importance of this formula is not just computational. In fact, it implies that every integral of the form (3.1.1) is an element in the image of the natural ring homomorphism

$$\rho : K_0(\mathfrak{Var}_X)[[\mathbb{P}^a_X]^{-1}]_{a\in\mathbb{N}} \to \widehat{\mathcal{M}}_X.$$

We let

$$\mathcal{N}_X := \operatorname{Im}(\rho) \subset \widehat{\mathcal{M}}_X. \tag{3.1.3}$$

All together, we have a commutative diagram

$$\begin{tikzcd} K_0(\mathfrak{Var}_X) \arrow[rr] \arrow[d] \arrow[drr, "\tau"] & & \mathcal{M}_X \arrow[d] \\ K_0(\mathfrak{Var}_X)[[\mathbb{P}^a_X]^{-1}]_{a\in\mathbb{N}} \arrow[rr, "\rho"'] & & \widehat{\mathcal{M}}_X. \end{tikzcd}$$

Note that the image of τ is contained in $\mathcal{N}_X$.

We close this section with the following remark on base change. Given a morphism $h : V \to X$ of complex varieties, we obtain a ring homomorphism

$$\psi_h : K_0(\mathfrak{Var}_X) \to K_0(\mathfrak{Var}_V), \quad [Y] \mapsto [Y \times_X V],$$

where Y is any X-variety (see [Loo, Section 4]). We have $\ker(\tau_X) \subseteq \ker(\tau_V \circ \psi_h)$, hence a commutative diagram of ring homomorphisms

$$\begin{tikzcd} K_0(\mathfrak{Var}_X) \arrow[r, "\psi_h"] \arrow[d, "\tau_X"'] & K_0(\mathfrak{Var}_V) \arrow[d, "\tau_V"] \\ \widehat{\mathcal{M}}_X \arrow[r] & \widehat{\mathcal{M}}_V. \end{tikzcd} \tag{3.1.4}$$

If h is an immersion, then for an element $\alpha \in \widehat{\mathcal{M}}_X$ we denote its image in $\widehat{\mathcal{M}}_V$ by $\alpha|_V$.

Exercise 3.1.3. Let V be an open subset of a complex variety X. Let D be an effective divisor on a smooth X-variety Y, and let $D_V := D \times_X V$ and $Y_V := Y \times_X V$. Observe that Y_V is a smooth V-variety and D_V is an effective divisor on Y_V. Then, using the commutative diagram

$$\begin{tikzcd} (Y_V)_\infty \arrow[r, hook] \arrow[d, "\sigma_m"'] & Y_\infty \arrow[d, "\pi_m"] \\ (Y_V)_m \arrow[r, hook] & Y_m, \end{tikzcd}$$

show that

$$\sigma_m(\mathrm{Cont}^p(D_V)) = \pi_m(\mathrm{Cont}^p(D)) \times_X V$$

for any $m, p \in \mathbb{N}$, hence conclude that

$$\left(\int_{Y_\infty} \mathbb{L}_X^{-\operatorname{ord}(D)} d\mu^X \right) \Big|_V = \int_{(Y_V)_\infty} \mathbb{L}_V^{-\operatorname{ord}(D_V)} d\mu^V. \tag{3.1.5}$$

3.2. **From the relative motivic ring to constructible functions.** In this section we present a natural, geometric way to read off constructible functions on a fixed variety X out of motivic integrals that are computed relative to X.

Fix a complex algebraic variety X, and let $F_*(X)$ be the group of constructible functions on X, and let $F_*(X)_{\mathbb{Q}} := F_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that, if Y and Y' are X-varieties and $x \in X$, then $(Y \times_X Y')_x = Y_x \times Y'_x$. By the multiplicativity property of the Euler characteristic (see Section 1.3), we have

$$\chi_c((Y \times_X Y')_x) = \chi_c(Y_x)\, \chi_c(Y'_x).$$

Therefore we obtain a ring homomorphism

$$\Phi_0 : K_0(\mathfrak{Var}_X) \to F_*(X), \quad [V \xrightarrow{g} X] \mapsto g_* \mathbf{1}_V.$$

Exercise 3.2.1. Use the additivity property of χ_c (see Section 1.3) to verify that Φ_0 is well-defined.

Proposition 3.2.2. *There is a unique ring homomorphism $\Phi : \mathcal{N}_X \to F_*(X)_{\mathbb{Q}}$ making*

$$\begin{tikzcd} K_0(\mathfrak{Var}_X) \arrow[r, "\Phi_0"] \arrow[d, "\tau"'] & F_*(X) \arrow[d, hook] \\ \mathcal{N}_X \arrow[r, "\Phi"] & F_*(X)_{\mathbb{Q}} \end{tikzcd}$$

a commutative diagram of ring homomorphisms.

Proof. Since $\Phi_0([\mathbb{P}^a_X]) = (a+1)\mathbf{1}_X$ is an invertible element in $F_*(X)_{\mathbb{Q}}$, Φ_0 extends, uniquely, to a ring homomorphism

$$\widetilde{\Phi} : K_0(\mathfrak{Var}_X)[[\mathbb{P}^a_X]^{-1}]_{a\in\mathbb{N}} \to F_*(X)_{\mathbb{Q}}.$$

We claim that

$$\ker(\rho) \subseteq \ker(\widetilde{\Phi}). \tag{3.2.1}$$

Let us grant this for now. We conclude that $\widetilde{\Phi}$ induces, and is uniquely determined by, a ring homomorphism $\Phi : \mathcal{N}_X \to F_*(X)_{\mathbb{Q}}$. The commutativity of the diagram in the statement is clear by the construction.

It remains to prove (3.2.1). By clearing denominators it is sufficient to show that, if α is in the kernel of τ, then $\Phi_0(\alpha) = 0$. When $X = \operatorname{Spec}\mathbb{C}$, this is proven in [DL1, (6.1)] using the degree of the Hodge-Deligne polynomial to control dimensions. In general, consider an arbitrary point $x \in X$ (we also denote by $x : \operatorname{Spec}\mathbb{C} \to X$ the corresponding morphism). By change of base, we obtain the commutative diagram (3.1.4) with $V = \{x\}$. Since $\tau(\alpha) = 0$, this gives $\tau_{\{x\}}(\psi_x(\alpha)) = 0$, hence $\chi_c(\psi_x(\alpha)) = 0$ by [DL1]. This means that $\Phi_0(\alpha)(x) = 0$. Varying x in X, we conclude that $\Phi_0(\alpha) = 0$. □

Given an effective divisor D on a smooth X-variety Y, we define

$$\Phi^X_{(Y,-D)} := \Phi\left(\int_{Y_\infty} \mathbb{L}_X^{-\operatorname{ord}(D)} d\mu^X\right) \in F_*(X)_{\mathbb{Q}}.$$

We will use the abbreviated notation $\Phi_{(Y,-D)}$ to denote this function anytime X is clear from the context. Moreover, if $D = 0$, then we write Φ^X_Y, or just Φ_Y.

Exercise 3.2.3. Does Φ "commute" with the integral? In other words, what happens if one applies the obvious extension of Φ_0, term by term, in the series defining the motivic integral before computing the sum that expresses it as an element in $\mathcal{N}_X$?

We get at once the following properties:

Proposition 3.2.4. *Consider two smooth X-varieties $Y \xrightarrow{f} X$ and $Y' \xrightarrow{f'} X$, and assume that there is a proper birational morphism $g : Y' \to Y$ over X. Let D be an effective divisor on Y. Then*

$$\Phi_{(Y,-D)} = \Phi_{(Y',-(K_{Y'/Y}+g^*D))}$$

in $F_*(X)_{\mathbb{Q}}$, where $K_{Y'/Y}$ is the relative canonical divisor of g.

Proof. Apply Φ to both sides of the formula in Theorem 3.1.2. □

Proposition 3.2.5. *If $D = \sum_{i\in J} a_i E_i$ is a simple normal crossing divisor on a smooth X-variety $Y \xrightarrow{f} X$, then*

$$\Phi_{(Y,-D)} = \sum_{I\subseteq J} \frac{f_*\mathbf{1}_{E_I^0}}{\prod_{i\in I}(a_i+1)} \tag{3.2.2}$$

in $F_*(X)_{\mathbb{Q}}$. In particular, if $D = 0$, then $\Phi_Y = f_*\mathbf{1}_Y$.

Proof. Apply Φ to both sides of (3.1.2). □

Exercise 3.2.6. Using Proposition 3.2.5, verify explicitly the identity stated in Proposition 3.2.4 when $X = Y = \mathbb{A}^2$, Y' is the blowup of $\mathbb{A}^2$ at the origin, and $D = aL$, where $L \subset A^2$ is a line through the origin and $a \in \mathbb{N}$.

Exercise 3.2.7. Let D an effective divisor on a smooth X-variety, and let V be an open subset of X. Letting $Y_V = Y \times_X V$ and $D_V = D|_{Y_V}$, prove that

$$\Phi^V_{(Y_V,-D_V)} = \Phi^X_{(Y,-D)}\big|_V. \tag{3.2.3}$$

3.3. **Constructible functions arising from klt pairs.** In the following we will use some terminology coming from the theory of singularities of pairs; standard references are [Kol, KoM]. As in the previous sections, we fix a complex variety X. The goal of this section is to generalize the construction introduced in the previous section and define a way to associate a constructible function on X to any Kawamata log-terminal pair (Y, Δ), namely, a pair consisting of a normal X-variety Y and a $\mathbb{Q}$-Weil divisor Δ on it such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier and the pair has Kawamata log-terminal singularities.

We start considering the case in which Y smooth and Δ is a simple normal crossing $\mathbb{Q}$-divisor on Y. Write

$$D := -\Delta = \sum_{i\in J} a_i E_i,$$

where E_i are the irreducible components of Δ and $a_i \in \mathbb{Q}$. Fix a positive integer r such that $ra_i \in \mathbb{Z}$ for every i, and define the ring $\widehat{\mathcal{M}}^{1/r}_X$ to be the completion of

$$K_0(\mathfrak{Var}_X)[\mathbb{L}_X^{\pm 1/r}]$$

with respect to a similar dimensional filtration as the one used in the case $r = 1$. Here $\mathbb{L}_X^{1/r}$ is a formal variable with $(\mathbb{L}_X^{1/r})^r = \mathbb{L}_X$, and we assign to it dimension $\frac{1}{r} + \dim X$. Then define

$$\int_{Y_\infty} \mathbb{L}_X^{-\operatorname{ord}(D)} d\mu^X := \sum_p \mu^X(\mathrm{Cont}^p(rD)) \cdot (\mathbb{L}_X^{1/r})^{-p}. \tag{3.3.1}$$

One can think that one is integrating $(\mathbb{L}_X^{1/r})^{-\operatorname{ord}(rD)}$ instead of $\mathbb{L}_X^{-\operatorname{ord}(D)}$. Since $\mathrm{Cont}^p(rD)$ is non-empty only for integral values of p, the summation appearing in the right hand side of (3.3.1) is taken over $\mathbb{Z}$. In fact, an explicit (and rather standard) computation shows that the summation is taken over $\mathbb{N}$ (this is not clear *a priori* because D needs not be effective), and moreover gives us the following formula for the integral:

$$\int_{Y_\infty} \mathbb{L}_X^{-\operatorname{ord}(D)} d\mu^X = \sum_{I\subseteq J} [E_I^0] \prod_{i\in I} \frac{\sum_{t=0}^{r-1} (\mathbb{L}_X^{1/r})^t}{\sum_{t=0}^{r(a_i+1)-1} (\mathbb{L}_X^{1/r})^t}. \tag{3.3.2}$$

Exercise 3.3.1. The assumption that (X, Δ) is Kawamata log-terminal is equivalent, in this case, to the expression in the right side of (3.3.2) makes sense. Why?

Exercise 3.3.2. Suppose that $X = \mathbb{A}^1$ and $E = \{0\} \subset X$, and let $D = aE$ for an arbitrary rational number $a > -1$. Then verify in this case that the series appearing in (3.3.1) indeed runs over $\mathbb{N}$, and that it converges to the corresponding expression determined by (3.3.2).

The expression (3.3.2) is an element in the image of the ring homomorphism

$$K_0(\mathfrak{Var}_X)\left[\left(\sum_{t=0}^{rd-1}(\mathbb{L}_X^{1/r})^t\right)^{-1}\right]_{d\in\frac{1}{r}\mathbb{N}^*} \longrightarrow \widehat{\mathcal{M}}_X^{1/r}.$$

Let $\mathcal{N}_X^{1/r}$ be the image of this homomorphism. The ring homomorphism $\Phi_0 : K_0(\mathfrak{Var}_X) \to F_*(X)$, defined in the previous section, extends to a ring homomorphism

$$\Phi_0 : K_0(\mathfrak{Var}_X)[\mathbb{L}_X^{1/r}] \to F_*(X)$$

by setting $\Phi_0(\mathbb{L}_X^{1/r}) = \mathbf{1}_X$. Observing that

$$\Phi_0(\sum_{t=0}^{b}(\mathbb{L}_X^{1/r})^t) = (b+1)\mathbf{1}_X,$$

we conclude (as in the proof of Proposition 3.2.2) that Φ_0 induces a ring homomorphism

$$\Phi : \mathcal{N}_X^{1/r} \to F_*(X)_{\mathbb{Q}}.$$

Note that, for every rational number $a > -1$ and any choice of r such that $ra \in \mathbb{Z}$,

$$\Phi\left(\frac{\sum_{t=0}^{r-1}(\mathbb{L}_X^{1/r})^t}{\sum_{t=0}^{r(a+1)-1}(\mathbb{L}_X^{1/r})^t}\right) = \frac{r}{r(a+1)}\mathbf{1}_X = \frac{\mathbf{1}_X}{a+1},$$

which, in particular, does not depend on the choice of r. Therefore, we can define

$$\Phi^X_{(Y,\Delta)} := \Phi\left(\int_{Y_\infty} \mathbb{L}_X^{-\operatorname{ord}(D)} d\mu^X\right) \in F_*(X)_{\mathbb{Q}}.$$

By (3.3.2) and the above discussion, this function does not depend on the choice of the integer r needed to compute it.

Remark 3.3.3. Using (3.3.2), we can extend the formula stated in (3.2.2) to this setting, namely allowing a_i to be rational numbers larger than -1.

Bearing in mind [KoM, Lemma 2.30], Proposition 3.2.4 also extends to this setting, giving us the following property.

Proposition 3.3.4. *Consider a Kawamata log-terminal pair (Y, Δ), with Y a smooth X-variety and Δ a simple normal crossing divisor. Let $g : Y' \to Y$ be a proper birational morphism such that Y' is smooth and $\Delta' := -K_{Y'/Y} + g^*\Delta$ is a simple normal crossing divisor. Then $(Y', \Delta'))$ is a Kawamata log-terminal pair, and*

$$\Phi^X_{(Y,\Delta)} = \Phi^X_{(Y',\Delta')}.$$

We are now ready to consider the general setting: we start with a Kawamata log-terminal pair (Y, Δ) over X, namely, a pair consisting of a normal X-variety Y and a $\mathbb{Q}$-Weil divisor Δ on Y such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier and the pair (Y, Δ) has Kawamata log-terminal singularities [KoM, Definition 2.34]. We can find a resolution of singularities

$$g : Y' \to Y$$

such that, if $\operatorname{Ex}(g)$ is the exceptional locus of g and $\Delta' \subset Y'$ is the proper transform of Δ, then $\operatorname{Ex}(g) \cup \Delta'$ is a simple normal crossing divisor on Y' (such a resolution is called a *log-resolution of the pair* (Y, Δ)). Fix an integer m such that $m(K_Y + \Delta)$ is Cartier. Then there exists a unique $\mathbb{Q}$-divisor Γ on Y' such that

$$\mathcal{O}_Y(m\Gamma) \cong \mathcal{O}_Y(-mK_{Y'} + g^*(m(K_Y + \Delta))) \quad \text{and} \quad \mathrm{Supp}(\Gamma + \Delta') \subseteq \mathrm{Ex}(g);$$

furthermore, Γ does not depend on the choices of K_Y and m [KoM, Section 2.3]. Note that Γ is a simple normal crossing $\mathbb{Q}$-divisor by our assumption on the resolution, and that the pair (Y', Γ) is Kawamata log-terminal [KoM, Lemma 2.30]). Thus we can define

$$\Phi^X_{(Y,\Delta)} := \Phi^X_{(Y',\Gamma)} \in F_*(X)_{\mathbb{Q}}.$$

By Proposition 3.3.4, this definition is independent of the choice of resolution. Similar abbreviations of the notation as in the previous section will be used.

Exercise 3.3.5. Let $X = \mathbb{A}^2$, let $C = \{y^2 = x^3\} \subset X$, and let $\Delta = \frac{1}{2}C$. Then compute $\Phi^X_{(X,\Delta)}$.

Lecture 4. Stringy Chern classes and their properties

4.1. **Invariance of Chern classes of K-equivalent manifolds.** It has been discovered more and more evidence that, among birational manifolds, those that are K-equivalent share many common properties and features. We recall that two manifolds X and X' are said to be *K-equivalent* if there exists a smooth variety Y and proper, birational morphisms $f : Y \to X$ and $f' : Y \to X'$ such that $K_{Y/X}$ is linearly equivalent to $K_{Y/X'}$ [Kaw].

Examples of K-equivalent manifolds are:

(a) Two manifolds related by a flop:

$$\begin{tikzcd} X \arrow[rr, dashed, "\text{flop}"] \arrow[dr, "h"'] & & X^+ \arrow[dl, "h^+"] \\ & Z. & \end{tikzcd}$$

(b) Two crepant resolutions of a normal, Gorenstein variety W:

$$K_{X/W} = 0 \qquad \begin{tikzcd} X \arrow[dr, "g"'] & & X' \arrow[dl, "g'"] \\ & W. & \end{tikzcd} \qquad K_{X'/W} = 0$$

(c) Two birationally equivalent Calabi-Yau manifolds:

$$\mathcal{O}_X(K_{Y/X}) \cong \omega_Y \qquad \begin{tikzcd} & Y \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ X \arrow[rr, dashed, "\text{bir}"] & & X'. \end{tikzcd} \qquad \mathcal{O}_X(K_{Y/X'}) \cong \omega_Y$$

In the last example we have carefully avoided the use of the symbol K_Y, as this is only defined as a divisor class, whereas it is important to regard the relative canonical divisors as divisors.

Exercise 4.1.1. Let X and X' be K-equivalent manifolds. Show that the condition on the relative canonical divisors is satisfied for every choice of Y, f and f'.

Chern classes of K-equivalent manifolds are related as follows.

Theorem 4.1.2 ([Alu1]). *Let X and X' be smooth K-equivalent varieties. Let Y be a smooth variety with proper birational morphisms $f : Y \to X$ and $f' : Y \to X'$. Then there is a class $\alpha \in A_*(Y)_{\mathbb{Q}}$ such that $f_*\alpha$ equal the image of $c(X)$ in $A_*(X)_{\mathbb{Q}}$ and $f'_*\alpha$ equals the image of $c(X')$ in $A_*(X')_{\mathbb{Q}}$.*

The notion of K-equivalence can be extended to a large class of singular varieties, namely, to normal $\mathbb{Q}$-Gorenstein varieties. We recall that a normal variety X is said to be $\mathbb{Q}$-*Gorenstein* if some multiple mK_X of the canonical divisor K_X (which is always well-defined as a Weil divisor on X) is Cartier. We will give a general definition of K-equivalence for this class of varieties in Section 4.3, and prove the above result in a more general setting (see Theorem 4.3.2 below). Under some assumptions on the morphisms f and f', we will also give an explicit formula for the class α pushing forward to the two Chern classes in terms of the discrepancies of the exceptional divisors.

A generalization of Theorem 4.1.2 to singular varieties presupposes a notion of Chern class for such varieties. It turns out that the generalizations of Chern classes discussed in the previous lectures do not behave well under K-equivalence. This is the motivation for what comes next.

Exercise 4.1.3. Give an example of a normal, Gorenstein variety X admitting a crepant resolution $f : Y \to X$, such that $f_*c_{SM}(Y) \neq c_{SM}(X)$ in $A_*(X)$ as well as in $A_*(X)_{\mathbb{Q}}$. Conclude that the Schwartz-MacPherson class cannot enjoy a similar property as the one that Chern classes of smooth varieties are stated to enjoy in Theorem 4.1.2.

4.2. **Definition and basic properties of stringy Chern classes.** We use the results of the previous lectures to define an alternative generalization of total Chern class of a singular variety, more in the spirit of the newly introduced "stringy" invariants arising from motivic integration.

Let X be a normal variety, and assume that the canonical class K_X (which, we recall, is defined as a Weil divisor) is $\mathbb{Q}$-Cartier, namely, a positive multiple mK_X of it is Cartier. Consider a resolution of singularities $f : Y \to X$ whose exceptional locus is a divisor in simple normal crossing. In this setting, the relative canonical divisor $K_{Y/X}$ of f is defined as the unique f-exceptional $\mathbb{Q}$-divisor of Y such that $mK_{Y/X}$ is Cartier and

$$\mathcal{O}_Y(mK_{Y/X}) \cong \mathcal{O}_Y(mK_Y) \otimes f^*\mathcal{O}_X(-mK_X)$$

(see Section 3.3). Then we say that X has at most *canonical* (resp. *log-terminal*) *singularities* if the divisor $K_{Y/X}$ (resp. the divisor $\lceil K_{Y/X} \rceil$) is effective. Standard arguments show that this definition is idependent of the choice of the resolution f. For short, we will say that X is a *canonical* (resp. *log-terminal*) variety to mean all the above.

Associated to an arbitrary log terminal variety X, we obtain the function

$$\Phi_X := \Phi^X_{(X,0)} = \Phi^X_{(Y,-K_{Y/X})} \in F_*(X)_{\mathbb{Q}}.$$

Then the *stringy Chern class* of X is the class

$$c_{st}(X) := c_*\Phi_X \in A_*(X)_{\mathbb{Q}},$$

where $A_*(X)_{\mathbb{Q}} := A_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. If Φ_X is a integral-valued function, then the class $c_{st}(X)$ can actually be defined in $A_*(X)$.

Exercise 4.2.1. Show that there is always an integer m (depending on X) such that

$$\Phi_X \in F_*(X) \otimes_{\mathbb{Z}} \tfrac{1}{m}\mathbb{Z},$$

and conclude that the stringy Chern class can be defined in $A_*(X) \otimes_{\mathbb{Z}} \frac{1}{m}\mathbb{Z}$.

Proposition 4.2.2. *If X admits a crepant resolution $f : Y \to X$, then $c_{st}(X) = f_* c(Y)$ in $A_*(X)$. In particular, if X is smooth, then $c_{st}(X) = c(X)$ in $A_*(X)$.*

Proof. Since $K_{Y/X} = 0$, f is proper, and Y is smooth, we have

$$c_{st}(X) = c_*\Phi_X = c_*\Phi_Y^X = c_* f_* \mathbf{1}_Y = f_* c_* \mathbf{1}_Y = f_* c(Y).$$

□

Exercise 4.2.3. Fix $k \geq 1$. Let $X \subset \mathbb{P}^3$ be the surface defined by the equation $x_0x_1x_2 = x_3^k$, and let $f : Y \to X$ be its minimal resolution of singularities. Recall that the exceptional locus of f, which is given by the fiber over the point $p = (1, 0, 0, 0) \in X$, consists of a chain of $k - 1$ rational curves. Compute $f_*\mathbf{1}_Y$, and then use MacPherson theorem to determine $c_{SM}(X)$ in terms of $f_*c(Y)$. Then compute Φ_X, hence determine the difference between $c_{SM}(X)$ and $c_{st}(X)$.

The previous exercise can be generalized as follows.

Exercise 4.2.4. Let X be a projective surface with a DuVal singularity at a point $p \in X$ and smooth elsewhere. Then, for each type of singularity (A_k, D_k, E_k), determine the difference $c_{SM}(X) - c_{st}(X)$ in $A_*(X)$.

Let X be a log-terminal variety, and let $f : Y \to X$ be a resolution of singularities such that $K_{Y/X} = \sum_i k_i E_i$ is a simple normal crossing $\mathbb{Q}$-divisor. The *stringy Euler number* of X is by definition

$$\chi_{st}(X) := \sum_{I \subseteq J} \frac{\chi_c(E_I^0)}{\prod_{i\in I}(k_i + 1)}.$$

(see [Bat2]). If X is proper, then we have

$$\int_X c_{st}(X) = \chi_{st}(X). \tag{4.2.1}$$

Exercise 4.2.5. Keeping in mind that it is only allowed to take degrees of cycles on proper varieties, prove (4.2.1).

It is possible to carry out a general theory of stringy Chern classes for pairs by defining

$$c_{st}(X, \Delta) := c_*\Phi_{(X,\Delta)} \in A_*(X)_{\mathbb{Q}}$$

for any Kawamata log-terminal pair (X, Δ).

Exercise 4.2.6. Assume that Y is a smooth variety, and let $-\Delta = \sum_{i\in J} a_i E_i$ be a simple normal crossing divisor on Y with rational coefficients $a_i > -1$. For each i, let $e_i := c_1(\mathcal{O}_Y(E_i))$. Using the adjunction formula [Ful, Example 3.2.12], show that

$$c_{st}(Y, \Delta) = \left(\prod_{i\in J} \frac{1 + \frac{1}{a_i+1} e_i}{1 + e_i} \right) \cap c(X) \tag{4.2.2}$$

in $A_*(Y)_{\mathbb{Q}}$.

Remark 4.2.7. The expression in the right side of (4.2.2) can also be obtained from the orbifold elliptic class, defined in [BL2, Definition 3.2], by taking the coefficients of a certain Laurent expansion in one of the two variables after a limiting process in the other variable. in particular, if X is a log-terminal variety and $Y \to X$ is a resolution of singularities with exceptional locus equal to a simple normal crossing divisor, then one sees by the previous observation (applied with $\Delta = -K_{Y/X}$) that the stringy Chern class of X can be reconstructed from its orbifold elliptic class.

4.3. **Invariance of stringy Chern classes of K-equivalent varieties.** Kawamata's definition of K-equivalence [Kaw, Definition 1.1] naturally extends to the class of normal varieties with $\mathbb{Q}$-Cartier canonical class, as follows. Two normal varieties X and X' whose canonical classes are $\mathbb{Q}$-Cartier are said to be *K-equivalent* if there exists a smooth variety Y and proper and birational morphisms $f : Y \to X$ and $f' : Y \to X'$ such that

$$K_{Y/X} = K_{Y/X'}$$

as divisors on Y. As in the smooth case, the definition does not depend on the choice of Y, f and f'.

It is a nice application of the negativity lemma to verify that, if the singularities are mild, then it suffices to require that the relative canonical classes be $\mathbb{Q}$-linearly equivalent (or even just numerically equivalent) in the definition of K-equivalence. This shows, in particular, that the definition of K-equivalence given here coincides with the one given in the previous section when we restrict it to manifolds.

Lemma 4.3.1. *Two canonical varieties X and X' are K-equivalent if and only if, for any common resolution Y as above, we have*

$$K_{Y/X} \equiv K_{Y/X'}.$$

Proof. We first recall the Negative Lemma [KoM, Lemma 3.39], with states that if $g : V \to Z$ is a proper birational morphism between normal varieties and $-B$ is a g-nef $\mathbb{Q}$-divisor on Y such that g_*B is effective, then B is effective.

One direction is obvious, so we can assume that $K_{Y/X} \equiv K_{Y/X'}$. Let $B = K_{Y/X'} - K_{Y/X}$. Since $B \equiv 0$, we certainly have that $-B$ is f-nef. On the other hand

$$f_*B = f_*(K_{Y/X'} - K_{Y/X}) = f_*K_{Y/X'},$$

since $K_{Y/X}$ is f-exceptional. But X is canoical, hence $f_*K_{Y/X'}$ is effective, and therefore B is effective by the negativity lemma. Repeating the argument with $-B$ in place of B, and projecting down to X', yields the effectiveness of $-B$. Therefore $B = 0$, as claimed. □

The following result extends Theorem 4.1.2.

Theorem 4.3.2 ([Alu2, dFLNU1])**.** *Let X and X' be K-equivalent log-terminal varieties, and consider a diagram*

$$\begin{tikzcd} & Y \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ X & & X', \end{tikzcd}$$

with Y a smooth variety and f and f' proper and birational morphisms. Then:

(a) *There is a class* $\alpha \in A_*(Y)_{\mathbb{Q}}$ *such that* $f_*\alpha = c_{st}(X)$ *in* $A_*(X)_{\mathbb{Q}}$ *and* $f'_*\alpha = c_{st}(X')$ *in* $A_*(X')_{\mathbb{Q}}$.

(b) *If*

$$K_{Y/X} = K_{Y/X'} = \sum_{i \in J} k_i E_i$$

is a simple normal crossing divisor (here the E_i *are the irreducible components), then* α *is the class*

$$\alpha = c_* \sum_{I \subseteq J} \frac{\mathbf{1}_{E_I^0}}{\prod_{i \in I}(k_i + 1)}.$$

Proof. By definition of K-equivalence, we have $K_{Y/X} = K_{Y/X'}$ as $\mathbb{Q}$-divisors. Let K denote this divisor. It is enough to prove the theorem assuming that K has simple normal crossings. Indeed, by further blowing up Y, we can always reduce to this case, and push-forward on Chow rings is functorial for proper morphisms. Then, defining α as in part (b) of the statement, we have

$$f_*\alpha = c_* \sum_{I \subseteq J} \frac{f_*\mathbf{1}_{E_I^0}}{\prod_{i \in I}(k_i + 1)} = c_*\Phi^X_{(Y,-K)} = c_*\Phi_X = c_{st}(X),$$

where we have applied the functoriality of c with respect to proper morphisms for the first equality, used (3.2.2) for the second one, and applied Proposition 3.2.4 for the third. Similarly, $f'_*\alpha = c_{st}(X')$. $\square$

Exercise 4.3.3. Give an explicit example to show that MacPherson class does not enjoy this invariance property under K-equivalence.

Lecture 5. McKay correspondence for stringy Chern classes of quotient varieties

5.1. **The classical McKay correspondence and the Euler orbifold number.** A striking correspondence between the representation theory of finite subgroups G of $\mathrm{SL}_2(\mathbb{C})$ and the geometry of the minimal resolution of singularities of their quotients $\mathbb{C}^2/G$ was observed by McKay. We will discuss here in some details the simplest case, namely when $G \cong \mathbb{Z}/k\mathbb{Z}$, acting on $\mathbb{C}^2$ by

$$\alpha : (u, v) \mapsto (\zeta u, \zeta^{-1} v).$$

Here α is a generator of G, and ζ is a primitive k-th root of unity.

By observing that the subring of invariant polynomials in $\mathbb{C}[u, v]$ is generated by u^k, v^k and uv, we can realize the quotient X as a subvariety of $\mathbb{C}^3$ defined by

$$xy = z^k.$$

The minimal resolution of singularities $f : Y \to X$ is a crepant resolution (that is, $K_{Y/X} = 0$), and the exceptional locus of f (the fiber over the origin) is a chain of $k - 1$ rational curves. The incidence graph of this fiber is A_{k-1}.

What is interesting to observe at this point is that G has $k - 1$ two-dimensional irreducible representations given by

$$V_i \cong \mathbb{C}^2, \quad \text{with action} \quad \alpha \mapsto \begin{pmatrix} \zeta^i & 0 \\ 0 & \zeta^{-i} \end{pmatrix} \quad \text{for } i = 1, \ldots, k-1.$$

The first one, namely V_1, corresponds to the inclusion $G \subset \mathrm{SL}_2(\mathbb{C})$. We include to this list the two-dimensional trivial representation V_0 of G. It is easy to check that

$$V_1 \otimes V_i = V_{i-1} \oplus V_{i+1} \quad \text{for every } 0 < i < k.$$

In particular each V_i is a direct summand of $V_1 \otimes V_{i-1}$ and $V_1 \otimes V_{i+1}$. The *McKay graph* of the representation V_1 of G is defined by having an vertex for each irreducible representation V_i of G, including V_0, and putting an edges V_i——V_j whenever V_i is a direct summand of $V_1 \otimes V_j$ or V_j is a direct summand of $V_1 \otimes V_i$. This graph is the extended Dynkin diagram $\widetilde{A}_{k-1}$; if we remove the trivial representation, then we obtain the incidence graph of the exceptional fiber of f.

Exercise 5.1.1. Check that $V_1 \otimes V_i = V_{i-1} \oplus V_{i+1}$ by writing down the corresponding action of G on a basis $\{e_i \otimes e'_j\}$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

As mentioned, analogous correspondence occurs between the irreducible representations of the other finite groups of $\mathrm{SL}_2(\mathbb{C})$ and the minimal resolutions of the corresponding DuVal singularities of the quotients. This is known as the *McKay correspondence.*

This correspondence has also an interpretation in terms of Euler numbers. Coming back to our example, let

$$E = E_1 \cup \cdots \cup E_{k-1}$$

be the exceptional divisor of f (we assume that the E_i are numbered progressively along the chain). Then we compute

$$\chi_c(Y) = \sum_{i=1}^{k-1} \chi_c(E_i) - \sum_{i=1}^{k-2} \chi_c(E_i \cap E_{i+1}) = k.$$

On the other hand, if we add the Euler characteristics of the quotients of the fixed point sets $(\mathbb{C}^2)^g \subseteq \mathbb{C}$, as g runs in G, we obtain

$$\sum_{g \in G} \chi_c((\mathbb{C}^2)^g/G) = \chi_c(X) + (k-1)\,\chi_c(\{0\}) = k.$$

It is not a coincidence that the two numbers are the same.

In order to generalize this correspondence, we need to modify the second computation when G is not abelian. In general, given a finite group G acting on a manifold M, we add the Euler characteristics of the quotients $M^g/C(g)$, where $M^g \subseteq M$ is the fixed locus of g, $C(g) \subseteq G$ is the centralizer of g, and g runs in a set of representatives $\mathcal{C}(G)$ of the conjugacy classes of G. This sum is what defines the *orbifold Euler number* of the pair (M, G):

$$\chi_{\mathrm{orb}}(M, G) := \sum_{g \in \mathcal{C}(G)} \chi_c(M^g/C(g)), \tag{5.1.1}$$

Exercise 5.1.2. Check that $C(g)$ does act on M^g, and that $\chi_{\mathrm{orb}}(M, G)$ does not depend on the choice of the representative g of each conjugacy class (g).

The orbifold Euler number defined in (5.1.1) was for the first time introduced by physicists Dixon, Harvey, Vafa, and Witten [DHVW]. In their original formulation, motivated by considerations in string theory, this invariant was defined for projective manifolds by

$$\chi_{\mathrm{orb}}(M,G) := \frac{1}{|G|}\sum_{gh=hg}\chi(M^{g,h}), \tag{5.1.2}$$

where the sum runs through all the (unordered) pairs of commuting elements g, h of G and $M^{g,h}$ is the set of points in M that are fixed both by g and by h. The formulation given in (5.1.1) was then deduced by Hirzebruch and Höfer [HH].

Exercise 5.1.3. For any finite group H acting on a projective manifold N, we have

$$\chi(N/H) = \frac{1}{|H|}\sum_{h\in H}\chi(N^h).$$

Using this formula, show that (5.1.1) follows from (5.1.2).

5.2. **The motivic McKay correspondence.** Let us illustrate another interesting correspondence, observed by Ito and Reid [**?**], in the special case when

$$G = \left\{ e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, g_1 = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}, g_2 = \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta \end{pmatrix} \right\} \subset \mathrm{GL}_2(\mathbb{C}),$$

where ζ is a fixed primitive 3-rd root of unity. Let $X = \mathbb{C}^2/G$ and let $f : Y \to X$ be the minimal resolution. The correspondence here is between between the two nontrivial elements g_1, g_2 of G (which, as we will see below, are "junior") and the two exceptional divisors E_1 and E_2 (which, as we know, are crepant). We can interpret this correspondence by considering the fiber over zero of $\pi : X_\infty \to X$. Note that X is defined by $xy = z^3$ in $\mathbb{A}^3$. The components of an arc

$$\gamma(t) = (x(t), y(t), z(t)) = (a_1t + a_2t^2 + \dots, b_1t + b_2t^2 + \dots, c_1t + c_2t^2 + \dots) \in \pi^{-1}(0),$$

must satisfy the relation

$$x(t)y(t) = z(t)^2.$$

Thus there are two components $\pi^{-1}(0) = C_1 \cup C_2$ given by $C_1 = \{\gamma \mid b_1 = 0\}$ and $C_1 = \{\gamma \mid a_1 = 0\}$. Let pick the following arcs:

$$\gamma_1 = (t, t^2, t) \in C_1 \quad \text{and} \quad \gamma_2 = (t^2, t, t) \in C_2.$$

By taking the square root $s = t^{1/2}$ of the parameter, we can lift these arcs to $\mathbb{C}^2$:

$$\begin{array}{ccc} \operatorname{Spec}\mathbb{C}[[s]] & \xrightarrow{\widetilde{\gamma}_i} & \mathbb{C}^2 \\ {\scriptstyle t=s^2}\downarrow & & \downarrow{\scriptstyle (x,y,z)=(u^2,v^2,uv)} \\ \operatorname{Spec}\mathbb{C}[[t]] & \xrightarrow{\gamma_i} & X. \end{array}$$

Note that $\widetilde{\gamma}_1(s) = (s, s^2)$ and $\widetilde{\gamma}_2(s) = (s^2, s)$, hence

$$\widetilde{\gamma}_1(\zeta s) = (\zeta s, \zeta^2 s^2) = g_1 \cdot \widetilde{\gamma}_1(s) \quad \text{and} \quad \widetilde{\gamma}_2(\zeta s) = (\zeta^2 s^2, \zeta s) = g_2 \cdot \widetilde{\gamma}_2(s).$$

In this way, each γ_i determines a g_i. In we let γ_i varying generically in C_i, the correspondence does not change. On the other hand, in this example the Nash correspondence that associates to each prime exceptional divisor E_i of f an irreducible component C_i is surjective. We conclude that for to each E_i there is a corresponding g_i, and conversely.

In more general setup, by studying the arc space of the quotient and stratifying this according to the conjugacy classes of the elements in the groups determined by the lifting of the arcs, Batyrev [Bat2] and Denef and Loeser [DL2] were able to compute the motivic integral determined by the relative canonical divisor of a resolution of singularities in terms of the measures of these strata and the corresponding numerical data associated to the action of the corresponding elements in the group. The formula they obtained is described next.

To fix notation, let M be a smooth quasi-projective complex variety of dimension n, and let G be a finite group with an action on M. Let $X = M/G$, with projection $p : M \to X$. We assume that there is a global non-vanishing G-invariant section of ω_M. This implies that X is a normal variety, K_X is Cartier, and $p^*\mathcal{O}_X(K_X) = \omega_M$ (e.g., see [DK, Proposition 2.3.11] and [Rei, Subsection 1.3]).

We stratify X according to the stabilizers of the points on M. For any subgroup H of G, let $M^H \subseteq M$ denote the locus of points with stabilizer equal to H, and let $X^H = p(M^H)$ (note that M^H may be different by the locus of M fixed by H).

Exercise 5.2.1. Show that the stabilizers of any two points $y, y' \in M$ lying over the same point in X are conjugated in G.

If we let H run in a set $\mathcal{S}(G)$ of representatives of conjugacy classes of subgroups of G, we obtain a stratification of X:

$$X = \bigsqcup_{H \in \mathcal{S}(G)} X^H,$$

For simplicity, we will assume that each X^H is connected.

Let $r = |G|$, and fix a primitive r-th root of unity ζ. For each $y \in M^H$ and $h \in H$, the action of h on $T_y M$ can be diagonalized as $h|_{T_y M} = \mathrm{diag}(\zeta^{a_1}, \dots, \zeta^{a_n})$, with $0 \le a_i < n$. Then we define the *age* of h at y to be the number

$$\mathrm{age}(h) := \tfrac{1}{r}(a_1, \dots, a_n)$$

(a *junior* element is an element of age 1). As long as we assume that X^H is connected, the age of h is the same at each point $h \in H$.

Consider now a resolution of singularities $f : Y \to X$. The have the following diagram:

$$\begin{tikzcd} & M \arrow[loop right, "G"] \arrow[d, "p"] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

To write the following formula, one needs a further quotient of the motivic ring. For every $x \in X$, every $\mathbb{C}^k_x := (\mathbb{C}^k \to \{x\} \hookrightarrow X)$, and every finite group L acting linearly on $\mathbb{C}^k_x$, we identify $[\mathbb{C}^k_x/L]$ with $[\mathbb{C}^k_x]$. We will denote by $\widehat{\mathcal{M}}_{X/}$ the resulting ring and by $\mathcal{N}_{X/}$ the image of $\mathcal{N}_X$ in there. Note that Φ factors through this quotient, defining a ring homomorphism $\Phi : \mathcal{N}_{X/} \to F_*(X)_{\mathbb{Q}}$.

Theorem 5.2.2 ([Bat2, DL2], see also [Rei, Loo, Yas2])**.** *For every resolution of singularities $f : Y \to X$, we have*

$$\int_{Y_\infty} \mathbb{L}_X^{-\operatorname{ord}(K_{Y/X})} d\mu^X = \sum_{H \in \mathcal{S}(G)} [X^H] \sum_{h \in \mathcal{C}(H)} \mathbb{L}_X^{\operatorname{age}(h)} \quad \text{in } \hat{\mathcal{M}}_{X/}, \tag{5.2.1}$$

where $\mathcal{C}(H)$ is a set of representatives of conjugacy classes in H.

The proof of this theorem goes beyond the purpose of these notes. Already a technical difficulty is that the arc spaces of Y and M are compared over X, and we have so far carefully avoided to perform motivic integration over arc spaces of singular varieties. In the proof, the formula in (5.2.1) breaks into two parts. The first part consists of computing the left hand side as an integral over X_∞. Since X is singular, the change of variable formula for motivic integration needs to be corrected (even when K_X is Cartier). This is done by twisting the motivic measure over X_∞ by the order of vanishing of the ideal $J \subseteq \mathcal{O}_X$ determined by the natural homomorphism

$$(\wedge^n \Omega_X^1)/\operatorname{torsion} \to \omega_X.$$

Another difficulty of working with singular varieties is that the projections $X_{m+1} \to X_m$ are not locally trivial affine bundles anymore (they may not even be surjective), so one needs to proceed cautiously with the definition of motivic measure. In any case, once things are suitably fixed, the change of variable formula gives

$$\int_{X_\infty} \mathbb{L}_X^{\operatorname{ord}(J)} d\mu^X = \int_{Y_\infty} \mathbb{L}_X^{-\operatorname{ord}(K_{Y/X})} d\mu^X \quad \text{in } \hat{\mathcal{M}}_X.$$

The core of the proof is then to show that

$$\int_{X_\infty} \mathbb{L}_X^{\operatorname{ord}(J)} d\mu^X = \sum_{H \in \mathcal{S}(G)} [X^H] \sum_{h \in \mathcal{C}(H)} \mathbb{L}_X^{\operatorname{age}(h)} \quad \text{in } \hat{\mathcal{M}}_{X/}.$$

The ages appear by stratifying

$$\pi^{-1}(X^H) = \bigsqcup_{h \in \mathcal{C}(H)} X_\infty^{H,h}$$

and computing the contribution that each of this piece brings to the integral. We refer to [DL2] for details.

5.3. **Stringy Chern classes of quotient varieties.** Consider the notation as in the previous section, so that M is a smooth quasi-projective complex variety of dimension d, G is a finite group with an action on M such that there is a global non-vanishing G-invariant section of ω_M, and $X = M/G$, with projection $p : M \to X$.

Recall that we have fixed a set of representatives $\mathcal{S}(G)$ of subgroups of G. For every $g \in \mathcal{C}(G)$, consider the fixed-point set $M^g \subseteq M$. There is a commutative diagram

$$\begin{tikzcd} M^g \arrow[r, hook] \arrow[d] & M \arrow[d, "p"] \\ M^g/C(g) \arrow[r, "p_g"] & X, \end{tikzcd}$$

where p_g is a proper morphism.

Warning 5.3.1. Given a finite group H acting on a X-variety V, we will denote by $[V/H]$ the element in the Grothendieck ring $K_0(\mathfrak{Var}_X)$ (or the element that this induces in $\widehat{\mathcal{M}}_X$) determined by the quotient variety V/H. We warn the reader that same notation is commonly used in literature to denote the Deligne-Mumford stack associated to the action of H on V.

Exercise 5.3.2. Check that $[M^g/C(g)]$, as an element in $K_0(\mathfrak{Var}_X)$, is independent of the representative g chosen for its conjugacy class in G.

Theorem 5.3.3 ([dFLNU1]). *With the notation as in the beginning of this section, we have*

$$\Phi_X = \sum_{g \in \mathcal{C}(G)} p_{g*} \mathbf{1}_{M^g/C(g)} \quad \text{in } F_*(X). \tag{5.3.1}$$

Sketch of the proof. We introduce the following notation. For each subgroup H of G we fix a set $\mathcal{C}(H)$ of representatives of conjugacy classes of elements of H. For each element $g \in G$ and each subgroup $L \subseteq G$ containing g, we denote by $C_L(g)$ the centralizer of g in L (if $L = G$, then we just write $C(g)$) and by $N_L(g)$ the normalizer of g in L.

Applying Φ to both sides of (5.2.1), we obtain

$$\Phi_X = \sum_{H \in \mathcal{S}(G)} |\mathcal{C}(H)| \cdot \mathbf{1}_{X^H} \quad \text{in } F_*(X). \tag{5.3.2}$$

In order to show that the right hand side of (5.3.2) is equal to that of (5.3.1), we start by observing that

$$\sum_{g \in \mathcal{C}(G)} [M^g/C(g)] = \sum_{H \in \mathcal{S}(G)} \left(\sum_{h \in \mathcal{C}(N_H) \cap H} [M^H/C_{N_H}(h)] \right) \quad \text{in } \hat{\mathcal{M}}_X.$$

This identity follows from certain formal identities in the Grothendieck ring of Deligne-Mumford stacks over X, and a natural homomorphism from this ring to $K_0(\mathfrak{Var}_X)$. For this step, we address the interested reader to [dFLNU1, Section 5], warning him of the different notation adopted there to denote the elements in the Grothendieck ring.

Then the proof boils down to show that, for every $H \in \mathcal{S}(G)$ and $h \in H$, there is an étale morphism

$$\nu_{H,h} : M^H/C_{N_H}(h) \to X^H$$

commuting with the other various quotient maps, and that

$$\sum_{h \in \mathcal{C}(N_H) \cap H} \deg \nu_{H,h} = |\mathcal{C}(H)|.$$

The details of this computation are contained in [dFLNU1, Section 6]. $\square$

Exercise 5.3.4. Note that for every open subset $V \subseteq X$, the restriction of f to $Y_V := f^{-1}(V)$ is a resolution of singularities of V, and that $K_{Y/X}|_{Y_V} = K_{Y_V/V}$. Then show that Theorem 5.3.3 can be extended to the situation in which we only assume that K_X is Cartier and $p^*\mathcal{O}_X(K_X) = \omega_M$, rather than requiring that ω_M admits a nowhere vanishing G-invariant section.

We can deduce from Theorem 5.3.3 formulas computing the stringy Euler number and the stringy Chern classes of quotient varieties.

Theorem 5.3.5 ([Bat2])**.** *With the notation as in the beginning of this section, we have*

$$\chi_{st}(X) = \chi_{\mathrm{orb}}(M, G).$$

In particular, the orbifold Euler characteristic of (M, G) is equal to the ordinary Euler characteristic of the resolution of M/G, if the latter is crepant.

Proof. Apply $g_* : F_*(X) \to F_*(\mathrm{pt}) \cong \mathbb{Z}$ to both sides of (5.3.1). □

Theorem 5.3.6 ([dFLNU1])**.** *With the notation as in the beginning of this section, we have*

$$c_{st}(X) = \sum_{g \in \mathcal{C}(G)} p_{g*} c_{SM}(M^g/C(g))$$

in $A_*(X)$.

Proof. Applying $c : F_*(X) \to A_*(X)$ to the first and last members of the formula in Theorem 5.3.3, we obtain

$$c_{st}(X) = \sum_{g \in \mathcal{C}(G)} c_* p_{g*} \mathbf{1}_{M^g/C(g)}.$$

Hence the statement follows by recalling that c_* commutes with p_{g*}. □

Remark 5.3.7. We remark that in general the quotients $M^g/C(g)$ may have several irreducible components, but the construction of MacPherson extends to this case, hence there is no problem in defining the Chern-Schwartz-MacPherson class of $M^g/C(g)$.

Exercise 5.3.8. Verify the formula in Theorem 5.3.6 for each finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, using the computation of $c_{st}(X) - c_{SM}(X)$ done in Exercise 4.2.4.

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