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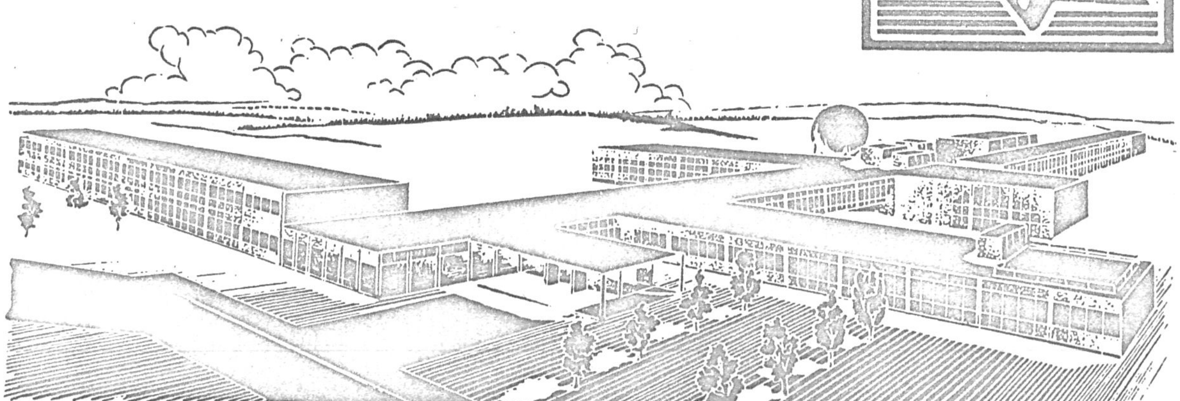

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FOREWORD

This interim report presents the results of an investigation conducted during the period September 1960 through November 1961 by members of the Fracture Mechanics Group in the Firestone Flight Sciences Laboratory of the Graduate Aeronautical Laboratories of the California Institute of Technology (GALCIT). Dr. D. D. Ang is currently Chairman of the Department of Mathematics, University of Saigon, South Viet-Nam. The work was performed for the Aeronautical Research Laboratory under Contract No. AF 33(616)-7806, "Research on Mechanics of Crack Initiation," Task No. 70524, "Structures Research at Elevated Temperatures" of Project No. 7063, "Mechanics of Flight." The notes and data for this report are recorded in GALCIT File No. SM 62-4.

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ABSTRACT

The behavior of a thin sheet having initial curvature is shown to be associated with that of an initially flat plate resting upon an elastic foundation of modulus k. Classical Kirchhoff bending solutions for a normally loaded elastically supported flat plate containing a semi-infinite straight crack are obtained using an integral equation formulation. The explicit nature of the stresses near the crack point is found to depend upon the inverse half power of the non-dimensional distance from the point, $r/(D/k)^{1/4}$, where D is the flexural rigidity of the plate and k the foundation modulus. The particular case of an infinite strip containing the crack along the negative x-axis and loaded by constant moments M* along $y = \pm y^*$ is presented to illustrate this part of the solution. The inverse half-power decay of stress is additionally damped by an exponential factor of the form $\exp(-\lambda y^*/\sqrt{2})$, where $\lambda^4 \equiv 12(1-\nu^2)/R^2h^2$. For the case of a spherical cap of radius R, the modulus $k = Eh/R^2$ and the bending stress singularity is proportional to $(r/\sqrt{Rh})^{-1/2}$.

The coupled Reissner equations are then solved for the in-plane stresses to complete the solution. The character of the combined extension-bending stress field near the crack tip is investigated for the special situation of a radial crack in a spherical cap which is subjected to normal uniform pressure, q_o, and constant membrane stress, N_o, at its simply supported boundary, r_o. Pending a complete study of the solution, an approximate result for the combined surface stress near the crack tip normal to the line of crack prolongation, in terms of the nominal membrane stress (Rq/2h), is of the form

$$\frac{\sigma_y(\varepsilon, 0)}{(Rq_o/2h)} = \frac{1}{\sqrt{\pi}\,\sqrt[8]{12(1-\nu^2)}}\;\frac{1}{\sqrt{\varepsilon/\sqrt{Rh}}}\left\{1 - \frac{1}{2}\left(1 - \frac{2N_o}{q_o R}\right) H(\lambda r_o)\right\} + \cdots$$

where $H(\lambda r_o)$ is a known function.

TABLE OF CONTENTS

I. INTRODUCTION

One of the problems which has elicited continual interest in the field of fracture mechanics is that related to the possible connection between the behavior of curved and flat sheets. As a practical matter it is obvious that there would be considerable advantage to knowing such a relation for then the test data accumulated on the flat specimens – which is significantly easier to obtain – could be used to predict the fracture of sheets of arbitrary curvature – which is more difficult to carry out experimentally.

The general theoretical problem is one of combined stress, and one inquires into the limits of validity of superimposing the separate bending and extensional stresses for flat sheets using small deflection theory, as was carried out earlier by Ang and Williams[(1)] and investigated experimentally by Swedlow and Liu[(7)]. In the present analysis the plate is assumed to possess a finite initial curvature which automatically leads to a combined stress field, that is, the curvature causes any stretching to introduce bending and vice versa. Hence superposition by itself is not in question but, as will be demonstrated, instead it turns out that the solution for increasingly large curvature does not tend toward the separate solutions for initially flat plates. This fact makes it difficult to deduce a simple factor to relate the two cases and requires further investigation, although several important features of the initial curvature situation are deduced.

The analysis is broken into several parts, paralleling the natural divisions of the problem. It will be shown that the effect of initial curvature is qualitatively equivalent to providing an elastic foundation for an initially flat plate such that as the radius of curvature increases, the foundation modulus becomes weaker and weaker. The mathematical difficulty at present pertains to showing that as the weak spring actually vanishes, the initially flat plate solutions are obtained as a limit case. It is also interesting to note that the analogy between curvature and foundation modulus is not new, leading to the well known discontinuity stress in a circular cylinder with internal pressure. This association has also been demonstrated experimentally by Sechler and Williams[(6)] for a cylinder containing a crack and hence provides some motivation for the present study.

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II. FORMULATION OF THE PROBLEM

Consider the deflection and stress situation in a thin flat plate supported by an elastic foundation and governed by classical equation of Kirchhoff plate bending, namely

$$D \nabla^4 w(x, y) + k\, w(x, y) = q(x, y) \tag{1}$$

where the flexural rigidity is $D = Eh^3/12\,(1-\nu^2)$ and the foundation modulus is k. For the time being restrict the attention to homogeneous solutions of (1) which can be taken as the sum of two solutions of the homogeneous equations.

$$(\nabla^2 \pm i\sqrt{k/D})\, w(x, y) = 0 \tag{2}$$

Denoting these solutions as w_1 and w_2, and defining $\lambda^4 \equiv k/D$, construct the representations

$$W_1(x, y^{\pm}) = \int_\gamma \{P_1(s) \mp Q_1(s)\} \exp\left[-\lambda(s^2-\alpha^2)^{1/2}|y| + i\lambda s x\right] ds \tag{3}$$

$$W_2(x, y^{\pm}) = \int_\gamma \{P_2(s) \mp Q_2(s)\} \exp\left[-\lambda(s^2-\beta^2)^{1/2}|y| + i\lambda s x\right] ds \tag{4}$$

where $\alpha = (i)^{1/2}$ and $\beta = (-i)^{1/2}$, with positive real parts of the roots being taken.

Suppressing for the moment a definition of the path γ, consider the specific situation resulting when there exists a crack along the negative real axis of the elastically supported plate. One must require that the moment and equivalent shear vanish. Suppose however that one has already found a particular solution to (1) which is satisfactory except that there is a residual moment, M_y, and equivalent shear, V_y, along the negative real axis, $x < 0$, of the general Fourier type, say for a particular term

$$M_y^{(P)} = -D\, m_o\, e^{i a x} \tag{5}$$

$$V_y^{(P)} = -D\, v_o\, e^{i a x} \tag{6}$$

where m_o and v_o are complex constants. Hence the homogeneous solution, providing it satisfies certain physical conditions far from the crack will be required to equal the negative of (5) - (6) along $x < 0$, i.e.

$$M_y(x,0) = -D\left[\frac{\partial^2}{\partial y^2} + \nu \frac{\partial^2}{\partial x^2}\right](w_1 + w_2) = D\, m_o\, e^{i a x} \quad ; \quad x < 0 \tag{7}$$

$$V_y(x,0) = -D\left[\frac{\partial^3}{\partial y^3} + (2-\nu)\frac{\partial^3}{\partial x^2 \partial y}\right](w_1 + w_2) = D\, v_o\, e^{i a x} \quad ; \quad x < 0 \tag{8}$$

Assuming that the integrals (3) - (4) can be differentiated under the integral sign and defining $\nu_o = 1 - \nu$, (7) - (8) are equivalent to

$$\int_\gamma \left\{(P_1 \mp Q_1)(\nu_o s^2 - \alpha^2) + (P_2 \mp Q_2)(\nu_o s^2 - \beta^2)\right\} \exp i\lambda s x \, ds = -\frac{m_o}{\lambda^2}\, e^{i a x} \tag{9}$$

$$\pm \int_\gamma \left\{(P_1 \mp Q_1)(s^2 - \alpha^2)^{1/2}(\nu_o s^2 + \alpha^2) + (P_2 \mp Q_2)(s^2 - \beta^2)^{1/2}(\nu_o s^2 + \beta^2)\right\} \exp i\lambda s x \, ds = \tag{10}$$

$$= -\frac{v_o}{\lambda^3}\, e^{i a x}$$

which must hold along the crack, $x < 0$. On the other hand, for $x > 0$ the deflection and its derivatives must be continuous across $y = 0$. The conditions

$$\lim_{y\to 0}\left[\frac{\partial^n}{\partial y^n}\left(w_1^+ + w_2^+\right) - \frac{\partial^n}{\partial y^n}\left(w_1^- + w_2^-\right)\right] = 0 \quad ; \quad n = 0,1,2,3 \tag{11}$$

may all be satisfied by taking, for $x > 0$

$$\int_\gamma Q_1 \exp i\lambda s x \, ds = 0 \quad ; \quad \int_\gamma Q_2 \exp i\lambda s x \, ds = 0 \tag{12}$$

$$\int_\gamma (s^2-\alpha^2)^{1/2} P_1 \exp i\lambda s x \, ds \quad ; \quad \int_\gamma (s^2-\beta^2)^{1/2} P_2 \exp i\lambda s x \, ds = 0 \tag{13}$$

Proceeding with the construction, arbitrarily let the following combinations in (9) and (10) vanish,

$$\int_\gamma \left\{ (\nu_0 s^2-\alpha^2)\, Q_1 + (\nu_0 s^2 - \beta^2)\, Q_2 \right\} \exp i\lambda s x \, ds = 0 \quad ; \quad x < 0 \tag{14}$$

$$\int_\gamma \left\{ (\nu_0 s^2+\alpha^2)(s^2-\alpha^2)^{1/2} P_1 + (\nu_0 s^2+\beta^2)(s^2-\beta^2)^{1/2} P_2 \right\} \exp i\lambda s x \, ds = 0 \quad ; \quad x < 0 \tag{15}$$

which are evidentally satisfied by taking

$$Q_1 = -(\nu_0 s^2 - \beta^2) Q(s) \tag{16}$$

$$Q_2 = (\nu_0 s^2 - \alpha^2) Q(s) \tag{17}$$

$$P_1 = -(\nu_o s^2 + \beta^2)(s^2 - \beta^2)^{1/2}\, P(s) \tag{18}$$

$$P_2 = (\nu_o s^2 + \alpha^2)(s^2 - \alpha^2)^{1/2}\, P(s) \tag{19}$$

where P(s) and Q(s) are new still largely arbitrary functions, leaving in (9) and (10)

$$\int_\gamma \left\{ (\nu_o s^2 - \alpha^2) P_1 + (\nu_o s^2 - \beta^2) P_2 \right\} \exp i\lambda s x \, ds = -\frac{m_o}{\lambda^2} e^{iax} \quad ; \quad x < 0 \tag{20}$$

$$-\int_\gamma \left\{ (s^2 - \alpha^2)^{1/2} (\nu_o s^2 + \alpha^2) Q_1 + (s^2 - \beta^2)^{1/2} (\nu_o s^2 + \beta^2) Q_2 \right\} \exp i\lambda s x \, ds = \dot{-}\, \frac{v_o}{\lambda^3} e^{iax} \quad ; \quad x < 0 \tag{21}$$

which using the new functions P(s), Q(s) from (16) - (19) reduce respectively to

$$\int_\gamma K(s)\, P(s) \exp i\lambda s x \, ds = -\frac{m_o}{\lambda^2} e^{iax} \quad ; \quad x < 0 \tag{22}$$

$$\int_\gamma K(s)\, Q(s) \exp i\lambda s x \, ds = \overset{+}{\underset{?}{-}} \left(-\frac{v_o}{\lambda^3} \right) e^{iax} \quad ; \quad x < 0 \tag{23}$$

where the kernel is

$$K(s) = (s^2 - i)^{1/2} (\nu_o s^2 + i)^2 - (s^2 + i)^{1/2} (\nu_o s^2 - i)^2 \tag{24}$$

$$= (s^2 - \alpha^2)^{1/2} (\nu_o s^2 + \alpha^2)^2 - (s^2 - \beta^2)^{1/2} (\nu_o s^2 + \beta^2)^2 \tag{24a}$$

Returning to the conditions of continuity across $y = 0$ for $x > 0$, introduce (16) - (19) into (12) and (13) to find

$$-\int_{\gamma} (\nu_0 s^2+i)\, Q(s) \exp i\lambda s x\, ds = 0 \quad ; \quad \int_{\gamma} (\nu_0 s^2-i)\, Q(s) \exp i\lambda s x\, ds = 0 \tag{25}$$

$$-\int_{\gamma} (s^4+1)^{1/2} (\nu_0 s^2-i)\, P(s) \exp i\lambda s x\, ds = 0 \quad ; \quad \int_{\gamma} (s^4+1)^{1/2} (\nu_0 s^2+i)\, P(s) \exp i\lambda s x\, ds = 0 \tag{26}$$

which can be satisfied by setting

$$\int_{\gamma} Q(s) \exp i\lambda s x\, ds = 0 \quad ; \quad x > 0 \tag{27}$$

$$\int_{\gamma} (s^4+1)^{1/2} P(s) \exp i\lambda s x\, ds = 0 \quad ; \quad x > 0 \tag{28}$$

taking into account that the second derivatives of (27) and (28) with respect to x are also zero.

Equations (22), (23) and (27), (28) are therefore the dual integral equations to be solved for the unknown functions $P(s)$ and $Q(s)$, which, when substituted into (16) - (19) and subsequently into (3) and (4) along with the particular solution producing (5) and (6), give the deflection function which satisfies the Kirchhoff conditions for a free edge along the crack of the elastically supported plate.

SOLUTION OF THE INTEGRAL EQUATIONS

First of all the path of integration, γ, is taken along the real axis except at the point $s = a/\lambda$ which is circled from above. The functions $(s^2 - \alpha^2)^{1/2}$ and $(s^2 - \beta^2)^{1/2}$ are made single valued by introducing branch cuts as shown in Figure 1. Specifically $(s^2 - \alpha^2)^{1/2}$ leads to the insertion of branch cuts $|\mathrm{Im}\ s| = |\mathrm{Im}\ \alpha|$; with $\mathrm{Re}\ s > \mathrm{Re}\ \alpha$ for $\mathrm{Im}\ s > 0$, and $\mathrm{Re}\ s < -\mathrm{Re}\ \alpha$ for $\mathrm{Im}\ s < 0$ as shown hatched in the figure.

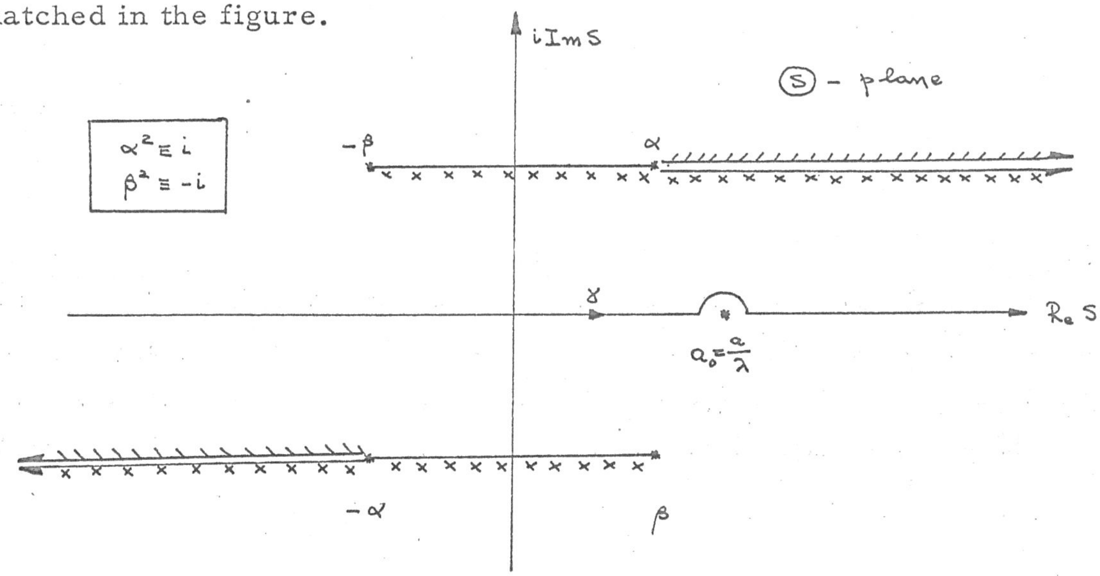

Figure 1.

Similarly, $(s^2 - \beta^2)^{1/2}$ also leads to cuts along $|\mathrm{Im}\ s| = |\mathrm{Im}\ \alpha|$, which are taken as $\mathrm{Re}\ s > -\mathrm{Re}\ \beta$ for $\mathrm{Im}\ s > 0$ and $\mathrm{Re}\ s < \mathrm{Re}\ \beta$ for $\mathrm{Im}\ s < 0$, as shown cross hatched in the figure. It develops that the portions of the branch cuts for $\mathrm{Re}\ s > \mathrm{Re}\ \alpha$ and $\mathrm{Re}\ s < -\mathrm{Re}\ \alpha$ cancel each other leaving the K(s) analytic in the entire s-plane except for two cuts in the upper and lower half planes of length $|\mathrm{Re}\ s| < |\mathrm{Re}\ \alpha|$

We consider first the equations in P(s), namely (22) and (28). It is found convenient at this stage to define an auxilliary function

$$F(s) \equiv \frac{K(s)}{i(4-\nu_0)\,\nu_0\,(s^2 - s_1^2)\,(s^2 - \alpha^2)^{1/2}} \tag{29}$$

where $\pm s_1$ are the zeros of K(s) in the first and third quadrants respectively (this matter will be elucidated in the next section). The dual equations for P(s) will be solved by an application of the theory of functions of a complex variable following Clemmow (Ref. 2). Thus, the equation for P(s) of (22) is satisfied if:

$$i(4-\nu_0)\,\nu_0\,(s^2-s_1^2)(s^2-\alpha^2)^{1/2}\,F(s)\,P(s) = \frac{m_0}{2\pi i\lambda^2}\;\frac{L(s)}{L(a_0)}\;\frac{1}{s-a_0} \tag{30}$$

where

$$a_0 \equiv \frac{a}{\lambda}$$

and L(s) is a function free from zeros and singularities in the lower half of the s-plane inclusive of γ, and furthermore of algebraic behavior at infinity. That (30) solves (22) results from Jordan's Lemma and the theorem of residues. By the same argument (28) is satisfied if

$$(s^4+1)^{1/2}\;P(s) = U(s) \tag{31}$$

where U(s) is the counter part of L(s) in the upper half-plane. Eliminating P(s) from (30)-(31) and after some rearrangement we obtain:

$$\frac{U(s)}{L(s)} = -\frac{m_0}{2\pi(4-\nu_0)\,\nu_0\lambda^2}\;\frac{1}{L(a_0)}\left\{\frac{(s-\beta)^{1/2}}{(s-a_0)(s+s_1)\,F_U(s)}\right\}\left\{\frac{(s+\beta)^{1/2}}{(s-s_1)\;F_L(s)}\right\} \tag{32}$$

where $F_{U,\,L}(s)$ are respectively U-type and an L-type functions, and such that

$$F_U(s)\cdot F_L(s) = F(s) \tag{33}$$

(This latter factorization of (29) will be carried out in the next section) A solution of (32) is

$$L(s) = \frac{(s-s_1)\,F_L(s)}{(s+\beta)^{1/2}} \tag{34}$$

$$U(s) = -\frac{m_0}{2\pi(4-\nu_0)\,\nu_0\lambda^2}\left[\frac{(a_0+\beta)^{1/2}}{(a_0-s_1)\,F_L(a_0)}\right]\frac{(s-\beta)^{1/2}}{(s-a_0)}\;\frac{1}{(s+s_1)\,F_u(s)} \tag{35}$$

Where the bracketted term follows from (34) evaluated at $s = a_0$. It follows then, from (31) and (35), that P(s) is given by:

$$P(s) = -\frac{m_0}{2\pi(4-\nu_0)\,\nu_0\lambda^2}\left[\frac{(a_0+\beta)^{1/2}}{(a_0-s_1)\,F_L(a_0)}\right]\left[\frac{(s-\beta)^{1/2}}{(s-a_0)\,(s+s_1)\,(s^4+1)^{1/2}\,F_u(s)}\right] \tag{36}$$

Next, by following exactly the same steps as above, we find for Q(s):

$$Q(s) = \frac{-\,v_0}{2\pi(4-\nu_0)\,\nu_0\lambda^3}\;\frac{1}{(a_0-s_1)\,F_L(a_0)\,(a_0-\alpha)^{1/2}}\left\{\frac{1}{(s-a_0)\,(s+s_1)\,(s+\alpha)^{1/2}\,F_u(s)}\right\} \tag{37}$$

So that finally $P_{1,2}(s)$ and $Q_{1,2}(s)$ can be deduced from (16) - (19), and the problem is formally solved. The practical matter of determining the factorization of F(s) as implied by (33) will now be described.

A FACTORIZATION OF THE KERNEL FUNCTION F(s)

It is proposed to split F(s) as defined in (29) into a product of a U- and an L- type function, i. e.

$$F(s) = F_u(s) \cdot F_L(s) = \frac{K(s)}{i\,(4-\nu_0)\,\nu_0\,(s^2-s_1^2)\,(s^2-\alpha^2)^{1/2}}$$

Define first

$$G(s) = \ln F(s) \tag{38}$$

where for definiteness the logarithm is taken as a principal value. We decompose G(s) into the sum of a U- and an L- type function

$$G(s) = G_u(s) + G_L(s) = \ln F_u(s) + \ln F_L(s) \tag{39}$$

from which the product factorization follows immediately. The decomposition (39) can be accomplished once we know the singularities of G(s) — which are the singularities of F(s), namely the branch cuts $|\mathrm{Im}\, s| = |\mathrm{Im}\,\alpha|$, $-\mathrm{Re}\,\beta \leq \mathrm{Re}\, s \leq \mathrm{Re}\,\alpha$.

To find the zeros of F(s), we rationalize the equation K(s) = 0, using (24), obtaining a quadratic equation in s^4. Of the eight roots of this rationalized equation, only four satisfy the original equation K(s) = 0 , namely:

$$s = \pm s_1 = \pm e^{i\frac{\pi}{4}} \left\{ \frac{3\nu_0 - 2 + 2\left[2\nu_0^2 - 2\nu_0 + 1\right]^{1/2}}{(4-\nu_0)\,\nu_0^2} \right\}^{1/4} \tag{40a}$$

$$s = \pm s_2 = \pm e^{i\frac{3\pi}{4}} \, |s_1| \tag{40b}$$

From the way it was defined in (29) however, F(s) has only the two zeros in the second and fourth quadrants, namely

$$s = \pm s_2 \approx \pm (1.0004) \exp\left(\frac{3i\pi}{4}\right) \qquad \text{for} \qquad \nu_0 = \frac{3}{4} \quad \text{or} \quad \nu = \frac{1}{4}$$

where it may be noted that

$$1.000 \leq |s_1| \leq \sqrt[4]{1.0448}$$

because of the physical restrictions on the value of Poisson's ratio.

Thus, G(s) has a strip of regularity namely, $|\text{Im } s| < |\text{Im } \alpha|$. Since F(s) was defined in (29) to include the proper constant such that its asymptotic value for large s gives unity upon expansion of (29), i. e.

$$F(s) = 1 + O\left(\frac{1}{s^2}\right) \qquad \text{for} \quad |s| \to \infty \tag{41}$$

it follows that for s belonging to the strip of regularity, Cauchy's integral formula gives:

$$G(s) = -\frac{1}{2\pi i}\int_{-\infty+i\eta}^{\infty+i\eta} \frac{G(z)}{z-s}\, dz + \frac{1}{2\pi i}\int_{-\infty-i\delta}^{\infty-i\delta} \frac{G(z)}{z-s}\, dz \tag{42}$$

where $\text{Im } s < \eta < \text{Im } \alpha \;,\; -\text{Im } \alpha < -\delta < \text{Im } s.$
The first integral of (42) is identified with $G_L(s)$, the second integral with $G_U(s)$. We shall put $G_U(s)$ in a form convenient for numerical evaluation by a deformation of the path of integration. The function G(z) is made single - valued in the lower half plane by introducing a cut for the logarithmic singularity (corresponding to the zero $z = -s_2$) in addition to the cut for the function F(z) itself. The cut for the logarithmic

singularity is conveniently defined as a semi-infinite line drawn through $z = -s_2$ parallel to the real axis in the negative direction.

We wish first to evaluate $G_U(s)$ for s in the upper half-plane and then continue it analytically to the whole s-plane. For s in the upper half plane the path of integration can be deformed into the real axis:

$$G_u(s) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{G(z)}{z-s}\, dz \tag{43}$$

Since G(z) is an even function in z, hence (43) can be transformed into the more convenient form

$$G_u(s) = \frac{s}{2\pi i} \int_{-\infty}^{\infty} \frac{G(z)}{z^2-s^2}\, dz \tag{44}$$

which is equivalent to:

$$G_u(s) = \frac{s}{2\pi i} \oint \frac{G(z)}{z^2-s^2}\, dz^{*} - \frac{s}{2\pi i} \int_{C+C'} \frac{G(z)}{z^2-s^2}\, dz - \frac{s}{2\pi i} \int_{C''+C'''} \frac{G(z)}{z^2-s^2}\, dz \tag{45}$$

where the paths c, c', c'', c''' are shown in Figure 2 on the next page

---

*

$$\frac{s}{2\pi i} \oint \frac{G(z)}{z^2-s^2}\, dz = \frac{1}{4\pi i} \oint \frac{G(z)}{z-s}\, dz - \frac{1}{4\pi i} \oint \frac{G(z)}{z+s}\, dz$$

$$= 0 + \frac{G(s)}{2}$$

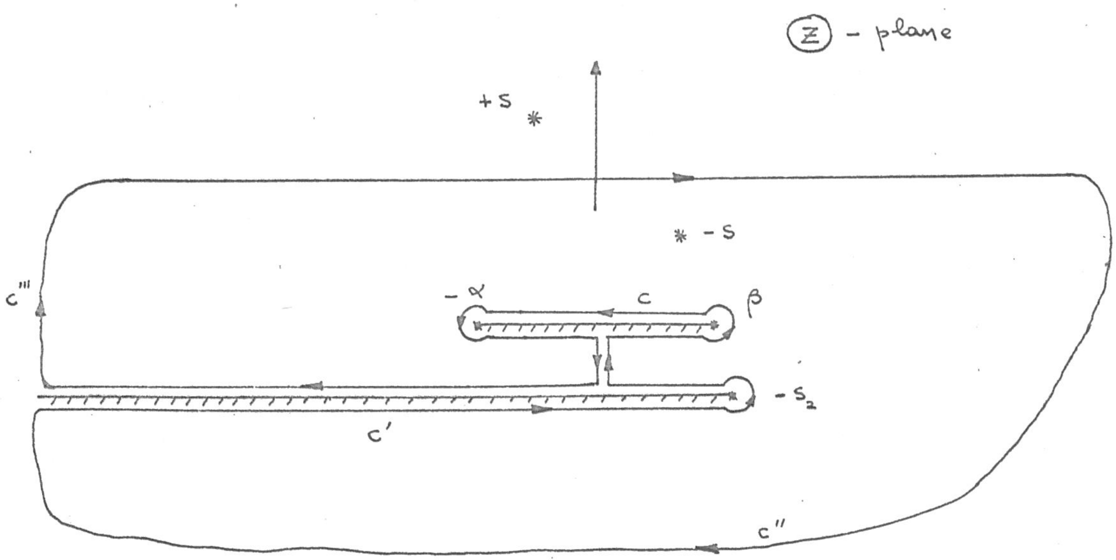

Figure 2.

hence

$$G_u(s) = \frac{G(s)}{2} - \frac{s}{2\pi i}\int_{c+c'} \frac{G(z)}{z^2-s^2}\, dz \tag{46}$$

From the properties of the logarithm we have:

$$\frac{1}{2\pi i}\int_{c'} \frac{G(z)}{z^2-s^2}\, dz = \frac{1}{2s}\ln\left[\frac{s_2-s}{s_2+s}\right]^{*} \tag{47}$$

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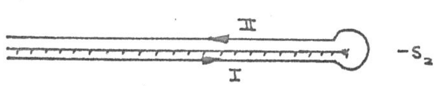


(path c')

$$I_I = \frac{1}{2\pi i}\int_{s_2}^{\infty} \frac{\ln[F(z)]_{bot}}{(s_2+\tau)^2-s^2}\, d\tau \quad ; \quad I_{II} = -\frac{1}{2\pi i}\int_{s_2}^{\infty} \frac{\ln[F(z)]_{top}}{(s_2+\tau)^2-s^2}\, d\tau$$

next recall that

$$\ln F(z) = \ln|F(z)| + i \arg F(z)$$

hence

$$-[\ln F(z)]_{top} + [\ln F(z)]_{bot.} = -2\pi i$$

etc.

Therefore,

$$F_u(s) = \exp\left\{ G_u(s) \right\} = \left\{ \frac{s_2+s}{s_2-s} F(s) \right\}^{1/2} \exp\left\{ -s\, I(s) \right\} \tag{48}$$

where

$$I(s) \equiv \frac{1}{2\pi i} \int_c \frac{G(z)}{z^2-s^2}\, dz \tag{48a}$$

In view of numerical computations, the integral (48a) is transformed into a sum of real integrals

$$2\pi i\, I(s) = \frac{1}{2} \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{\ln\left[\frac{A_-^2+B_-^2}{A_+^2+B_+^2} \right]}{\left(x-\frac{i}{\sqrt{2}}\right)^2 - s^2}\, dx + i \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{\tan^{-1}\left(\frac{B_-}{A_-}\right) - \tan^{-1}\left(\frac{B_+}{A_+}\right)}{\left(x-\frac{i}{\sqrt{2}}\right)^2 - s^2}\, dx \tag{49}$$

where

$$\begin{aligned}
&A_\pm = A_1 \pm A_2 R \pm B_2 I \qquad ; \qquad B_\pm = B_1 \pm B_2 R \mp A_2 I \\
&A_{1,2} = \nu_0\left(x^2-\tfrac{1}{2}\right)^2 - \left(\pm 1-\nu_0 x\sqrt{2}\right)^2 \qquad ; \qquad B_{1,2} = 2\nu_0\left(\pm 1-\nu_0 x\sqrt{2}\right)\left(x^2-\tfrac{1}{2}\right) \\
&R = \left[\frac{\sqrt{M^2+N^2} + M}{2} \right]^{1/2} \qquad ; \qquad I = \left[\frac{\sqrt{M^2+N^2} - M}{2} \right]^{1/2} \\
&M = \frac{(2x^2-1)(3+2x^2)}{(2x^2+1)^2+4\left(1+\sqrt{2}\,x\right)^2} \qquad ; \qquad N = \frac{4\,(2x^2-1)}{(2x^2+1)^2+4\left(1+\sqrt{2}\,x\right)^2}
\end{aligned} \tag{50}$$

We shall need the values of I(s) for large $|s|$. Its expansion gives for $|s| > 1$ * :

$$I(s) = -\sum_{n=0}^{\infty} C_n \; s^{-2(n+1)} \tag{51}$$

where

$$\begin{aligned} 2\pi i C_n &= \frac{1}{2}\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left(x - \frac{i}{\sqrt{2}}\right)^{2n} \ln\left[\frac{A_-^2 + B_-^2}{A_+^2 + B_+^2}\right] dx \; + \\ &+ i\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left(x - \frac{i}{\sqrt{2}}\right)^{2n} \left\{\tan^{-1}\frac{B_-}{A_-} - \tan^{-1}\frac{B_+}{A_+}\right\} dx \end{aligned} \tag{52}$$

In view of (51), an approximate expansion for $F_U(s)$ - c.f. (48) - is:

$$|s| \to \infty \quad : \quad F_u(s) = -i\left[1 + \frac{C_0 + S_2}{s} + O\left(\frac{1}{s^2}\right)\right] \tag{53}$$

and hence

$$|s| \to \infty \quad : \quad F_L(s) = i\left[1 - \frac{C_0 + S_2}{s} + O\left(\frac{1}{s^2}\right)\right] \tag{54}$$

---

* For future reference the reader should note that for small s

$$I(s) = \sum_{n=0} \bar{C}_n \; s^{2n}$$

where

$$2\pi i \bar{C}_n = \frac{1}{2}\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left(x - \frac{i}{\sqrt{2}}\right)^{-2n-2} \ln\left[\frac{A_-^2 + B_-^2}{A_+^2 + B_+^2}\right] dx + i\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left(x - \frac{i}{\sqrt{2}}\right)^{-2n-2} \left\{\tan^{-1}\frac{B_-}{A_-} - \tan^{-1}\frac{B_+}{A_+}\right\} dx$$

STRESS DISTRIBUTION NEAR THE CRACK POINT

From equations (16) - (19), (36), (37), (53) and (54), we see that in (3) - (4), $P_{1,2}(s)$ and $Q_{1,2}(s)$ go to zero, for $|s| \to \infty$, at least as fast as $s^{-1/2}$. Hence, the integrals (3), (4) converge and the differentiations under the integral signs are also justified, at least for $y \neq 0$. The values of the derivatives at $y = 0$, $x < 0$ can be obtained by a proper limiting process.

The stresses are given in terms of the bending deflection as

$$\tau_{xy} = -2Gz \frac{\partial^2 w}{\partial x \partial y} \tag{55}$$

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right] \tag{56}$$

$$\sigma_y = -\frac{Ez}{1-\nu^2}\left[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right] \tag{57}$$

where here z is the distance through the thickness, h, of the plate measured from the middle surface. No subsequent confusion with the complex variable will result. Then in terms of (3), (4), (16 - 19), (36), (37), (53), (54) the stresses (55) - (57) can be expressed as a linear combination of integrals of the form:

$$\int_{-\infty}^{\infty} R(s) \exp\left\{-\lambda |y| (s^2 \pm i)^{1/2} + i\lambda s x\right\} ds \tag{58}$$

where R(s) behaves for large s, i.e. $|s| \to \infty$ as:

$$R(s) = R_0\, s^{-1/2} + O(s^{-3/2})$$

and R_0 being a suitable constant. It is obvious, by changing variables $s' \sim sx$, that the stresses tend to infinity as $r^{-1/2}$ for $r \to 0$ which is characteristic of crack problems. Furthermore by expanding the integrands for large $|s|$ we can find the variation of stresses with angular position as $r \to 0$. Thus, we get expressions of the form:

$$M_n\, |s|^{-\frac{1}{2}-n} \exp\left\{-\lambda|y||s| + i\lambda x s\right\}$$

and

$$N_n\, |s|^{1/2-n}\, \lambda^{n+1}\, |y|^{n+1} \exp\left\{-\lambda|y||s| + i\lambda x s\right\}$$

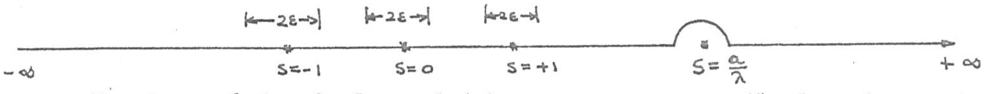
$$(n = 0, 1, 2, 3, \ldots\ldots)$$

Here, we should mention that, the exact expansion of the stress integrals should be of the form:

$$\lim_{\varepsilon \to 0} \Bigg[\int_{-\infty}^{-1-\varepsilon} (\text{expansion for large } s)\, ds + \int_{-1-\varepsilon}^{-1+\varepsilon} (\text{exp. for small } (s+1))\, ds +$$

$$+ \int_{-1+\varepsilon}^{1-\varepsilon} (\text{exp. for small } s)\, ds + \int_{1-\varepsilon}^{1+\varepsilon} (\text{exp. for small } (s-1))\, ds + \tag{58'}$$

$$+ \int_{1+\varepsilon}^{\infty} (\text{exp. for large } s)\, ds \Bigg].$$



The contribution of the 2nd and 4th terms can easily be shown to vanish. At the present we consider the singular part of the solution arising from the 1st and 5th term. Furthermore, for convenience, we integrate over $(-\infty, \infty)$ instead of $(-\infty, -1-\varepsilon)$ and $(1+\varepsilon, \infty)$ in order to express the integrals in terms of known functions, namely the Γ-functions.

It can be shown that the differences between (58) and the integrals

$$T_o(x,y) = \int_{-\infty}^{\infty} \left\{ M_o |s|^{-1/2} + N_o |s|^{1/2} \lambda |y| \right\} e^{-\lambda|y||s| + i\lambda xs} \, ds \tag{59}$$

are bounded for all r, in particular for small r, and by an appropriate deformation of the path of integration T_o (x, y) can be expanded in terms of Γ- functions. Without going into the details, we list the results for the stresses on the plate surface $z = \pm h/2$ below.

$$\tau_{xy} = \mp \frac{\Gamma(3/2)\, G h\, (1+i)\, \lambda^2 P_o}{i\sqrt{\lambda r}} \left\{ 4\left(\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2}\right) - \nu_o \cos\varphi \left(\cos\frac{3\varphi}{2} - \sin\frac{3\varphi}{2}\right) \right\}$$

$$- \frac{\Gamma(3/2)\, G h\, (1-i)\, \lambda^2 Q_o}{i\sqrt{\lambda r}} \left\{ 2(2-\nu_o)\left(\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\right) + \nu_o \cos\varphi \left(\cos\frac{3\varphi}{2} + \sin\frac{3\varphi}{2}\right) \right\} + \cdots \tag{60}$$

$$\sigma_x = - \frac{\Gamma(3/2)\, E h\, \nu_o\, (1+i)\, \lambda^2 P_o}{2i\,(1+\nu)\sqrt{\lambda r}} \left\{ 2\left(\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\right) - \cos\varphi \left(\cos\frac{3\varphi}{2} + \sin\frac{3\varphi}{2}\right) \right\}$$

$$\pm \frac{\Gamma(3/2)\, E h\, (1-i)\, \lambda^2 Q_o}{2i\,(1+\nu)\sqrt{\lambda r}} \left\{ 4(2-\nu_o)\left(\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2}\right) + \nu_o \cos\varphi \left(\cos\frac{3\varphi}{2} - \sin\frac{3\varphi}{2}\right) \right\} + \cdots \tag{61}$$

$$\sigma_y = \frac{\Gamma(3/2)\, E h\, (1+i)\, \lambda^2 P_o}{2i\,(1+\nu)\sqrt{\lambda r}} \left\{ 2(4-\nu_o)\left(\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\right) - \nu_o \cos\varphi \left(\cos\frac{3\varphi}{2} + \sin\frac{3\varphi}{2}\right) \right\}$$

$$\mp \frac{\Gamma(3/2)\, E h\, (1-i)\, \nu_o \lambda^2 Q_o}{i\,(1+\nu)\sqrt{\lambda r}} \cos\varphi \left(\cos\frac{3\varphi}{2} - \sin\frac{3\varphi}{2}\right) + \cdots \tag{62}$$

where the upper and lower signs correspond to $y > 0$ and $y < 0$ respectively, and the angular coordinate has been defined using

$$\varphi = \tan^{-1}\frac{x}{|y|} \quad ; \quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \tag{63}$$

Further, the general expression for the complex constants P_o and Q_o are deduced as

$$P_o = -\frac{i m_o (a_o+\beta)^{1/2}}{2\pi(4-\nu_o)\nu_o\lambda^2(a_o-s_1)F_L(a_o)} = -\frac{i m_o \left(\frac{a}{\lambda}+\beta\right)^{1/2}}{2\pi(4-\nu_o)\nu_o\lambda^2\left(\frac{a}{\lambda}-s_1\right)F_L\left(\frac{a}{\lambda}\right)} \quad (64)$$

$$Q_o = \frac{-i v_o}{2\pi(4-\nu_o)\nu_o\lambda^3(a_o-s_1)(a_o-\alpha)^{1/2}F_L(a_o)} = \frac{-i v_o}{2\pi(4-\nu_o)\nu_o\lambda^3\left(\frac{a}{\lambda}-s_1\right)\left(\frac{a}{\lambda}-\alpha\right)^{1/2}F_L\left(\frac{a}{\lambda}\right)} \quad (65)$$

along with certain limiting cases of interest, namely

Case i: $\lambda \to 0$ but $a \neq 0$

$$P_o = -\frac{m_o\sqrt{a\lambda}}{2\pi(4-\nu_o)\nu_o\lambda^2 a}\left\{1+\frac{s_1+s_2+c_o+\beta/2}{a}\lambda+O\left(\frac{\lambda^2}{a^2}\right)\right\} \quad (64a)$$

$$Q_o = \frac{-v_o\sqrt{\lambda}}{2\pi(4-\nu_o)\nu_o\lambda^2 a\sqrt{a}}\left\{1+\frac{s_1+s_2+c_o+\alpha/2}{a}\lambda+O\left(\frac{\lambda^2}{a^2}\right)\right\} \quad (65a)$$

Case ii:* $\lambda \neq 0$ but $a \to 0$

$$P_o = -\frac{i m_o e^{-i\pi/4}}{2\pi(4-\nu_o)\nu_o\lambda^2}\sqrt{\frac{(4-\nu_o)\nu_o}{\sqrt{2}}}\left\{1+\left(\frac{1}{2\beta}+\frac{1}{s_1}+\frac{1}{s_2}-\bar{c}_o\right)\frac{a}{\lambda}+O\left(\frac{a^2}{\lambda^2}\right)\right\} \quad (64b)$$

$$Q_o = \frac{-i v_o e^{-i\pi/4}}{2\pi(4-\nu_o)\nu_o\lambda^3}\sqrt{\frac{(4-\nu_o)\nu_o}{\sqrt{2}}}\left\{1+\left(\frac{1}{2\alpha}+\frac{1}{s_1}+\frac{1}{s_2}-\bar{c}_o\right)\frac{a}{\lambda}+O\left(\frac{a^2}{\lambda^2}\right)\right\} \quad (65b)$$

---

* For small s,

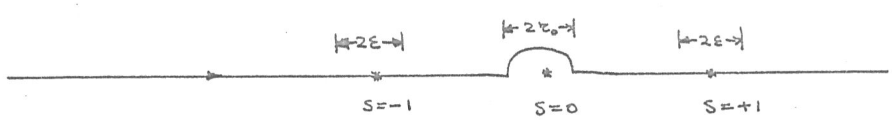
and $\sqrt{\beta} = e^{i3\pi/8}$; $\sqrt{-\alpha} = e^{i5\pi/8}$

$$F_L(s) = \xi e^{-i5\pi/8}[1+O(s)] \quad ; \quad \xi = \sqrt{\frac{\sqrt{2}}{(4-\nu_o)\nu_o|s_1|^2}} \quad (66) \; (67)$$

The expressions for the stresses given above are approximations to the first term of the homogeneous solution. To this the contribution of the particular should be added. It should be noted, that along the crack (i.e. $\varphi = -\pi/2$) equation (62) gives $\sigma_y = 0$. In addition to this we have from (5) the contribution of the particular solution namely the term:

$$+ \frac{E z m_o}{(1-\nu^2)} e^{iax}$$

It thus appears as if the sum of the σ_y contributions is not zero along the crack, which would not satisfy the boundary condition. Such a deduction however is not true because in the expansion of the stress integrals, we neglected the expansion for small s. For small s, point $s = 0$ is a pole, hence by taking the contribution around the



semi-circle we cancel half of the particular contribution. The other half must come from $\int_{-1+\varepsilon}^{-\tau_o} + \int_{\tau_o}^{1-\varepsilon}$. The integrand however is a power series with complex constants which makes an explicit analytical evaluation very difficult and computationally tedious.

III. A PARTICULAR SOLUTION

As an illustration of how the local solution may be combined in a particular case, consider a rectangular strip, infinitely long in the x - direction and of finite width y* in the y direction. Furthermore, let the plate be subjected to a constant moment M* and zero shear at $y = \pm y^*$, and simultaneously subjected to a uniform normal loading q_o. We proceed to write down a solution of (1) as follows

$$w^{(P)}(y) = \frac{q_o}{D\lambda^4} + A\cos\frac{\lambda y}{\sqrt{2}}\cosh\frac{\lambda y}{\sqrt{2}} + B\sin\frac{\lambda y}{\sqrt{2}}\sinh\frac{\lambda y}{\sqrt{2}} \tag{68}$$

and using

$$M_y^{(P)} = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right] \tag{69}$$

$$V_y^{(P)} = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right] \tag{70}$$

compute

$$-D^{-1} M_y^{(P)}(y^*) = -\lambda^2\left[A\sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right) - B\cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right)\right] \tag{71}$$

$$\begin{aligned} -D^{-1} V_y^{(P)}(y^*) = \frac{\lambda^3}{\sqrt{2}}\Big\{ & -A\left[\cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right) + \sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right)\right] \\ & - B\left[\sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right) - \cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right)\right]\Big\} \end{aligned} \tag{72}$$

Equating (71) to $-D^{-1}M^*$ and (72) to zero and solving for the constants one finds

$$A = \frac{M^*}{D\lambda^2} \; \frac{\sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right) - \cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right)}{\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right) + \cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)} \tag{73}$$

$$B = -\frac{M^*}{D\lambda^2} \; \frac{\cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right) + \sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right)}{\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right) + \sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)} \tag{74}$$

along the crack, $y=0$, it is found that

$$M_y^{(P)}(x,0) = M^* \; \frac{\cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right) + \sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cosh\left(\lambda\frac{y^*}{\sqrt{2}}\right)}{\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right) + \sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)} \tag{75}$$

$$V_y^{(P)}(x,0) = 0 \tag{76}$$

from which, comparison with (5) and (6) indicates that

$$-D\cdot\left(m_{0r} + i\,m_{0j}\right)\left(\cos ax + i\sin ax\right) = M^* f\left(\frac{\lambda y^*}{\sqrt{2}}\right) \tag{77}$$

$$-D\left(v_{0r} + i\,v_{0j}\right)\left(\cos ax + i\sin ax\right) = 0 \tag{78}$$

or

$$a = v_{or} = v_{oj} = m_{oj} = 0 \quad . \tag{79}$$

$$m_{or} = - \frac{M^*}{D} f\left(\frac{\lambda y^*}{\sqrt{2}}\right) \tag{80}$$

where using (74),

$$f\left(\frac{\lambda y^*}{\sqrt{2}}\right) \equiv - \frac{BD\lambda^2}{M^*} \tag{81}$$

Returning now to the stresses along the crack prolongation, for example the normal stress σ_y (x, 0), one finds using (62) and (64b) that

$$\frac{\sigma_y(x,0)}{Eh} = - \frac{\frac{M^*}{D} f\left(\frac{\lambda y^*}{\sqrt{2}}\right)}{2\sqrt[4]{2}\,\sqrt{2\pi}\,\sqrt{(3+\nu)(1-\nu)}\,(1+\nu)} \cdot \frac{1}{\sqrt{\lambda x}} + \cdots \tag{82}$$

or in terms of the stress

$$\frac{\sigma_y(x,0)}{\sigma_y^*} = - \frac{f\left(\frac{\lambda y^*}{\sqrt{2}}\right)}{\sqrt[4]{2}\,\sqrt{2\pi}} \sqrt{\frac{1-\nu}{3+\nu}} \cdot \frac{1}{\sqrt{\lambda x}} + \cdots \tag{83}$$

It is interesting to compare this result to the one obtained for the classical bending stress obtained at the end of a finite crack of half length b, (Ref. 1) when the initially flat (non-elastically supported) plate is subjected to a uniform bending moment or stress σ_y^* far from the crack — for comparative purposes here, y^* large. It was found (Ref. 1) that

$$\frac{\sigma_y(x,0)}{\sigma_y^*} = -\frac{1-\nu}{3+\nu}\cdot\frac{1}{\sqrt{2x/b}} + \cdots \approx \bigg|_{\nu=\frac{1}{3}} -\frac{0.20}{\sqrt{x/b}} \tag{84}$$

In order to obtain a fair comparison for the elastically supported plate, consider (83) normalized on the average stress through the uncracked portion, viz $\sigma_y{}^* f(\lambda y^*/\sqrt{2})$

$$\frac{\sigma_y(x,0)}{\sigma_y^* f\left(\frac{\lambda y^*}{\sqrt{2}}\right)}\Bigg|_{\nu=\frac{1}{3}} \approx -\frac{0.18}{\sqrt{\lambda x}} = -\frac{0.18}{\sqrt{x/\left(\frac{D}{k}\right)^{1/4}}} \tag{85}$$

Hence with respect to local conditions near the crack point, one conjectures that the spring constant $(D/k)^{1/4}$ of the elastically supported plate plays the same role as an effective crack length b in an unsupported plate, providing the crack is reasonably far from the boundary at y^* and is sufficiently long compared to the plate thickness.

Having disposed now of the preliminary problem of a plate on an elastic foundation, we proceed to show how this problem is intimately connected with the more complicated one of an initially curved spherical cap containing a radial crack.

IV. FORMULATION OF THE COMBINED STRESS PROBLEM FOR A SHALLOW SPHERE

Consider a portion of a thin shallow spherical shell subjected to internal pressure, $q\,(x, y)$, which contains a radial crack. Following Reissner[5], the coupled differential equations governing the bending deflection, $w\,(x, y)$, and membrane stress function, $F\,(x, y)$, are

$$-\frac{E\ell}{R}\,\nabla^2 W(x,y) + \nabla^4 F(x,y) = 0 \tag{86}$$

$$\nabla^4 w(x,y) + \frac{1}{RD}\,\nabla^2 F(x,y) = \frac{q(x,y)}{D} \tag{87}$$

where R is the initial radius of curvature of the spherical segement and $D = Eh^3/12\,(1-\nu^2)$ is the flexural rigidity.

Along the edge of the unloaded crack, $x < 0$ the classical bending boundary conditions require

$$-\frac{M_y(x,0)}{D} = \frac{\partial^2 w}{\partial y^2} + \nu\,\frac{\partial^2 w}{\partial x^2} = 0 \tag{88}$$

$$-\frac{V_y(x,0)}{D} = \frac{\partial^3 w}{\partial y^3} + (2-\nu)\,\frac{\partial^3 w}{\partial y\,\partial x^2} = 0 \tag{89}$$

and vanishing normal and tangential membrane stresses per unit length require, $x < 0$,

$$N_y(x,0) = \frac{\partial^2 F}{\partial x^2} = 0 \tag{90}$$

$$N_{xy}(x,0) = -\frac{\partial^2 F}{\partial x\,\partial y} = 0 \tag{91}$$

As we will be interested mainly in the local conditions at the crack point, we shall suppress detailed consideration of specific boundary conditions far from the crack, and merely require that they be physically reasonable.

The coupled equations may be separated by suitable multiplication of the Laplacian operator to yield

$$\nabla^4 w + \lambda^4 w = \frac{q}{D} + \Phi_1 \tag{92}$$

$$\nabla^4 F + \lambda^4 F = \Phi_2 \tag{93}$$

where Φ_1, is an arbitrary harmonic function, Φ_2 is the solution to a Laplace or Poisson equation depending upon the applied normal loading, namely $\nabla^2 \Phi_2 = \lambda^4 Rq(x, y)$, and the stiffness parameter

$$\lambda^4 \equiv \frac{12(1-\nu^2)}{R^2 h^2} \tag{94}$$

has been defined. It is now desired to solve the separated equations and their associated boundary conditions.

Within a particular solution, the bending part of the problem is identical to the problem of an initially flat cracked plate upon an elastic foundation of modulus $k = Eh/R^2$ such that $\lambda^4 = k/D$. The stress distribution in the vicinity of the crack point has been discussed, and it will now be shown that the membrane solution for the similar situation can be obtained from it by a simple association of parameters.

Following the previous solution, let $F = F_1 + F_2$ where

$$F_1(x, y^{\pm}) = \int_{\gamma} \left\{ \tilde{P}_1(s) \mp \tilde{Q}_1(s) \right\} \exp\left[-\lambda (s^2 - \alpha^2)^{1/2} |y| + i\lambda s x \right] ds \tag{95}$$

$$F_2(x, y^{\pm}) = \int_{\gamma} \left\{ \tilde{P}_2(s) \mp \tilde{Q}_2(s) \right\} \exp\left[-\lambda (s^2 - \beta^2)^{1/2} |y| + i\lambda s x \right] ds \tag{96}$$

and $\alpha = (i)^{1/2}$, $\beta = (-i)^{1/2}$ with positive real parts of the roots being taken.

If then we have a particular solution such that along $x < 0$

$$N_y^{(P)} = n_0 e^{iax} \tag{97}$$

$$N_{xy}^{(P)} = t_0 e^{iax} \tag{98}$$

where n_o and t_o are complex constants, then the homogeneous solution must equal the negative of (97) and (98) along the crack. Carrying out the derivatives of (95) and (96) according to (90) and (91) respectively, find that for $x < 0$

$$-\int_{\gamma} \lambda^2 \left[(\tilde{P}_1 \mp \tilde{Q}_1) + (\tilde{P}_2 \mp \tilde{Q}_2) \right] s^2 \exp i\lambda s x \, ds = -n_o e^{iax} \tag{99}$$

$$\pm \int_{\gamma} i\lambda^2 \left[(\tilde{P}_1 \mp \tilde{Q}_1)(s^2-\alpha^2)^{1/2} + (\tilde{P}_2 \mp \tilde{Q}_2)(s^2-\beta^2)^{1/2} \right] s \exp i\lambda s x \, ds = -t_o e^{iax} \tag{100}$$

or upon differentiating the latter with respect to x

$$\mp \int_{\gamma} \lambda^3 \left[(\tilde{P}_1 \mp \tilde{Q}_1)(s^2-\alpha^2)^{1/2} + (\tilde{P}_2 \mp \tilde{Q}_2)(s^2-\beta^2)^{1/2} \right] s^2 \exp i\lambda s x \, ds = -iat_o e^{iax} \tag{100a}$$

Continuity conditions along $x > 0$, $y = 0$ require that

$$\lim_{y \to 0} \left[\frac{\partial^n}{\partial y^n} (F_1^+ + F_2^+) - \frac{\partial^n}{\partial y^n} (F_1^- + F_2^-) \right] = 0 \quad ; \quad n = 0, 1, 2, 3. \tag{101}$$

which may be satisfied by setting for $x > 0$

$$\int_{\gamma} \tilde{Q}_1 \exp i\lambda s x \, ds = 0 \quad ; \quad \int_{\gamma} \tilde{Q}_2 \exp i\lambda s x \, ds = 0 \tag{102}$$

$$\int_{\gamma} (s^2-\alpha^2)^{1/2} \tilde{P}_1 \exp i\lambda s x \, ds = 0 \quad ; \quad \int_{\gamma} (s^2-\beta^2)^{1/2} \tilde{P}_2 \exp i\lambda s x \, ds = 0 \tag{103}$$

Equations (99) (100a), (102) and (103) are the integral equations to be solved for $\tilde{P}_i(s)$ and $\tilde{Q}_i(s)$ $(i = 1, 2)$.

At this point it may be noted that these are the same integral equations as were obtained for the elastically supported plate problem except that the integrands are slightly changed. It can be shown ** that the $\tilde{P}_i$, $\tilde{Q}_i$ can be constructed from the P_i, Q_i's as follows:

$$\text{(1) Consider } P_i \tag{104a}$$

$$\text{(2) define } \quad P_i^* \equiv \nu_0 P_i \tag{104b}$$

$$\text{(3) then } \quad \tilde{P}_i = \lim_{\nu_0 \to \infty} P_i^* \tag{104c}$$

A similar construction holds for the $\tilde{Q}_i$'s. For the present, physical restrictions on Poisson's ratio are not pertinent in taking $\nu_0 = 1-\nu \to \infty$, since ν_0 can be viewed as a parameter under the integral sign. Note also that D which is a function of ν (Poisson's ratio) has been absorbed into the parameter λ. In view of the above we can say:

$$F = \delta \lim_{\nu_0 \to \infty} W\left(P_i^*, Q_i^*\right) = \delta W\left(\tilde{P}_i, \tilde{Q}_i\right) \tag{105}$$

where δ is a constant.

** For example in the supported plate work, the moment expression can be written as

$$\lambda^2 \int_\gamma \left\{ (\nu_0 P_1 \mp \nu_0 Q_1)\left(s^2 - \frac{\alpha^2}{\nu_0}\right) + (\nu_0 P_2 \mp \nu_0 Q_2)\left(s^2 - \frac{\beta^2}{\nu_0}\right) \right\} \exp i\lambda s x \, ds = -m_0 e^{iax}$$

or upon setting $P^* = \nu_0 P_i$, $Q^* = \nu_0 Q_i$ and taking the limit for $\nu_0 \to \infty$ there results

$$\lambda^2 \int_\gamma \left\{ (\tilde{P}_1 \mp \tilde{Q}_1) + (\tilde{P}_2 \mp \tilde{Q}_2) \right\} s^2 \exp i\lambda s x \, ds = -m_0 e^{iax}$$

which is to be compared to (99).

Whence, insofar as the mathematical processes are concerned, the solution of the extensional problem may be written down from the bending solution by replacing

$$-m_o \text{ by } n_o \quad ; \quad -\nu_o \text{ by } i a t_o \tag{106}$$

Hence

$$\tilde{P}_o = \lim_{\nu_o \to \infty} \nu_o^2 P_o = \lim_{\nu_o \to \infty} \left\{ \frac{i n_o \nu_o \left(\frac{a}{\lambda} + \beta\right)^{1/2}}{2\pi (4-\nu_o) \lambda^2 \left(\frac{a}{\lambda} - s_1\right) F_L\left(\frac{a}{\lambda}\right)} \right\} \tag{107}$$

$$\tilde{Q}_o = \lim_{\nu_o \to \infty} \nu_o^2 Q_o = \lim_{\nu_o \to \infty} \left\{ \frac{a t_o \nu_o}{2\pi (4-\nu_o) \lambda^3 \left(\frac{a}{\lambda} - s_1\right) \left(\frac{a}{\lambda} - \alpha\right)^{1/2} F_L\left(\frac{a}{\lambda}\right)} \right\} \tag{108}$$

A consideration of the same two limit cases as before produces

Case i: $\lambda \to 0$ but $a \neq 0$.

$$\tilde{P}_o = -\frac{i n_o \sqrt{a\lambda}}{2\pi \lambda^2 a} \left\{ 1 + \frac{\tilde{c}_o + \beta/2}{a} \lambda + O\left(\frac{\lambda^2}{a^2}\right) \right\}^* \tag{109}$$

$$\tilde{Q}_o = \frac{t_o \sqrt{\lambda}}{2\pi \lambda^2 \sqrt{a}\, a} \left\{ 1 + \frac{\tilde{c}_o + \alpha/2}{a} \lambda + O\left(\frac{\lambda^2}{a^2}\right) \right\} \tag{110}$$

Case ii: $\lambda \neq 0$ but $a \to 0$.

In this case, note that we have two limiting processes, i.e. $\nu_o \to \infty$ and $a \to 0$. The first one implies $s_1 \to 0$, and the second implies $a_o \to 0$. A study of the related expression, e.g. (32) of the previous section, shows that the pole becomes a zero, and of third order, i.e. $L(a_o)$ should be $L''(a_o)$, and hence

$$\tilde{P}_o = -\frac{i n_o}{2\pi \lambda^2} e^{-i\pi/4} \tag{111}$$

$$\tilde{Q}_o = \frac{t_o}{2\pi \lambda^2} e^{-i\pi/4} \tag{112}$$

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* where

$$\tilde{c}_o = \lim_{\nu_o \to \infty} c_o \approx 0.027 + i\,0.022 \quad ; \quad \alpha = e^{i\pi/4} \quad ; \quad \beta = e^{i3\pi/4}$$

This method of solution therefore permits us to write down the membrane part of the solution for a shallow spherical cap providing the boundary conditions on the crack, imposed by the particular solution associated with the loading Φ_2, are expressible as components of a Fourier expansion, $\exp(ia_n x)$.

V. THE COMBINED STRESS DISTRIBUTION IN THE SHELL NEAR THE CRACK

MEMBRANE STRESSES

The membrane stresses are defined in terms of F(x, y) by

$$N_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \tag{113}$$

$$N_x = \frac{\partial^2 F}{\partial y^2} \tag{114}$$

$$N_y = \frac{\partial^2 F}{\partial x^2} \tag{115}$$

from which upon using

$$\varphi = \tan^{-1}\frac{x}{|y|} \quad ; \quad -\pi/2 \leq \varphi \leq \pi/2 \tag{116}$$

the stresses become

$$\begin{aligned} N_{xy\,shell} = &\mp \frac{i\lambda^2(1+i)\Gamma(3/2)\tilde{P}_o}{\sqrt{\lambda r}}\left\{\cos\varphi\left(\cos\frac{3\varphi}{2} - \sin\frac{3\varphi}{2}\right)\right\} \\ &- \frac{\lambda^2\Gamma(3/2)(1+i)\tilde{Q}_o}{\sqrt{\lambda r}}\left\{2\left(\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\right) - \cos\varphi\left(\cos\frac{3\varphi}{2} + \sin\frac{3\varphi}{2}\right)\right\} + \cdots \end{aligned} \tag{117}$$

$$\begin{aligned} N_{x\,shell} = &\frac{i\lambda^2\Gamma(3/2)(1+i)\tilde{P}_o}{\sqrt{\lambda r}}\left\{2\left(\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\right) - \cos\varphi\left(\cos\frac{3\varphi}{2} + \sin\frac{3\varphi}{2}\right)\right\} \\ &\mp \frac{\lambda^2\Gamma(3/2)(1+i)\tilde{Q}_o}{\sqrt{\lambda r}}\left\{4\left(\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2}\right) - \cos\varphi\left(\cos\frac{3\varphi}{2} - \sin\frac{3\varphi}{2}\right)\right\} + \cdots \end{aligned} \tag{118}$$

$$\begin{aligned} N_{y\,shell} = &\frac{i\lambda^2\Gamma(3/2)(1+i)\tilde{P}_o}{\sqrt{\lambda r}}\left\{2\left(\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\right) + \cos\varphi\left(\cos\frac{3\varphi}{2} + \sin\frac{3\varphi}{2}\right)\right\} \\ &\mp \frac{\lambda^2\Gamma(3/2)(1+i)\tilde{Q}_o}{\sqrt{\lambda r}}\left\{\cos\varphi\left(\cos\frac{3\varphi}{2} - \sin\frac{3\varphi}{2}\right)\right\} + \cdots \end{aligned} \tag{119}$$

where the upper and lower signs refer to y > 0 and y < 0 respectively.

BENDING STRESSES

In a similar fashion, the classical linear bending stresses are defined in terms of the deflection function w (x, y) by

$$\tau_{xy} = -2Gz\frac{\partial^2 w}{\partial x \partial y} \tag{120}$$

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right] \tag{121}$$

$$\sigma_y = -\frac{Ez}{1-\nu^2}\left[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right] \tag{122}$$

where z is the distance through the thickness of the plate, h. Repeating the results of the earlier section for convenience, we have for the stresses on the plate surfaces $z = \pm h/2$, and using the previously defined relations between (τ , φ) and (x, y),

$$\begin{aligned}\tau_{xy} = {} & \mp \frac{\Gamma(3/2)\, Gh\,(1+i)\,\lambda^2 P_0}{i\sqrt{\lambda\tau}}\left\{4\left(\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2}\right) - \nu_0 \cos\varphi\left(\cos\frac{3\varphi}{2} - \sin\frac{3\varphi}{2}\right)\right\} \\ & - \frac{\Gamma(3/2)\, Gh\,(1-i)\,\lambda^2 Q_0}{i\sqrt{\lambda\tau}}\left\{2(2-\nu_0)\left(\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\right) + \nu_0 \cos\varphi\left(\cos\frac{3\varphi}{2} + \sin\frac{3\varphi}{2}\right)\right\} + \cdots\end{aligned} \tag{123}$$

$$\begin{aligned}\sigma_x = {} & -\frac{\Gamma(3/2)\, Eh\,\nu_0\,(1+i)\,\lambda^2 P_0}{2i\,(1+\nu)\sqrt{\lambda\tau}}\left\{2\left(\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\right) - \cos\varphi\left(\cos\frac{3\varphi}{2} + \sin\frac{3\varphi}{2}\right)\right\} \\ & \pm \frac{\Gamma(3/2)\, Eh\,(1-i)\,\lambda^2 Q_0}{2i\,(1+\nu)\sqrt{\lambda\tau}}\left\{4(2-\nu_0)\left(\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2}\right) + \nu_0 \cos\varphi\left(\cos\frac{3\varphi}{2} - \sin\frac{3\varphi}{2}\right)\right\} + \cdots\end{aligned} \tag{124}$$

$$\begin{aligned}\sigma_y = {} & \frac{\Gamma(3/2)\, Eh\,(1+i)\,\lambda^2 P_0}{2i\,(1+\nu)\sqrt{\lambda\tau}}\left\{2(4-\nu_0)\left(\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\right) - \nu_0 \cos\varphi\left(\cos\frac{3\varphi}{2} + \sin\frac{3\varphi}{2}\right)\right\} \\ & \mp \frac{\Gamma(3/2)\, Eh\,(1-i)\,\nu_0\,\lambda^2 Q_0}{i\,(1+\nu)\sqrt{\lambda\tau}} \cos\varphi\left(\cos\frac{3\varphi}{2} - \sin\frac{3\varphi}{2}\right) + \cdots\end{aligned} \tag{125}$$

COMBINED STRESSES

In general the combined stresses will depend upon the contributions of the particular solutions reflecting the magnitude and distribution of the applied normal pressure. On the other hand the singular part of the solution, that is the terms producing infinite elastic stresses at the crack tip will depend only upon the local stresses existing along the locus of the crack before it is cut, which of course are precisely the stresses which must be removed or cancelled by the particular solutions described above in order to obtain the stress free edges as required physically. Hence the distribution of $q\ (x,\ y)$ does not — to the first order — affect the character of stress at the crack point.

ILLUSTRATIVE EXAMPLE

An obvious problem to illustrate the characteristics of the solution might be that of, say, a simply supported circular plate of radius r_o, containing a crack along $\varphi = -\pi/2$, and loaded by uniform normal pressure q_o, and a uniform radial extension $N_r(r_o) = N_o$. For this problem, Reissner[6] gives the solution of the coupled extension - bending equations for the uncracked plate as

$$W_p = C_1 \operatorname{ber}(\lambda r) + C_2 \operatorname{bei}(\lambda r) + C_5 \tag{126}$$

$$F_p = \frac{Eh}{\lambda^2 R}\left[C_1 \operatorname{bei}(\lambda r) - C_2 \operatorname{ber}(\lambda r)\right] \tag{127}$$

where

$$C_1 = C_2 \frac{\operatorname{ber}(\lambda r_o) - (1-\nu)(\lambda r_o)^{-1}\operatorname{bei}'(\lambda r_o)}{\operatorname{bei}(\lambda r_o) + (1-\nu)(\lambda r_o)^{-1}\operatorname{ber}'(\lambda r_o)} \tag{128}$$

$$C_5 = -C_2\left[\operatorname{bei}(\lambda r_o) + \frac{\operatorname{ber}(\lambda r_o) - (1-\nu)(\lambda r_o)^{-1}\operatorname{bei}'(\lambda r_o)}{\operatorname{bei}(\lambda r_o) + (1-\nu)(\lambda r_o)^{-1}\operatorname{ber}'(\lambda r_o)}\operatorname{ber}(\lambda r_o)\right] \tag{129}$$

$$C_2 = \frac{\left[\lambda r_o R/(Eh)\right]\left[N_o - \frac{1}{2} q_o R\right]}{\operatorname{bei}'(\lambda r_o)\dfrac{\operatorname{ber}(\lambda r_o) - (1-\nu)(\lambda r_o)^{-1}\operatorname{bei}'(\lambda r_o)}{\operatorname{bei}(\lambda r_o) + (1-\nu)(\lambda r_o)^{-1}\operatorname{ber}'(\lambda r_o)} - \operatorname{ber}'(\lambda r_o)} \tag{130}$$

Along any radial ray, and in particular along $\varphi = -\pi/2$, the bending and extensional shear vanish by symmetry, and the circumferential bending and stretching stresses are, respectively

$$M_{\theta}^{(P)} = \lambda^2 D \left\{ c_1 \left[\nu \, \mathrm{bei}(\lambda r) - (1-\nu)(\lambda r)^{-1} \mathrm{ber}'(\lambda r) \right] - c_2 \left[\nu \, \mathrm{ber}(\lambda r) + (1-\nu)(\lambda r)^{-1} \mathrm{bei}'(\lambda r) \right] \right\} \quad (131)$$

$$N_{\theta}^{(P)} = \frac{1}{2} q_0 R + \frac{Eh}{R} \left\{ c_1 \, \mathrm{bei}''(\lambda r) - c_2 \, \mathrm{ber}''(\lambda r) \right\} \quad (132)$$

The homogeneous solutions must therefore negate these values from the particular solution and, upon expanding in the Fourier series to obtain the typical trigonometric loading components, there would result

$$M_{\theta}^{(P)}\left(r, -\frac{\pi}{2}\right) = -D \sum_{n} m_0^{(n)} e^{i a_n r} \Big]_{r=x} \quad (133)$$

$$N_{\theta}^{(P)}\left(r, -\frac{\pi}{2}\right) = \sum_{n} n_0^{(n)} e^{i a_n r} \Big]_{r=x} \quad (134)$$

from which the coefficients $m_o^{(n)}$ and $n_o^{(n)}$ can be determined. The bending and extension shear coefficients, $v_o^{(n)}$ and $t_o^{(n)}$ respectively are zero for the prescribed set of boundary conditions.

In principle then, the problem has been solved. As a practical matter however, it may be recalled that the functions, e.g. P(s), were determined in terms of the loading coefficients, e.g. m_o. When the loading is general instead of a single Fourier component, one would have summations essentially of the form (see Part II, equation(64 a)).

$$P(s) = \sum_{n} P^{(n)}(s) \approx \sum_{n} P_0^{(n)}(a_n) \approx \sum_{n} a_n^{-1/2} \left\{ 1 + \gamma^{(n)} \lambda a_n^{-1} + O(\lambda^2 a_n^{-2}) \right\} \quad (135)$$

Now as in a Fourier expansion $a_n = n$, $n = 1, 2, 3, \cdots$, and as only the first terms of $P_o^{(n)}(a_n)$ have been computed due to the involved nature of $F^{(n)}(s)$, practical accuracy of the expansion, particularly for small n, may require more terms for P_o than have been given herein.

In order to obtain some idea of the stresses in the vicinity of the crack point however, consider the local conditions $r = \varepsilon \to 0$. Using the series expansion for the Bessel functions and taking the first term,

$$-D^{-1} M_\theta(\varepsilon, -\tfrac{\pi}{2}) = \tfrac{1}{2}(1+\nu)\lambda^2 c_2 + O(\varepsilon^2) + \cdots = m_o^{(o)} + \cdots \tag{136}$$

$$N_\theta(\varepsilon, -\tfrac{\pi}{2}) = q_o \frac{R}{2} + \frac{Eh}{2R} c_1 + O(\varepsilon^2) + \cdots = n_o^{(o)} \tag{137}$$

so that *

$$m_o^{(o)} = \tfrac{1}{2}(1+\nu)\lambda^2 c_2 \tag{138}$$

$$n_o^{(o)} = q_o \frac{R}{2} + \frac{Eh}{2R} c_1 \tag{139}$$

$$v_o^{(o)} = t_o^{(o)} = 0 \tag{140}$$

whereupon the combined stresses can now be found.

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* to be exact we should have:

$$m_o^{(o)} = -\int_0^R M_\theta^{(P)} dz = -\lambda^2 R c_1 \left[(\nu-1)\Lambda_2 + \nu\Lambda_1\right] + R\lambda^2 c_2\left[(\nu-1)\Lambda_4 + \nu\Lambda_3\right]$$

$$n_o^{(o)} = \tfrac{1}{2} q_o R^2 + Eh\left[c_1 \operatorname{bei}'(\lambda R) - c_2 \operatorname{ber}'(\lambda R)\right]$$

$$\Lambda_1 = \int_0^1 \operatorname{bei}(\lambda R\xi)\, d\xi \quad ; \quad \Lambda_2 = \int_0^1 \frac{\operatorname{ber}'(\lambda R\xi)}{(\lambda R\xi)} d\xi$$

$$\Lambda_3 = \int_0^1 \operatorname{ber}(\lambda R\xi)\, d\xi \quad ; \quad \Lambda_4 = \int_0^1 \frac{\operatorname{bei}'(\lambda R\xi)}{(\lambda R\xi)} d\xi$$

$$\lambda R = \sqrt[4]{12(1-\nu^2)} \sqrt{\frac{R}{h}}$$

For example, along the line of crack prolongation, $\varphi = \pi/2$,

$$\sigma_y\left(r, \frac{\pi}{2}\right) = \sigma_y(x>0,0) = \frac{\Gamma(3/2)\, E h\, (1+i)\, \lambda^2 P_0}{2i\,(1+\nu)\sqrt{\lambda x}} \left[\frac{4\,(4-\nu_0)}{\sqrt{2}}\right]_{\text{bending}} + \frac{\Gamma(3/2)\,(1+i)\, i \lambda^2 \tilde{P}_0}{\sqrt{\lambda x}\; h} \left[\frac{4}{\sqrt{2}}\right]_{\text{extension}} \tag{141}$$

whereupon substituting the limit values of P_0 and $\tilde{P}_0$ for $a \rightarrow 0$, $\lambda \neq 0$, one finds

$$\frac{\sigma_y(x>0,0)}{q_0} = \frac{(R/h)^{5/4}}{2\sqrt{\pi}\,\sqrt[8]{12(1-\nu^2)}}\; \frac{1}{\sqrt{x/h}} \left\{1 - \frac{1}{2}\left(1 - \frac{2N_0}{q_0 R}\right) H(\lambda r_0)\right\} + \dots \tag{142}$$

where

$$H(\lambda r_0) = (\lambda r_0)\, \frac{\left[\dfrac{\mathrm{ber}(\lambda r_0) - (1-\nu)(\lambda r_0)^{-1}\mathrm{bei}'(\lambda r_0)}{\mathrm{bei}(\lambda r_0) + (1-\nu)(\lambda r_0)^{-1}\mathrm{ber}'(\lambda r_0)}\right] \pm \sqrt{\dfrac{3(3+\nu)(1+\nu)}{\sqrt{2}}}}{\mathrm{bei}'(\lambda r_0)\left[\dfrac{\mathrm{ber}(\lambda r_0) - (1-\nu)(\lambda r_0)^{-1}\mathrm{bei}'(\lambda r_0)}{\mathrm{bei}(\lambda r_0) + (1-\nu)(\lambda r_0)^{-1}\mathrm{ber}'(\lambda r_0)}\right] - \mathrm{ber}'(\lambda r_0)} \tag{143}$$

The foregoing example illustrates the main features of the solution which has yet to be studied in detail. In conclusion however it is worth calling attention to a few specific points.

First, it is important to note that the preceeding typical result (142) was deduced assuming that only the leading term of the particular solution contributed to the solution, (rather than using the average value from zero to r_0) whereas a more precise computation of the leading Fourier coefficient can be determined readily by using the formulas of the footnote on the bottom of page 35.

Second, it is implicit in the solution that the singular solution dies out rapidly far from the crack. At the intersection of the crack with the circumferential boundary $r = r_0$ however, the y-distance from the crack approaches zero and deviations in the solution should be expected.

Third, the bending part of the solution is based upon classical linear bending theory wherein only the integrated Kirchhoff shear condition is satisfied along a free edge. One might therefore anticipate differences between the classical and higher order theory very similar to the changes in the Williams[7] linear bending problem for the crack as found by Knowles and Wang[8]. In this latter case, in which the bent plate was initially flat, the stress singularity remained of the inverse square root type in both solutions, but in the higher order theory, the distribution of stress around the crack point became identical with that predicted for an initially flat plate subjected solely to extension[9]. The similar improved solution for the non-Kirchhoff solution for an initially curved plate however is a considerably more difficult problem, and temporarily then one might assume that the difference reflected for the flat plate case will also hold if a more refined theory were developed for the curved sheet.

Finally an interesting paradox should be stressed. It has appeared from several pieces of indirect evidence that the solution for an initially curved plate will not smoothly approach the solution for an initially flat plate as the curvature in the former problem, i.e. λ, approaches zero. This seems strange because the solution for a beam resting upon an elastic foundation will approach that of a non-elastically supported beam as the foundation modulus becomes vanishingly weak.

These matters properly form the subject matter for continuing study.

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