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The Bending Stress in a Cracked Plate on an Elastic Foundation[1]

(2)

Classical Kirchhoff bending solutions for a normally loaded elastically supported flat plate containing a semi-infinite straight crack are obtained using an integral equation formulation. Because the effects of initial spherical plate curvature are related to those of an elastic foundation, the solution can be applied to the problem of a crack in an initially curved unsupported plate as well. The explicit nature of the stresses near the crack point is found to depend upon the inverse half power of the nondimensional distance from the point, $r/(D/k)^{1/4}$, *where* D *is the flexural rigidity of the plate and* k *the foundation modulus. The particular case of an infinite strip containing the crack along the negative x-axis and loaded by constant moments* M^* *along* $y = \pm y^*$ *is presented. The inverse half-power decay of stress is additionally damped by an exponential factor of the form* $\exp(-\lambda y^*/\sqrt{2})$.

One of the problems in fracture mechanics which apparently has not received extensive theoretical treatment is that concerning the effect of initial curvature upon the stress distribution in a thin sheet containing a crack. Considerable work has been carried out on initially flat sheets subjected to either extensional or bending stresses, and for small deformations the superposition of these separate effects [1][2] is permissible. On the other hand, if a thin sheet is initially curved, a bending load will generally produce both bending and extensional stresses, and similarly a stretching load will also induce both bending and extensional stresses. The subject of eventual concern therefore is that of the simultaneous stress fields produced in an initially curved sheet containing a crack.

Two geometries immediately come to mind: a spherical, and a cylindrical shell. In the latter case one of the principal radii of curvature is infinite and the other constant. It might appear therefore that this geometric simplicity leads to a rather straightforward analytical solution. However, the fact that the curvature varies between zero and a constant as one considers different angular positions—say around the point of a crack which is aligned parallel to the cylinder axis—more than obviates the initial geometric simplification. For this reason a spherical section of large radius of curvature is chosen for consideration.

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[2] Numbers in brackets indicate References at end of paper.

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The essential feature of the problem is immediately revealed by a study of the Reissner shallow shell equations [5, 6].

$$-(Eh/R)\nabla^2 w + \nabla^4 F = 0$$

$$\nabla^4 w + (RD)^{-1}\nabla^2 F = q/D$$

which upon suitable cross differentiation leads to the classical linear equation for plate bending

$$\nabla^4 w + \bar{\lambda}^4 w = \frac{q}{D} + \Phi_1$$

where $\bar{\lambda}^4 \equiv 12(1 - \nu^2)/R^2h^2$ and Φ_1 is an arbitrary harmonic function. It is evident, therefore, that small initial spherical curvature in a thin plate will play a part similar to an elastic support. Thus, attention will be focused in this paper upon the solution for a cracked plate supported by an elastic foundation which, aside from its connection with the fracture of initially curved pressure vessels, may have direct value in civil engineering applications such as roadways.

Formulation of the Problem

Consider the deflection and stress situation in a thin flat plate supported by an elastic foundation and governed by the classical equation of plate bending, namely

$$D\nabla^4 w(x, y) + kw(x, y) = q(x, y) \tag{1}$$

For the time being, attention is restricted to homogeneous solutions of equation (1) which can be taken as the sum of two solutions of the homogeneous equations.

$$[\nabla^2 \pm i(k/D)^{1/2}]w(x, y) = 0 \tag{2}$$

Denoting these solutions as w_1 and w_2 construct the representations

Nomenclature

a = constant determined by loading
a_0 = a/λ
D = $Eh^3/[12(1 - \nu^2)]$ = flexural rigidity of a plate
E = Young's modulus of elasticity
θ = shear modulus of elasticity
h = thickness of a plate
k = foundation modulus of a plate
M_y, M^* = bending moments as defined in text.
V_y = shear force as defined in text
r, φ = angular coordinates defined as:
$r = (x^2 + y^2)^{1/2}$; $\varphi = \tan^{-1} x/|y|$; $-\pi/2 \leq \varphi \leq \pi/2$
w = transverse deflection of a plate in bending
x, y = rectangular coordinates in middle plane of a plate
$\alpha \equiv (i)^{1/2} = e^{i\pi/4}$
$\beta \equiv (-i)^{1/2} = e^{i(7\pi)/4}$
γ = path of integration as defined in text
$\lambda \equiv \sqrt[4]{k/D}$
ν = Poisson's ratio
ν_0 = $1 - \nu$

$$w_1(x, y^\pm) = \int_\gamma [P_1(s) \mp Q_1(s)] \exp [-\lambda(s^2 - \alpha^2)^{1/2} |y| + i\lambda sx] ds \quad (3)$$

$$w_2(x, y^\pm) = \int_\gamma [P_2(s) \mp Q_2(s)] \exp [-\lambda(s^2 - \beta^2)^{1/2} |y| + i\lambda sx] ds \quad (4)$$

where positive real parts of the roots are taken.

Suppressing for the moment a definition of the path γ, consider the specific situation resulting when there exists a crack along the negative real axis of the elastically supported plate. One must require that the moment and equivalent shear vanish. Suppose, however, that one has already found a particular solution to equation (1) which is satisfactory except that there is a residual moment, M_y, and equivalent shear, V_y, along the negative real axis, $x < 0$, of the general Fourier type, say for a particular term

$$M_y^{(p)} = -Dm_0 e^{iax} \quad (5)$$

$$V_y^{(p)} = -Dv_0 e^{iax} \quad (6)$$

where m_0 and v_0 are complex constants. Hence the homogeneous solution, providing it satisfies certain physical conditions far from the crack, will be required to equal the negative of equations (5) and (6) along $x < 0$, i.e.,

$$M_y(x, 0) = -D\left(\frac{\partial^2}{\partial y^2} + \nu \frac{\partial^2}{\partial x^2}\right)(w_1 + w_2) = Dm_0 e^{iax}; \quad x < 0 \quad (7)$$

$$V_y(x, 0) = -D\left[\frac{\partial^3}{\partial y^3} + (2 - \nu)\frac{\partial^3}{\partial x^2 \partial y}\right](w_1 + w_2) = Dv_0 e^{iax}; \quad x < 0 \quad (8)$$

Assuming that the integrals of equations (3) and (4) can be differentiated under the integral sign, equations (7) and (8) are equivalent to

$$\int_\gamma [(P_1 \mp Q_1)(\nu_0 s^2 - \alpha^2) + (P_2 \mp Q_2)(\nu_0 s^2 - \beta^2)] \exp i\lambda sx\, ds = -m_0/\lambda^2 e^{iax} \quad (9)$$

$$\pm \int_\gamma [(P_1 \mp Q_1)(s^2 - \alpha^2)^{1/2}(\nu_0 s^2 + \alpha^2) + (P_2 \mp Q_2)\cdot(s^2 - \beta^2)^{1/2}(\nu_0 s^2 + \beta^2)] \exp i\lambda sx\, ds = (-v_0/\lambda^3)e^{iax} \quad (10)$$

which must hold along the crack, $x < 0$. On the other hand, for $x > 0$ the deflection and its derivatives must be continuous across $y = 0$. The conditions

$$\lim_{y \to 0} \left[\frac{\partial^n}{\partial y^n}(w_1^+ + w_2^+) - \frac{\partial^n}{\partial y^n}(w_1^- + w_2^-)\right] = 0; \quad n = 0, 1, 2, 3 \quad (11)$$

may all be satisfied by taking, for $x > 0$,

$$\int_\gamma Q_1 \exp i\lambda sx\, ds = 0; \quad \int_\gamma Q_2 \exp i\lambda sx\, ds = 0 \quad (12)$$

$$\int_\gamma (s^2 - \alpha^2)^{1/2} P_1 \exp i\lambda sx\, ds = 0;$$

$$\int_\gamma (s^2 - \beta^2)^{1/2} P_2 \exp i\lambda sx\, ds = 0 \quad (13)$$

Proceeding with the construction, arbitrarily let the following combinations in equations (9) and (10) vanish,

$$\int_\gamma [(\nu_0 s^2 - \alpha^2)Q_1 + (\nu_0 s^2 - \beta^2)Q_2] \exp i\lambda sx\, ds = 0; \quad x < 0 \quad (14)$$

$$\int_\gamma [(\nu_0 s^2 + \alpha^2)(s^2 - \alpha^2)^{1/2} P_1 + (\nu_0 s^2 + \beta^2)(s^2 - \beta^2)^{1/2} P_2] \exp i\lambda sx\, ds = 0; \quad x < 0 \quad (15)$$

which are evidentally satisfied by taking

$$Q_1 = -(\nu_0 s^2 - \beta^2)Q(s) \quad (16)$$

$$Q_2 = (\nu_0 s^2 - \alpha^2)Q(s) \quad (17)$$

$$P_1 = -(\nu_0 s^2 + \beta^2)(s^2 - \beta^2)^{1/2} P(s) \quad (18)$$

$$P_2 = (\nu_0 s^2 + \alpha^2)\cdot(s^2 - \alpha^2)^{1/2} P(s) \quad (19)$$

where $P(s)$ and $Q(s)$ are new, still largely arbitrary functions, leaving in equations (9) and (10)

$$\int_\gamma [(\nu_0 s^2 - \alpha^2)P_1 + (\nu_0 s^2 - \beta^2)P_2] \exp i\lambda sx\, ds = -\frac{m_0}{\lambda^2} e^{iax}; \quad x < 0 \quad (20)$$

$$\int_\gamma [(s^2 - \alpha^2)^{1/2}(\nu_0 s^2 + \alpha^2)Q_1 + (s^2 - \beta^2)^{1/2}(\nu_0 s^2 + \beta^2)Q_2] \exp i\lambda sx\, ds = (v_0/\lambda^3)e^{iax}; \quad x < 0 \quad (21)$$

which using the new functions $P(s)$, $Q(s)$ from equations (16) to (19) reduce, respectively, to

$$\int_\gamma K(s)P(s) \exp i\lambda sx\, ds = (-m_0/\lambda^2)e^{iax}; \quad x < 0 \quad (22)$$

$$\int_\gamma K(s)Q(s) \exp i\lambda sx\, ds = (-v_0/\lambda^3)e^{iax}; \quad x < 0 \quad (23)$$

where the kernel is

$$K(s) = (s^2 - i)^{1/2}(\nu_0 s^2 + i)^2 - (s^2 + i)^{1/2}(\nu_0 s^2 - i)^2 \quad (24)$$

$$= (s^2 - \alpha^2)^{1/2}(\nu_0 s^2 + \alpha^2)^2 - (s^2 - \beta^2)^{1/2}(\nu_0 s^2 + \beta^2)^2 \quad (24a)$$

Returning to the conditions of continuity across $y = 0$ for $x > 0$, introduce equations (16) to (19) into equations (12) and (13) to find

$$-\int_\gamma (\nu_0 s^2 + i)Q(s) \exp i\lambda sx\, ds = 0;$$

$$\int_\gamma (\nu_0 s^2 - i)Q(s) \exp i\lambda sx\, ds = 0 \quad (25)$$

$$-\int_\gamma (s^4 + 1)^{1/2}(\nu_0 s^2 - i)P(s) \exp i\lambda sx\, ds = 0;$$

$$\int_\gamma (s^4 + 1)^{1/2}(\nu_0 s^2 + i)P(s) \exp i\lambda sx\, ds = 0 \quad (26)$$

which can be satisfied by setting

$$\int_\gamma Q(s) \exp i\lambda sx\, ds = 0; \quad x > 0 \quad (27)$$

$$\int_\gamma (s^4 + 1)^{1/2} P(s) \exp i\lambda sx\, ds = 0; \quad x > 0 \quad (28)$$

taking into account that the second derivatives of equations (27) and (28) with respect to x are also zero.

Equations (22), (23) and (27), (28) are therefore the dual integral equations to be solved for the unknown functions $P(s)$ and $Q(s)$, which, when substituted into equations (16) to (19) and subsequently into equations (3) and (4) along with the particular solution producing equations (5) and (6), give the deflection function which satisfies the Kirchhoff conditions for a free edge along the crack of the elastically supported plate.

Solution of the Integral Equations. First of all the path of integration, γ, is taken along the real axis except at the point $s = a/\lambda$ which is circled from above. The functions $(s^2 - \alpha^2)^{1/2}$ and $(s^2 - \beta^2)^{1/2}$ are made single valued by introducing branch cuts as shown in Fig. 1. Specifically $(s^2 - \alpha^2)^{1/2}$ leads to the insertion of branch cuts $|\text{Im } s| = |\text{Im } \alpha|$; with Re $s >$ Re α for Im $s > 0$, and Re $s < -$Re α for Im $s < 0$ as shown hatched in the figure. Similarly, $(s^2 - \beta^2)^{1/2}$ also leads to cuts along $|\text{Im } s| = |\text{Im } \alpha|$, which are taken as Re $s > -$Re β for Im $s > 0$ and Re $s <$ Re β for Im $s < 0$, as shown cross hatched in the figure. It develops that

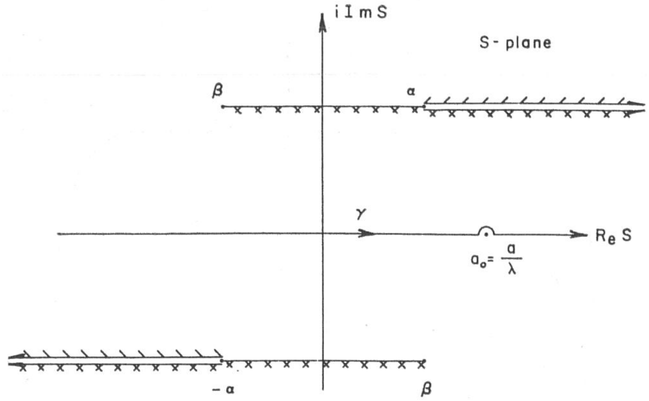

Fig. 1 (NOTE: Upper left β should read $-\beta$)

the portions of the branch cuts for $\operatorname{Re} s > \operatorname{Re}\alpha$ and $\operatorname{Re} s < -\operatorname{Re}\alpha$ cancel each other leaving the $K(s)$ analytic in the entire s-plane, except for two cuts in the upper and lower half planes of length $|\operatorname{Re} s| < |\operatorname{Re}\alpha|$.

We consider first the equations in $P(s)$, namely, equations, (22) and (28). It is found convenient at this stage to define an auxiliary function

$$F(s) \equiv \frac{K(s)}{i(4 - \nu_0)\nu_0(s^2 - s_1^2)(s^2 - \alpha^2)^{1/2}} \tag{29}$$

where $\pm s_1$ are the zeros of $K(s)$ in the first and third quadrants, respectively (this matter will be elucidated in the next section). The dual equations for $P(s)$ will be solved by an application of the theory of functions of a complex variable following Clemmow [2]. Thus, the equation for $P(s)$ of equation (22) is satisfied if:

$$i(4 - \nu_0)\nu_0(s^2 - s_1^2)(s^2 - \alpha^2)^{1/2}F(s)P(s) = \frac{m_0}{2\pi i\lambda^2}\frac{L(s)}{L(a_0)}\frac{1}{s - a_0} \tag{30}$$

where $L(s)$ is a function free from zeros and singularities in the lower half of the s-plane inclusive of γ, and furthermore of algebraic behavior at infinity. That equation (30) solves equation (22) results from Jordan's lemma and the theorem of residues. By the same argument equation (28) is satisfied if

$$(s^4 + 1)^{1/2}P(s) = U(s) \tag{31}$$

where $U(s)$ is the counter part of $L(s)$ is the upper half-plane. Eliminating $P(s)$ from equations (30) and (31), and after some rearrangement, we obtain:

$$\frac{U(s)}{L(s)} = \frac{-m_0}{2\pi(4 - \nu_0)\nu_0\lambda^2}\frac{1}{L(a_0)}\left[\frac{(s - \beta)^{1/2}}{(s - a_0)(s + s_1)F_U(s)}\right] \times \left[\frac{(s + \beta)^{1/2}}{(s - s_1)F_L(s)}\right] \tag{32}$$

where $F_{U,L}(s)$ are, respectively, a U-type and an L-type function, such that

$$F_U(s)\cdot F_L(s) = F(s) \tag{33}$$

(This latter factorization of equation (29) will be carried out in the next section.) A solution of equation (32) is

$$L(s) = \frac{(s - s_1)F_L(s)}{(s + \beta)^{1/2}} \tag{34}$$

$$U(s) = \frac{-m_0}{2\pi(4 - \nu_0)\nu_0\lambda^2}\left[\frac{(a_0 + \beta)^{1/2}}{(a_0 - s_1)F_L(a_0)}\right] \times \frac{(s - \beta)^{1/2}}{(s - a_0)}\frac{1}{(s + s_1)F_U(s)} \tag{35}$$

Where the bracketed term follows from equation (34) evaluated at $s = a_0$. It follows then, from equations (31) and (35), that $P(s)$ is given by:

$$P(s) = \frac{-m_0}{2\pi(4 - \nu_0)\nu_0\lambda^2}\left[\frac{(a_0 + \beta)^{1/2}}{(a_0 - s_1)F_L(a_0)}\right] \times \left[\frac{(s - \beta)^{1/2}}{(s - a_0)(s + s_1)(s^4 + 1)^{1/2}F_U(s)}\right] \tag{36}$$

Next, by following exactly the same steps as above, we find for $Q(s)$:

$$Q(s) = \frac{-v_0}{2\pi(4 - \nu_0)\nu_0\lambda^3}\frac{1}{(a_0 - s_1)F_L(a_0)(a_0 - \alpha)^{1/2}} \times \left[\frac{1}{(s - a_0)(s + s_1)(s + \alpha)^{1/2}F_U(s)}\right] \tag{37}$$

so that finally $P_{1,2}(s)$ and $Q_{1,2}(s)$ can be deduced from equations (16) to (19), and the problem is formally solved. The practical matter of determining the factorization of $F(s)$ as implied by equation (33) will now be described.

A Factorization of the Kernel Function $F(s)$. It is proposed to split $F(s)$ as defined in equation (29) into a product of a U-type and an L-type function, i.e.,

$$F(s) = F_U(s)\cdot F_L(s) = \frac{K(s)}{i(4 - \nu_0)\nu_0(s^2 - s_1^2)(s^2 - \alpha^2)^{1/2}}$$

Define first

$$G(s) = \ln F(s) \tag{38}$$

where for definiteness the logarithm is taken as a principal value. We decompose $G(s)$ into the sum of a U-type and an L-type function

$$G(s) = G_U(s) + G_L(s) = \ln F_U(s) + \ln F_L(s) \tag{39}$$

from which the product factorization follows immediately. The decomposition of equation (39) can be accomplished once we know the singularities of $G(s)$ which are the singularities of $F(s)$, namely, the branch cuts $|\operatorname{Im} s| = |\operatorname{Im}\alpha|$, $-\operatorname{Re}\beta \leq \operatorname{Re} s \leq \operatorname{Re}\alpha$.

To find the zeros of $F(s)$, we rationalize the equation $K(s) = 0$, using equation (24), obtaining a quadratic equation in s^4. Of the eight roots of this rationalized equation, only four satisfy the original equation $K(s) = 0$, namely:

$$s = \pm s_1 = \pm e^{i\pi/4}\left[\frac{3\nu_0 - 2 + 2[2\nu_0^2 - 2\nu_0 + 1]^{1/2}}{(4 - \nu_0)\nu_0^2}\right]^{1/4} \tag{40a}$$

$$s = \pm s_2 = \pm e^{i(3\pi/4)}|s_1| \tag{40b}$$

From the way it was defined in (29), however, $F(s)$ has only the two zeros in the second and fourth quadrants, namely

$$s = \pm s_2 \approx \pm(1.0004)\exp\left(\frac{3i\pi}{4}\right) \quad \text{for} \quad \nu_0 = \frac{3}{4} \quad \text{or} \quad \nu = \frac{1}{4}$$

where it may be noted that

$$1.000 \leq |s_1| \leq \sqrt[4]{1.0448}$$

because of the physical restrictions on the value of Poisson's ratio.

Thus, $G(s)$ has a strip of regularity, namely, $|\operatorname{Im} s| < |\operatorname{Im}\alpha|$. Since $F(s)$ was defined in equation (29) to include the proper constant such that its asymptotic value for large s gives unity upon expansion of equation (29), i.e.

$$F(s) = 1 + 0(1/s^2) \quad \text{for} \quad |s| \to \infty \tag{41}$$

it follows that for s belonging to the strip of regularity, Cauchy's integral formula gives:

$$G(s) = -\frac{1}{2\pi i}\int_{-\infty+i\eta}^{\infty+i\eta}\frac{G(z)}{z - s}dz + \frac{1}{2\pi i}\int_{-\infty-i\delta}^{\infty-i\delta}\frac{G(z)}{z - s}dz \tag{42}$$

where $\operatorname{Im} s < \eta < \operatorname{Im} \alpha$, $-\operatorname{Im} \alpha < -\delta < \operatorname{Im} < s$. The first integral of equation (42) is identified with $G_L(s)$, the second integral with $G_U(s)$. We shall put $G_U(s)$ in a form convenient for numerical evaluation by a deformation of the path of integration. The function $G(z)$ is made single-valued in the lower half plane by introducing a cut for the logarithmic singularity (corresponding to the zero $z = -s_2$) in addition to the cut for the function $F(z)$ itself. The cut for the logarithmic singularity is conveniently defined as a semi-infinite line drawn through $z = -s_2$ parallel to the real axis in the negative direction. We wish first to evaluate $G_U(s)$ for s in the upper half-plane and then continue it analytically to the whole s-plane. For s in the upper half plane the path of integration can be deformed into the real axis:

$$G_U(s) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{G(z)}{z - s} dz \tag{43}$$

Since $G(z)$ is an even function in z, hence equation (43) can be transformed into the more convenient form

$$G_U(s) = \frac{s}{2\pi i} \int_{-\infty}^{\infty} \frac{G(z)}{z^2 - s^2} dz \tag{44}$$

which is equivalent to:

$$G_U(s) = \frac{s}{2\pi i} \oint \frac{G(z)}{z^2 - s^2} dz - \frac{s}{2\pi i} \int_{c+c'} \frac{G(z)}{z^2 - s^2} dz - \frac{s}{2\pi i} \int_{c''+c'''} \frac{G(z)}{z^2 - s^2} dz \quad (45)^3$$

where the paths c, c', c'', c''' are shown in Fig. 2, hence

$$G_U(s) = \frac{G(s)}{2} - \frac{s}{2\pi i} \int_{c+c'} \frac{G(z)}{z^2 - s^2} dz \tag{46}$$

From the properties of the logarithm we have:

$$\frac{1}{2\pi i} \int_{c'} \frac{G(z)}{z^2 - s^2} dz = \frac{1}{2s} \ln \left(\frac{s_2 - s}{s_2 + s} \right) \tag{47}$$

because, noting in Fig 3,

$$I_I = \frac{1}{2\pi i} \int_{s_2}^{\infty} \frac{\ln [F(z)]_{\text{bot}}}{(s_2 + r)^2 - s^2} dr;$$

$$I_{II} = -\frac{1}{2\pi i} \int_{s_2}^{\infty} \frac{\ln [F(z)]_{\text{top}}}{(s_2 + r)^2 - s^2} dr,$$

and next recall that

$$\ln F(z) = \ln |F(z)| + i \arg F(z)$$

hence

$$-[\ln F(z)]_{\text{top}} + [\ln F(z)]_{\text{bot}} = -2\pi i$$

Therefore,

$$F_U(s) = \exp [G_U(s)] = \left[\frac{s_2 + s}{s_2 - s} F(s) \right]^{1/2} \exp [-sI(s)] \tag{48}$$

where

$$I(s) \equiv \frac{1}{2\pi i} \int_c \frac{G(z)}{z^2 - s^2} dz \tag{48a}$$

In view of numerical computations, the integral of equation (48*a*) is transformed into a sum of real integrals

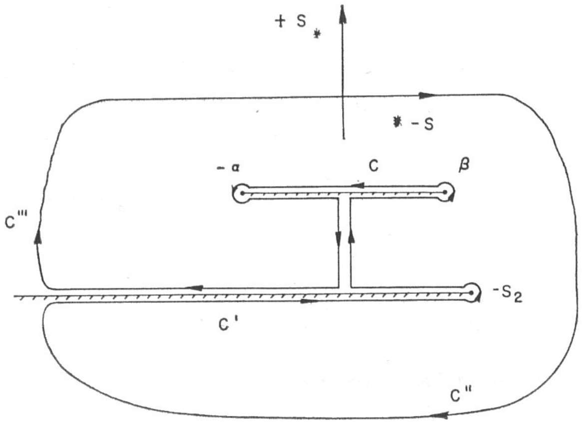

Fig. 2

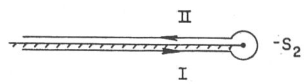

Fig. 3

$$2\pi i I(s) = \frac{1}{2} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{\ln \left[\dfrac{A_-^2 + B_-^2}{A_+^2 + B_+^2} \right]}{\left(x - \dfrac{i}{\sqrt{2}} \right)^2 - s^2} dx + i \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{\tan^{-1} \left(\dfrac{B_-}{A_-} \right) - \tan^{-1} \left(\dfrac{B_+}{A_+} \right)}{\left(x - \dfrac{i}{\sqrt{2}} \right)^2 - s^2} dx \tag{49}$$

where

$$A_\pm = A_1 \pm A_2 R \pm B_2 I; \quad B_\pm = B_1 \pm B_2 R \mp A_2 I$$

$$A_{1,2} = \nu_0 \left(x^2 - \frac{1}{2} \right)^2 - (\pm 1 - \nu_0 x \sqrt{2})^2;$$

$$B_{1,2} = 2\nu_0 (\pm 1 - \nu_0 x \sqrt{2}) \left(x^2 - \frac{1}{2} \right)$$

$$R = \left[\frac{\sqrt{M^2 + N^2} + M}{2} \right]^{1/2}; \quad I = \left[\frac{\sqrt{M^2 + N^2} - M}{2} \right]^{1/2} \tag{50}$$

$$M = \frac{(2x^2 - 1)(3 + 2x^2)}{(2x^2 + 1)^2 + 4(1 + \sqrt{2}\, x)^2}; \quad N = \frac{4(2x^2 - 1)}{(2x^2 + 1)^2 + 4(1 + \sqrt{2}\, x)^2}$$

We shall need the values of $I(s)$ for large $|s|$. Its expansion gives for $|s| > 1$[4]:

$$I(s) = -\sum_{n=0}^{\infty} c_n s^{-2(n+1)} \tag{51}$$

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[3] $$\frac{s}{2\pi i} \oint \frac{G(z)}{z^2 - s^2} dz = \frac{1}{4\pi i} \oint \frac{G(z)}{z - s} dz - \frac{1}{4\pi i} \oint \frac{G(z)}{z + s} dz = 0 + G(s)/2$$

[4] For future reference the reader should note that for small s

$$I(s) = \sum_{n=0}^{\infty} \bar{c}_n s^{2n}$$

where

$$2\pi i \bar{c}_n = \frac{1}{2} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(x - \frac{i}{\sqrt{2}} \right)^{-2n-2} \ln \left(\frac{A_-^2 + B_-^2}{A_+^2 + B_+^2} \right) dx + i \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(x - \frac{i}{\sqrt{2}} \right)^{-2n-2} \left(\tan^{-1} \frac{B_-}{A_-} - \tan^{-1} \frac{B_+}{A_+} \right) dx$$

where

$$2\pi i c_n = \frac{1}{2}\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(x - \frac{i}{\sqrt{2}}\right)^{2n} \ln\left(\frac{A_-{}^2 + B_-{}^2}{A_+{}^2 + B_+{}^2}\right) dx$$

$$+ i\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(x - \frac{i}{\sqrt{2}}\right)^{2n} \left(\tan^{-1}\frac{B_-}{A_-} - \tan^{-1}\frac{B_+}{A_+}\right) dx \quad (52)$$

In view of equation (51), an approximate expansion for $F_U(s)$, c.f. equation (48) is:

$$|s| \to \infty: \quad F_U(s) = -i\,[1 + (c_0 + s_2)/s + 0(1/s^2)] \quad (53)$$

and hence

$$|s| \to \infty: \quad F_L(s) = i[1 - (c_0 + s_2)/s + 0(1/s^2)] \quad (54)$$

Stress Distribution Near the Crack Point. From equations (16) to (19), (36), (37), (53), and (54), we see that in equations (3) and (4), $P_{1,2}(s)$ and $Q_{1,2}(s)$ go to zero, for $|s| \to \infty$, at least as fast as $s^{-1/2}$. Hence, the integrals of equations (3) and (4) converge and the differentiations under the integral signs are also justified, at least for $y \neq 0$. The values of the derivatives at $y = 0$, $x < 0$ can be obtained by a proper limiting process.

The stresses are given in terms of the bending deflection as

$$\tau_{xy} = -2Gz\,\frac{\partial^2 w}{\partial x \partial y} \quad (55)$$

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right) \quad (56)$$

$$\sigma_y = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\,\frac{\partial^2 w}{\partial x^2}\right) \quad (57)$$

where here z is the distance through the thickness, h, of the plate measured from the middle surface. No subsequent confusion with the complex variable will result. Then, in terms of equations (3), (4), (16) to (19), (36), (37), (53), and (54), the stresses of equations (55) to (57) can be expressed as a linear conbination of integrals of the form:

$$\int_{-\infty}^{\infty} R(s)\exp\left[-\lambda|y|(s^2 \pm i)^{1/2} + i\lambda s x\right] ds \quad (58)$$

where $R(s)$ behaves for large s, i.e., $|s| \to \infty$ as:

$$R(s) = R_0 s^{-1/2} + 0(s^{-3/2})$$

and R_0 being a suitable constant. It is obvious, by changing variables $s' \sim sx$, that the stresses tend to infinity as $r^{-1/2}$ for $r \to 0$ which is characteristic of crack problems. Furthermore, by expanding the integrands for large $|s|$ we can find the variation of stresses with angular position as $r \to 0$. Thus we get expressions of the form:

$$M_n|s|^{-1/2-n}\exp(-\lambda|y||s| + i\lambda x s)$$

and

$$N_n|s|^{1/2-n}\lambda^{n+1}|y|^{n+1}\exp(-\lambda|y||s| + i\lambda x s)$$

$$(n = 0, 1, 2, 3, \ldots\ldots)$$

Here, we should mention that the exact expansion of the stress integrals should be of the form (Fig. 4):

$$\lim_{\epsilon \to 0}\Bigg[\int_{-\infty}^{-1-\epsilon} (\text{expansion for large } s)ds + \int_{-1-\epsilon}^{-1+\epsilon} [\text{exp for small } (s+1)]ds$$

$$+ \int_{-1+\epsilon}^{1-\epsilon} (\text{exp for small } s)ds + \int_{1-\epsilon}^{1+\epsilon} [\text{exp for small } (s-1)]ds$$

$$+ \int_{1+\epsilon}^{\infty} (\text{exp for large } s)ds\Bigg] \quad (58')$$

The contribution of the 2nd and 4th terms can easily be shown to vanish. At the present we consider the singular part of the solution arising from the 1st and 5th term. Furthermore, for convenience, we integrate over $(-\infty, \infty)$ instead of $(-\infty, -1-\epsilon)$ and $(1+\epsilon, \infty)$ in order to express the integrals in terms of known functions, namely, the Γ-functions.

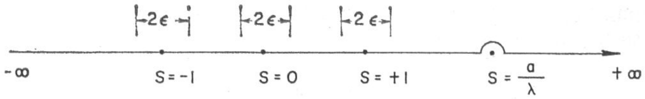

Fig. 4

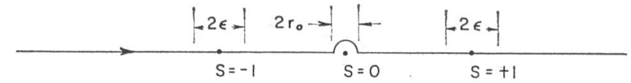

Fig. 5

It can be shown that the differences between equation (58) and the integrals

$$T_0(x, y) \equiv \int_{-\infty}^{\infty} (M_0|s|^{-1/2} + N_0|s|^{1/2}\lambda|y|)e^{-\lambda|y||s| + i\lambda x s}\,ds \quad (59)$$

are bounded for all r, in particular for small r, and by an appropriate deformation of the path of integration $T_0(x, y)$ can be expanded in terms of Γ-functions. Without going into the details, we list the results for the stresses on the plate surface $z = \pm h/2$ below.

$$\tau_{xy} = \mp\frac{\Gamma(3/2)Gh(1+i)\lambda^2 P_0}{i\sqrt{\lambda r}}\left[4\left(\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2}\right) - \nu_0\cos\varphi\left(\cos\frac{3\varphi}{2} - \sin\frac{3\varphi}{2}\right)\right]$$

$$-\frac{\Gamma(3/2)Gh(1-i)\lambda^2 Q_0}{i\sqrt{\lambda r}}\left[2(2-\nu_0)\left(\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\right) + \nu_0\cos\varphi\left(\cos\frac{3\varphi}{2} + \sin\frac{3\varphi}{2}\right)\right] + \ldots \quad (60)$$

$$\sigma_x = -\frac{\Gamma(3/2)Eh\nu_0(1+i)\lambda^2 P_0}{2i(1+\nu)\sqrt{\lambda r}}\left[2\left(\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\right) - \cos\varphi\left(\cos\frac{3\varphi}{2} + \sin\frac{3\varphi}{2}\right)\right]$$

$$\pm\frac{\Gamma(3/2)Eh(1-i)\lambda^2 Q_0}{2i(1+\nu)\sqrt{\lambda r}}\left[4(2-\nu_0)\left(\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2}\right) + \nu_0\cos\varphi\left(\cos\frac{3\varphi}{2} - \sin\frac{3\varphi}{2}\right)\right] + \ldots \quad (61)$$

$$\sigma_y = \frac{\Gamma(3/2)Eh(1+i)\lambda^2 P_0}{2i(1+\nu)\sqrt{\lambda r}}\left[2(4-\nu_0)\left(\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\right) - \nu_0\cos\varphi\left(\cos\frac{3\varphi}{2} + \sin\frac{3\varphi}{2}\right)\right]$$

$$\mp\frac{\Gamma(3/2)Eh(1-i)\nu_0\lambda^2 Q_0}{i(1+\nu)\sqrt{\lambda r}}\left[\cos\varphi\left(\cos\frac{3\varphi}{2} - \sin\frac{3\varphi}{2}\right)\right] + \ldots. \quad (62)$$

where the upper and lower signs correspond to $y > 0$ and $y < 0$, respectively. Further, the general expression for the complex constants P_0 and Q_0 are deduced as

$$P_0 = \frac{-im_0(a_0 + \beta)^{1/2}}{2\pi(4-\nu_0)\nu_0\lambda^2(a_0 - s_1)F_L(a_0)} = -\frac{im_0[(a/\lambda) + \beta]^{1/2}}{2\pi(4-\nu_0)\nu_0\lambda^2[(a/\lambda) - s_1]F_L(a/\lambda)} \quad (64)$$

$$Q_0 = \frac{-iv_0}{2\pi(4-\nu_0)\nu_0\lambda^3(a_0 - s_1)(a_0 - \alpha)^{1/2}F_L(a_0)} = \frac{-iv_0}{2\pi(4-\nu_0)\nu_0\lambda^3[(a/\lambda) - s_1][(\alpha/\lambda) - a]^{1/2}F_L(a/\lambda)} \quad (65)$$

along with certain limiting cases of interest, namely

Case i: $\lambda \to 0$ but $a \neq 0$

$$P_0 = \frac{-m_0\sqrt{a\lambda}}{2\pi(4-\nu_0)\nu_0\lambda^2 a}\left[1 + \frac{s_1+s_2+c_0+\beta/2}{a}\lambda + 0\left(\frac{\lambda^2}{a^2}\right)\right] \quad (64a)$$

$$Q_0 = \frac{-v_0\sqrt{\lambda}}{2\pi(4-\nu_0)\nu_0\lambda^2 a\sqrt{a}}\left[1 + \frac{s_1+s_2+c_0+\alpha/2}{a}\lambda + 0\left(\frac{\lambda^2}{a^2}\right)\right] \quad (65a)$$

Case ii:[5] $\lambda \neq 0$ but $a \to 0$

$$P_0 = \frac{-im_0e^{-i\pi/4}}{2\pi(4-\nu_0)\nu_0\lambda^2}\left[\frac{(4-\nu_0)\nu_0}{\sqrt{2}}\right]^{1/2}\left[1+\left(\frac{1}{2\beta}+\frac{1}{s_1}+\frac{1}{s_2}-\bar{c}_0\right)\frac{a}{\lambda}+0\left(\frac{a^2}{\lambda^2}\right)\right] \quad (64b)$$

$$Q_0 = \frac{-iv_0e^{-i\pi/4}}{2\pi(4-\nu_0)\nu_0\lambda^3}\left[\frac{(4-\nu_0)\nu_0}{\sqrt{2}}\right]^{1/2}\left[1+\left(\frac{1}{2\alpha}+\frac{1}{s_1}+\frac{1}{s_2}-\bar{c}_0\right)\frac{a}{\lambda}+0\left(\frac{a^2}{\lambda^2}\right)\right] \quad (65b)$$

The expressions for the stresses given above are approximations to the first term of the homogeneous solution. To this the contribution of the particular should be added. It should be noted, that along the crack (i.e. $\varphi = -\pi/2$) equation (62) gives $\sigma_y = 0$. In addition to this we have from equation (5) the contribution of the particular solution, namely, the term:

$$-\frac{Ezm_0}{(1-\nu^2)}e^{iax}$$

It thus appears as if the sum of the σ_y contributions is not zero along the crack, which would not satisfy the boundary condition. Such a deduction, however, is not true because in the expansion of the stress integrals, we neglected the expansion for small s. For small s, point $s = 0$ is a pole, hence by taking the contribution around the semicircle, we cancel half of the particular contribution. The other half must come from $\int_{-1+\epsilon}^{-r_0}(\ ) + \int_{r_0}^{1-\epsilon}(\ )$. The integrand, however, is a power series with complex constants which makes an explicit analytical evaluation very difficult and computationally tedious.

A Particular Solution

As an illustration of how the local solution may be combined in a particular case, consider a rectangular strip, infinitely long in the x-direction and of finite width y^* in the y-direction. Furthermore, let the plate be subjected to a constant moment M^* and zero shear at $y = \pm y^*$, and simultaneously subjected to a uniform normal loading q_0. We proceed to write down a solution of (1) as follows

$$w^{(p)}(y) = \frac{q_0}{D\lambda^4} + A\cos\frac{\lambda y}{\sqrt{2}}\cosh\frac{\lambda y}{\sqrt{2}} + B\sin\frac{\lambda y}{\sqrt{2}}\sinh\frac{\lambda y}{\sqrt{2}} \quad (68)$$

[5] For small s,

$$\sqrt{\beta} = e^{-i\pi/8};\quad \sqrt{-\alpha} = e^{i(5\pi/8)} \quad (66)$$

and

$$F_L(s) = \xi e^{-i5\pi/8}[1 + 0(s)];\quad \xi = \left[\frac{\sqrt{2}}{(4-\nu_0)\nu_0|s_1|^2}\right]^{1/2} \quad (67)$$

and using

$$M_y^{(p)} = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right] \quad (69)$$

$$V_y^{(p)} = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2\partial y}\right] \quad (70)$$

compute

$$-D^{-1}M_y^{(p)}(y^*) = -\lambda^2\left[A\sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right) - B\cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right)\right] \quad (71)$$

$$-D^{-1}V_y^{(p)}(y^*) = \frac{\lambda^3}{\sqrt{2}}\left\{-A\left[\cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right) + \sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right)\right] - B\left[\sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right) - \cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right)\right]\right\} \quad (72)$$

Equating equation (71) to $-D^{-1}M^*$ and equation (72) to zero and solving for the constants, one finds

$$A = \frac{M^*}{D\lambda^2}\,\frac{\sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right) - \cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right)}{\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right) + \cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)} \quad (73)$$

$$B = -\frac{M^*}{D\lambda^2}\,\frac{\cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right) + \sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right)}{\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right) + \sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)} \quad (74)$$

along the crack, $y = 0$, it is found that

$$M_y^{(p)}(x, 0) = M^*\,\frac{\cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right) + \sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cosh\left(\lambda\frac{y^*}{\sqrt{2}}\right)}{\cosh\left(\frac{\lambda y^*}{\sqrt{2}}\right)\sinh\left(\frac{\lambda y^*}{\sqrt{2}}\right) + \sin\left(\frac{\lambda y^*}{\sqrt{2}}\right)\cos\left(\frac{\lambda y^*}{\sqrt{2}}\right)} \quad (75)$$

$$V_y^{(p)}(x, 0) = 0 \quad (76)$$

from which comparison with equations (5) and (6) indicates that

$$-D(m_{0r} + im_{0j})(\cos ax + i\sin ax) = M^*f(\lambda y^*/\sqrt{2}) \quad (77)$$

$$-D(v_{0r} + iv_{0j})(\cos ax + i\sin ax) = 0 \quad (78)$$

or

$$a = v_{0r} = v_{0j} = m_{0j} = 0 \quad (79)$$

$$m_{0r} = -M^*D^{-1}f(\lambda y^*/\sqrt{2}) \quad (80)$$

where using equation (74),

$$f(\lambda y^*/\sqrt{2}) \equiv -(BD\lambda^2)/M^* \quad (81)$$

Returning now to the stresses along the crack prolongation, for example the normal stress $\sigma_y(x, 0)$, one finds using equations (62) and (64b) that

$$\frac{\sigma_y(x, 0)}{Eh} = -\frac{M^*D^{-1}f(\lambda y^*/\sqrt{2})}{2\sqrt[4]{2}\sqrt{2\pi}\sqrt{(3+\nu)(1-\nu)}(1+\nu)}\cdot\frac{1}{\sqrt{\lambda x}} + \ldots \quad (82)$$

or, in terms of the stress,

$$\frac{\sigma_y(x,0)}{\sigma_y{}^*} = -\frac{f(\lambda y^*/\sqrt{2}\,)}{\sqrt[4]{2}\,\sqrt{2\pi}}\left(\frac{1-\nu}{3+\nu}\right)^{1/2}\cdot\frac{1}{\sqrt{\lambda x}} + \ldots \quad (83)$$

It is interesting to compare this result to the one obtained for the classical bending stress obtained at the end of a finite crack of half length b, [1] when the initially flat (non-elastically supported) plate is subjected to a uniform bending moment or stress $\sigma_y{}^*$ far from the crack—for comparative purposes here, y^* large. It was found [1] that

$$\frac{\sigma_y(x,0)}{\sigma_y{}^*} = -\frac{1-\nu}{3+\nu}\,\frac{1}{\sqrt{2x/b}} + \ldots \approx\Bigg|_{\nu=1/3} -\frac{0.20}{\sqrt{x/b}} \quad (84)$$

In order to obtain a fair comparison for the elastically supported plate, consider equation (83) normalized on the average stress through the uncracked portion, viz. $\sigma_y{}^*f(\lambda y^*/\sqrt{2}\,)$

$$\frac{\sigma_y(x,0)}{G_y{}^*f(\lambda y^*/\sqrt{2}\,)}\Bigg|_{\nu=1/3} \approx -\frac{0.18}{\sqrt{\lambda x}} = -\frac{0.18}{[x(D/k)^{-1/4}]^{1/2}} \quad (85)$$

Hence with respect to local conditions near the crack point, one conjectures that the spring constant $(D/k)^{1/4}$ of the elastically supported plate plays the same role as an effective crack length b in an unsupported plate, providing the crack is reasonably far from the boundary at y^* and is sufficiently long compared to the plate thickness.

Conclusion

It is worth emphasizing that classical bending theory has been used in deducing the foregoing results. Hence it is inherent that only the Kirchhoff equivalent shear free condition is satisfied along the crack, and not the vanishing of both individual shearing stresses. One might therefore expect the same type of discrepancy near the crack point to exist between Kirchhoff and Reissner bending results for this elastically supported case as found for the unsupported case [3, 7], wherein the singularity remained unchanged but the circumferential distribution around the crack tip did change. Outside this local region, Kirchhoff results are satisfactory.

Finally, if the foundation modulus is identified with an initial curvature as indicated in the Introduction, it is relatively straightforward to deduce the combined stress field, although these results will be reported separately.

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$$\sigma_y{}^* = \frac{6M^*}{h^2}$$

$$\frac{\sigma_y(x,0)}{\sigma_y{}^*} = \frac{f(\lambda y^*/\sqrt{2})}{\sqrt[4]{2}\,\sqrt{\pi}}\sqrt{(3+\nu)(1-\nu)}\,\frac{1}{\sqrt{\lambda x}} + O(x^0) \quad (83)$$

$$\frac{\sigma_y(x,0)}{\sigma_y{}^* f(\;)}\Bigg|_{\nu=1/3} \approx \frac{1}{\sqrt{2\lambda x}} \quad (85)$$

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