is much more complicated and leads to slowly convergent or even divergent expressions for the stress resultants.

2 The circular support's action on the plate is equivalent to a line load, and so the discontinuity in the elastic constants certainly make the deflection function singular at $r = a$. This is reflected in the fact that two different expressions are needed to describe the deflection function in the region $0 \leq r < \infty$, and that the functions w_1 and w_2 are not analytic continuations of each other across $r = a$, even for the uniform plate.

3 The second equation following (8) is only intended to indicate *one* convenient way for obtaining $\nabla^2 w^*$. If x,y; x_1,y_1; and x_2,y_2 are coordinate systems in the plane with different origins, and ∇^2, ∇_1^2, and ∇_2^2 denote Laplace's operators in these coordinate systems, it is clear that

$$\nabla^2(f + g) = \nabla^2 f + \nabla^2 g = \nabla_1^2 f + \nabla_2^2 g$$

In addition, it makes no difference whether Δ_1^2 and Δ_2^2 are expressed in their Cartesian or polar forms, and the equation in question follows directly from (5).

4 That the longer expression gives the shorter can be checked using the following intermediate results:

$$\nabla^2 r^2 \log r = 4(\log r + 1),$$

$$\nabla^2 r \log r \cos\theta = 2\cos\theta/r,$$

$$\nabla^2 \log r = 0,$$

$$\nabla^2 r^2 = 4.$$

5 Let us consider the case of the load being inside the circular support. For a vanishing rigidity of the exterior in *comparison* to the rigidity of the interior, the interior part deflects as if the exterior part were absent, and w_1 becomes the deflection function for a simply supported plate. The exterior part, however, is forced to deflect so as to maintain continuity in the radial slope at $r = a$. Moreover, the radial bending moment at $r = a$ required to deflect the exterior part causes only negligible deflections in the interior part because of the vastly different rigidities. Thus (28) constitutes the solution for the exterior part with the following boundary conditions at $r = a$: (*a*) Zero deflection, (*b*) radial slope equal to that of the simply supported interior plate (Reissner's solution). A similar interpretation can be given to (38).

The Bending Stress in a Cracked Plate on an Elastic Foundation[1]

G. C. SIH[2] and D. E. SETZER.[3] This paper is a valuable addition to the existing literature on the analysis of stress distribution near a crack point. It is of importance in the discussion of the remaining strength of bodies containing cracks. The authors are to be complimented not only for having solved a difficult crack problem, but also for indicating a possible relation between the solution of an elastically supported flat plate and a plate with initial curvature. This information suggests possibilities for extending some of the current fracture-mechanics theories to shell-like structures.

Upon a detailed examination of the authors' results, it should be pointed out that the effect of elastic foundation does not alter the qualitative character of the ordinary bending stresses near the crack point in an unsupported plate. Specifically, the circumferential stress variation in equations (60) through (62) of the paper should be the same as that obtained for the plate without an elastic foundation. The foundation modulus only enters into the problem in the intensity of the local stress field. Because of the coordinate system employed in the paper, this conclusion is not readily observable.

After lengthy manipulations, it can be shown that the local stress distribution in a cracked plate on an elastic foundation may be expressed in polar form as follows:

$$\sigma_r = \frac{7+\nu}{2(3+\nu)}\frac{K_1 z}{h(2r)^{1/2}}\left[\frac{3+5\nu}{7+\nu}\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right] + \frac{5+3\nu}{2(3+\nu)}\frac{K_2 z}{h(2r)^{1/2}}\left[\sin\frac{3\theta}{2} - \frac{3+5\nu}{5+3\nu}\sin\frac{\theta}{2}\right] + 0(r^{1/2}) \quad (1)$$

$$\sigma_\theta = \frac{7+\nu}{2(3+\nu)}\frac{K_1 z}{h(2r)^{1/2}}\left[\cos\frac{3\theta}{2} + \frac{5+3\nu}{7+\nu}\cos\frac{\theta}{2}\right] - \frac{5+3\nu}{2(3+\nu)}\frac{K_2 z}{h(2r)^{1/2}}\left[\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}\right] + 0(r^{1/2}) \quad (2)$$

$$\tau_{r\theta} = \frac{7+\nu}{2(3+\nu)}\frac{K_1 z}{h(2r)^{1/2}}\left[\sin\frac{3\theta}{2} - \frac{1-\nu}{7+\nu}\sin\frac{\theta}{2}\right] + \frac{5+3\nu}{2(3+\nu)}\frac{K_2 z}{h(2r)^{1/2}}\left[\cos\frac{3\theta}{2} - \frac{1-\nu}{5+3\nu}\cos\frac{\theta}{2}\right] + 0(r^{1/2}) \quad (3)$$

where z is the thickness coordinate measured from the middle plane of the plate. The formal appearance of these equations is indeed in agreement with that derived in an earlier paper by Williams [1][4] for a plate without elastic support. In making comparison with equations (60) through (62) of the paper, it should be noted that the angle θ, measured from the line of crack extension, is the complement of the angle φ. In addition, the real constants K_1 and K_2 are related to the complex constants P_0 and Q_0, respectively, by the following expressions

$$K_1 = \frac{2\pi^{1/2}(3+\nu)(1+i)Gh\lambda^2}{i\lambda^{1/2}} P_0 \quad (4)$$

$$K_2 = -\frac{2\pi^{1/2}(3+\nu)(1-i)Gh\lambda^2}{i\lambda^{1/2}} Q_0 \quad (5)$$

Here, K_1 and K_2 may be regarded as the crack-tip stress-intensity factors [2] for symmetric and skew-symmetric bending-stress distributions, respectively.

From these definitions of $K_i (i = 1, 2)$, the authors' results for the two limiting cases of interest may also be rearranged into the form previously stated, equations (4) and (5). They are as follows:

Case *i*: $\lambda \to 0$ but $a \neq 0$

$$K_1 = -\frac{(1-i)Gh}{(1-\nu)(a\pi)^{1/2}} m_0 \quad (4a)$$

$$K_2 = -\frac{(1+i)Gh}{(1-\nu)(a\pi)^{1/2}}\left(\frac{v_0}{a}\right) \quad (5a)$$

Case *ii*: $\lambda \neq 0$ but $a \to 0$

$$K_1 = -\frac{Gh}{(2\pi\lambda)^{1/2}}\left[\frac{3+\nu}{1-\nu}\right]^{1/2} m_0 \quad (4b)$$

$$K_2 = -\frac{iGh}{(2\pi\lambda)^{1/2}}\left[\frac{3+\nu}{1-\nu}\right]^{1/2}\left(\frac{v_0}{\lambda}\right) \quad (5b)$$

In connection with the Griffith-Irwin theory of fracture, it may be stated that the onset of rapid crack extension will correspond to reaching some critical values of the combination of K_1 and K_2 for a given material. Presumably, equations (4) and (5) may be employed directly in the fracture analysis of plates on elastic foundations.

[1] By D. D. Ang, E. S. Folias, and M. L. Williams, published in the June, 1963, issue of the JOURNAL OF APPLIED MECHANICS, vol. 30, TRANS. ASME, vol. 85, Series E, pp. 245–251.

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[4] Numbers in brackets designate References at end of discussion.

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As a specific example, the authors' problem of a cracked rectangular strip, subjected to bending moment M^* and uniform normal loading q_0, may be used. Substituting equations (79) and (80) of the paper into equations (4b) and (5b), it is found that

$$K_1 = \sigma_y{}^* f(\lambda y^*/\sqrt{2}) \left[\frac{(3+\nu)(1-\nu)}{2\pi\lambda}\right]^{1/2} \tag{6}$$

$$K_2 = 0$$

where $\sigma_y{}^* = 6M^*/h^2$. Note that q_0 does not appear in the solution of K_1. This result is as expected, since a uniform compression of the plate on an elastic foundation does not affect the singular stress components near the crack point.

Aside from its importance in fracture theories, equation (6) may be conveniently used to compute the stresses along the line of crack extension. For instance, inserting equation (6) into (2), it gives

$$\frac{\sigma_y(x, 0)}{\sigma_y{}^*} = \frac{f(\lambda y^*/\sqrt{2})}{2^{1/4}\,\pi^{1/2}} (3+\nu)^{1/2}(1-\nu)^{1/2} \frac{1}{(\lambda x)^{1/2}} + 0(x^{1/2})$$

This result differs from that of equation (83) obtained in the paper. It follows then that by these considerations equation (85) should be of the form

$$\left.\frac{\sigma_y(x,0)}{\sigma_y{}^* f(\lambda y^*/\sqrt{2})}\right|_{\nu=\frac{1}{3}} \approx \frac{1}{(2\lambda x)^{1/2}}$$

As a final point of interest, it is worthwhile noting that the simultaneous extension-bending stress field in a shallow spherical shell with a radial crack follows immediately from the observations made in the earlier part of this discussion. At the same time, the discussers would like to take this opportunity to clarify the interpretation of some of the results given by the authors in a previous report [3], which is intimately related to the work in this paper. In the report, it was not clear that the shell solution for increasingly large curvature tends toward the solution for initially flat plates.

In the introduction to the paper, the effect of initial curvature, R, is shown to be qualitatively equivalent to having an elastic foundation with modulus k for a flat plate. Hence, the bending part of the shell solution is identical to the supported-plate solution upon recognizing the equivalency relation $k = Eh/R^2 = D\lambda^4$. This implies that equations (1) through (3) may also represent the bending stresses in a shell, where $K_i(i = 1, 2)$ are now dependent upon the initial curvature, R.

Similar results for the membrane part of the problem may be observed also. Restricting attention to a small region around the crack tip, the Reissner [4] shallow-shell equations show that both w and F satisfy basically the same type of differential equation. Moreover, the stress solution given by equations (1) through (3) reveals that for small values of r, $\nabla^4 w = 0$, and thus F must also behave as the biharmonic function in this vicinity. By specifying the free crack surface conditions of vanishing normal and tangential membrane stresses

$$N_\theta = \frac{\partial^2 F}{\partial r^2} = 0, \qquad N_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right) = 0, \qquad \text{for } \theta = \pm\pi \tag{7}$$

the local membrane stresses may be written as[5]

$$N_r = \frac{1}{2(2r)^{1/2}}\left[k_1 \cos\frac{\theta}{2}(3 - \cos\theta) + k_2 \sin\frac{\theta}{2}(3\cos\theta - 1)\right] + 0(r^{1/2}) \tag{8}$$

$$N_\theta = \frac{1}{2(2r)^{1/2}}\left[k_1 \cos\frac{\theta}{2}(1 + \cos\theta) - k_2\left(3 \sin\theta \cos\frac{\theta}{2}\right)\right] + 0(r^{1/2}) \tag{9}$$

$$N_{r\theta} = \frac{1}{2(2r)^{1/2}}\left[k_1 \sin\theta \cos\frac{\theta}{2} + k_2 \cos\frac{\theta}{2}(3\cos\theta - 1)\right] + 0(r^{1/2}) \tag{10}$$

Comparing equations (8) through (10) with the authors' equations (117) through (119) of the report [3], it is observed that they are indeed identical if

$$k_1 = \frac{2i\pi^{1/2}(1+i)\lambda^2}{\lambda} \tilde{P}_0 \tag{11}$$

$$k_2 = \frac{2\pi^{1/2}(1+i)\lambda^2}{\lambda^{1/2}} \tilde{Q}_0 \tag{12}$$

Similar to the bending solutions, k_1 and k_2 may be regarded as the crack-tip stress-intensity factors for the symmetric and skew-symmetric portions of the membrane stress field, respectively.

While the local angular distribution of the extension-bending stress field in a spherical shell appears to be identical with those obtained by superimposing the separate extensional and bending stress of a flat plate, it must be emphasized that the stress-intensity factors $k_i(i = 1, 2)$ and $K_i(i = 1, 2)$ for the shell problem are interconnected through the loading. More precisely, if a plate is initially curved, a stretching load will generally produce both types of stress-intensity factors; namely, k_i and K_i, and similarly a bending load will also yield both k_i and K_i, i.e., the curvature causes stretching to introduce bending and vice versa.

It is now clear that as the initial curvature of the shell becomes increasingly large, the circumferential stress distribution near the crack point remains unchanged, while the stress-intensity factors k_i and K_i approach the separate solutions for the initially flat plate.

In closing, the discussers wish to congratulate the authors for the excellent piece of analytical work. The results of the paper are most useful to those interested in the field of fracture mechanics.

[5] For details, the reader may refer to a paper by Williams [5].

References

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Authors' Closure

The authors wish to thank the discussers for their interest in the paper and for calling attention to corrections for equation (82), and hence (83) and (85) which follow. Incidentally, it is believed that the discusser's equations (4b), (5b), and (6) should be corrected by the multiplication factor 2 and three fourths on the right-hand side of the respective equations.

In connection with the report (Ref. [3] of the paper) to which the discussers refer, consisting of the problem of a shallow spherical shell containing a semi-infinite crack, there are, as in the case of a cracked plate on an elastic foundation, three special cases of interest:

$$\text{(i) } a \to 0, \quad \lambda \neq 0$$
$$\text{(ii) } a \neq 0, \quad \lambda \to 0$$
$$\text{(iii) } a \to 0, \quad \lambda \to 0$$

From our solution we can *only* recover the first two limits.

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This is characteristic of mixed-boundary-value problems, which are solved by the Wiener-Hopf technique. In order to obtain the third case we must relax our boundary conditions at infinity. Thus the method used may, for some loadings, prevent one from recovering the flat-sheet behavior from that of the curved sheet. For these reasons, and as discussed orally during the presentation, the problem of a shallow spherical shell containing a finite crack has now been solved using a different mathematical approach (E. S. Folias, "The Stresses in a Spherical Shell Containing a Crack," GALCIT SM 63-20, November, 1963, doctorate dissertation).

In closing, the authors wish to again emphasize the distinction between the Kirchhoff and Reissner type bending boundary conditions [Refs. [3] and [7] of the paper) and urge caution in the use of stress-intensity factors which, in contrast to the extensional solution, have been deduced from a local (bending) stress distribution incorporating incomplete boundary conditions.

The Influence of the Equilibrium Dissociation of a Diatomic Gas on Brayton-Cycle Performance[1]

R. SABERSKY.[2] This paper is an extension of the authors' previous work in which they analyzed the effect of changes in specific heat of the working substance on the efficiency of the Brayton cycle. The results of the present paper show that the effect of chemical equilibrium on this efficiency may, under certain conditions, be significant. The effect comes about, presumably, by the dissociation causing changes in the heat capacity in such a way that the Brayton cycle approaches the Carnot cycle more closely.

The range in which significant improvements are indicated is limited, however, to very low overall temperature ratios (T_3/T_1), and to a restricted range of the initial temperatures. In order to estimate whether or not the indicated improvement in efficiency is of practical significance, it will be necessary to carry out the indicated calculations for specific substances and to compute the actual temperatures which would occur in the cycle.

Furthermore, one may also point out that if the temperature ratios in question are small, changes in component efficiency also lead to large percentage changes in efficiency. Facts of this type will have to be taken into account in any optimization study, particularly as one could imagine that the component efficiency might be affected by the initial temperature T_1, which has to be selected within certain limits in order to take advantage of the shifting chemical equilibrium.

F. A. WILLIAMS.[3] This is the second of two papers concerning the maximum theoretical thermal efficiency of the Brayton cycle for a working substance with a variable specific heat. The first paper dealt with an ideal gas with partially excited vibrational modes; this paper concerns Lighthill's model of an ideal dissociating diatomic gas. It may be worth emphasizing that chemical and thermodynamic equilibrium is postulated throughout the cycle in both analyses. The thermal efficiency is maximized, subject to given compressor and turbine efficiencies, a given turbine-inlet temperature and given compressor-inlet conditions. This maximum efficiency is compared with the maximum efficiency obtainable by utilizing an ideal gas with a constant specific heat as a working substance. It is shown that, for low turbine-inlet temperatures, the maximum thermal efficiency can be increased by more than a factor of 2 by choosing a dissociating gas with an optimum value of the dissociation energy. However, it is, of course, found that this higher efficiency is achieved only at the expense of employing considerably higher pressure ratios. Although the thermal efficiency is not the only parameter of importance in heat-engine design, the authors have certainly justified their conclusion that the dissociating working fluid merits further study with a view toward application.

[1] By T. A. Jacobs and J. R. Lloyd, published in the June, 1963, issue of the JOURNAL OF APPLIED MECHANICS, vol. 30, TRANS. ASME, vol. 85, Series E, pp. 288–290.

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Force Singularities of Shallow Cylindrical Shells[1]

W. FLÜGGE[2] and D. A. CONRAD.[3] The author is to be congratulated for his success in finding a solution for the concentrated force in terms of integrals of products of cylindrical and circular functions. We attempted several years ago to find such a solution, but were unsuccessful. The author, referring to our paper[4] states that we concluded "that unlike the thermal singularities, the force singularities are not expressible in terms of moderately simple solutions of field equations." In actuality, we developed a particular set of singular solutions to the shallow-shell equations, interpreted two of them as thermal singularities, and showed that the force singularity was not included among them. It was not implied or stated that other sets of solutions, perhaps containing the force singularities, did not exist. Our only conclusion was that we did not find such a solution and were therefore forced to use other methods in dealing with the concentrated force.

In a later note,[5] we called attention to a convenient method for the calculation of shallow shells in general and presented a solution for the concentrated force on the cylinder. It would have been of interest to see comparable numerical results using the new solution. It would appear that the series approach is more convenient for direct calculation, but that the author's solution may provide some advantages for use as a Green's function.

Author's Closure

I thank Professor Flügge and Dr. Conrad for their kind comments and remarks. I would like to point out that partly what I had hoped to be emphasized in the paper was the organic relation between the singular solutions, that is, the fact that once a singular solution of a partial differential equation is known in general, other singularities can be identified by differentiation or integration.

[1] By A. Jahanshahi, published in the September, 1963, issue of the JOURNAL OF APPLIED MECHANICS, vol. 30, TRANS. ASME, vol. 85, Series E, pp. 342–346.

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[3] Senior Staff Engineer, Hughes Aircraft Company, Los Angeles, Calif. Assoc. Mem. ASME.

[4] W. Flügge and D. A. Conrad, "Thermal Singularities for Cylindrical Shells," *Proceedings of the Third U. S. National Congress of Applied Mechanics*, 1958, p. 321.

[5] W. Flügge and D. A. Conrad, "A Note on the Calculation of Shallow Shells," JOURNAL OF APPLIED MECHANICS, vol. 26, TRANS. ASME, vol. 81, Series E, 1959, p. 683.

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