A CIRCUMFERENTIAL CRACK IN A PRESSURIZED CYLINDRICAL SHELL

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ABSTRACT

Following an earlier analysis of an axial line crack in a cylindrical shell, the stresses for a finite circumferential crack are presented. The inverse square root singularity of the stresses peculiar to crack problems is obtained in both the extensional and bending components. Furthermore, the initial curvature may be related to that found in an initially flat plate by a factor of the form

$$\frac{\sigma_{shell}}{\sigma_{plate}} \approx 1 + \left(a + b\,\ell n \frac{c}{\sqrt{Rh}}\right)\frac{c^2}{Rh} + \dots$$

where the quantity in parenthesis is positive.

INTRODUCTION

This paper is the third in a series concerned with the study of the effect of an initial curvature on the stress distribution of a thin sheet containing a crack. The inherent consequence of initial curvature is the presence of an interaction between bending and stretching. Therefore, the subject of eventual concern is the simultaneous stress fields produced in initially curved cracked sheets.

For the two simple geometries which come to mind, a spherical shell, and a cylindrical shell, the author has discussed the results in two recent papers.[1,2] The former is concerned with a line crack in a spherical cap, while the latter discusses a finite axial crack in a pressurized cylindrical shell. It is the intent of this paper to extend the work of Ref. 2 by considering the conjugate problem, namely, that of a finite circumferential crack in a cylindrical shell.

FORMULATION OF THE PROBLEM

Consider a portion of a thin, shallow cylindrical shell of constant thickness h, subjected to an internal pressure q. This material of the shell is assumed to be homogeneous and isotropic; perpendicular to the axis there exists a cut of length 2c. Following Marguerre,[3] the coupled differential equations governing the displacement function w and the stress function F, with x and y as dimensionless rectangular coordinates of the base plane (see Figure 1), are given by:

$$\frac{Ehc^2}{R}\frac{\partial^2 w}{\partial x^2} + \nabla^4 F = 0 \tag{1}$$

$$\nabla^4 w - \frac{c^2}{RD}\frac{\partial^2 F}{\partial x^2} = \frac{q}{D}c^4 \tag{2}$$

where R is the radius of the cylinder. As to boundary conditions, one must require that the normal moment, equivalent vertical shear, and normal and tangential in-plane stresses vanish along the crack. However, suppose that one has already found a particular solution** satisfying eqns 1 and 2,

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** As an illustration of how the local solution may be combined in a particular case, see ref. 4.

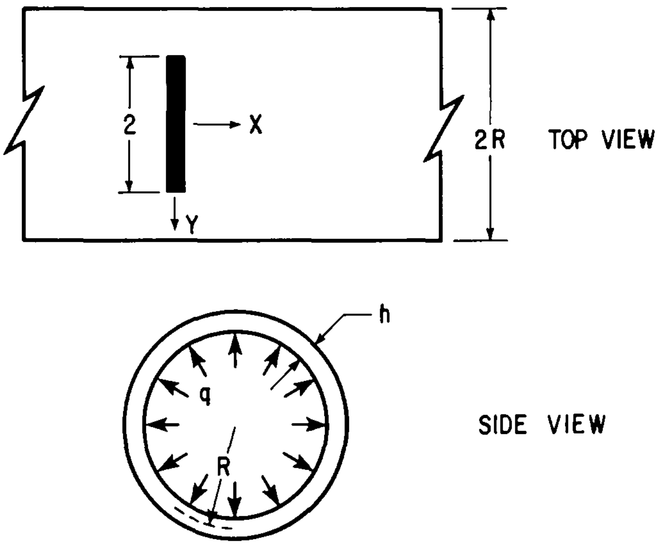

Fig. 1. Geometry and coordinates.

but that there is a residual normal moment M_x, equivalent vertical shear V_x, normal in-plane stress N_x, and in-plane tangential stress N_{xy}, along the crack $|y| < 1$ of the form:

$$M_x^{(P)} = -D\,\frac{m_o}{c^2},\quad V_x^{(P)} = 0,\quad N_x^{(P)} = -\frac{n_o}{c^2},\quad N_{xy}^{(P)} = 0 \qquad (3\text{-}6)$$

where m_o and n_o will be considered constants for simplicity.

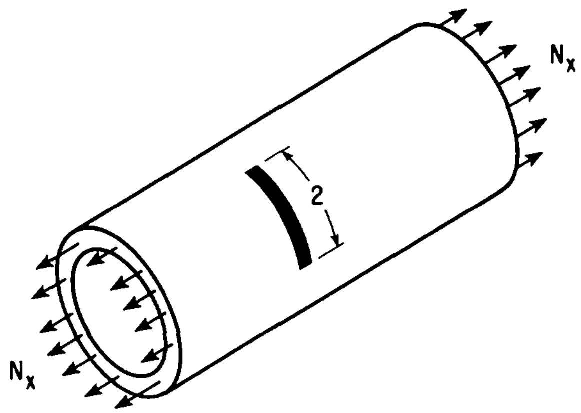

Fig. 2. Cracked shell under uniform axial extension N_x and internal pressure q_0.

MATHEMATICAL STATEMENT OF THE PROBLEM

Assuming, therefore, that a particular solution has been found, we need to find two functions of the dimensionless coordinates (x, y), W(x, y) and F(x, y), such that they satisfy the homogeneous partial differential equations 1 and 2 and the following boundary conditions.

At $x = 0$ and $|y| < 1$:

$$M_x(0,y) = -\frac{D}{c^2}\left[\frac{\partial^2 W}{\partial x^2} + \nu\frac{\partial^2 W}{\partial y^2}\right] = \frac{Dm_o}{c^2} \tag{7}$$

$$V_x(0,y) = -\frac{D}{c^3}\left[\frac{\partial^3 W}{\partial x^3} + (2-\nu)\frac{\partial^3 W}{\partial y^2 \partial x}\right] = 0 \tag{8}$$

$$N_x(0,y) = \frac{1}{c^2}\frac{\partial^2 F}{\partial y^2} = \frac{n_o}{c^2} \tag{9}$$

$$N_{xy}(0,y) = -\frac{1}{c^2}\frac{\partial^2 F}{\partial x \partial y} = 0 \tag{10}$$

At $x = 0$ and $|y| > 1$ we must satisfy the continuity requirements, namely

$$\lim_{|x|\to 0}\left[\frac{\partial^n}{\partial x^n}(W^+) - \frac{\partial^n}{\partial x^n}(W^-)\right] = 0 \tag{11}$$

$$\lim_{|x|\to 0}\left[\frac{\partial^n}{\partial x^n}(F^+) - \frac{\partial^n}{\partial x^n}(F^-)\right] = 0 \tag{12}$$

for $n = 0, 1, 2, 3$. Furthermore, we shall limit ourselves to large radii of curvature, i.e., small deviations from flat sheets; we thus require that the displacement function W and the stress function F together with their first derivatives be finite far away from the crack. In this manner, we avoid infinite stresses and infinite displacements in the region far away from the crack. These restrictions at infinity simplify the mathematical complexities of the problem considerably. Furthermore, the first corresponds to the usual expectations of St. Venant's principle.

METHOD OF SOLUTION

We construct the following integral representations which have the proper symmetrical behavior with respect to y, with $\lambda^4 \equiv Ehc^4/R^2D$:

$$\begin{aligned} W(x^{\pm}, y) = \int_0^\infty & P_1 \exp\left[\left(\frac{\alpha\lambda}{2} - \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\right)|x|\right] + P_2 \exp\left[-\left(\frac{\alpha\lambda}{2} + \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\right)|x|\right] \\ & + P_3 \exp\left[-\left(\frac{\beta\lambda}{2} + \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\right)|x|\right] \\ & + P_4 \exp\left[\left(\frac{\beta\lambda}{2} - \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\right)|x|\right]\Bigg\} \cos ys \, ds \end{aligned} \tag{13}$$

$$\begin{aligned} F(x^{\pm}, y) = \frac{\alpha^2 Ehc^2}{\lambda^2 R} \int_0^\infty & \Bigg\{ P_1 \exp\left[\left(\frac{\alpha\lambda}{2} - \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\right)|x|\right] \\ & + P_2 \exp\left[-\left(\frac{\alpha\lambda}{2} + \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\right)|x|\right] \\ & - P_3 \exp\left[-\left(\frac{\beta\lambda}{2} + \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\right)|x|\right] \\ & - P_4 \exp\left[\left(\frac{\beta\lambda}{2} - \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\right)|x|\right]\Bigg\} \cos ys \, ds \end{aligned} \tag{14}$$

where the P_i (i = 1, 2, 3, 4) are arbitrary functions of s to be determined from the boundary conditions. The ± signs refer to x > 0 and x < 0, respectively, and $\alpha \equiv i^{\frac{1}{2}}$, $\beta \equiv (-i)^{\frac{1}{2}}$.

Imposition of the boundary condition requirements eqns 7-10, using eqns 8 and 10 to determine P_3 and P_4,* give for x = 0 | y | < 1

$$- \lim_{|x| \to 0} \int_0^\infty \left\{ \left[(1-\nu)s^2 + \frac{\alpha^2\lambda^2}{2} \right] (P_1 + P_2) \exp\left[- \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\, |x| \right] \right.$$

$$- \alpha\lambda \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\, (P_1 - P_2) \exp\left[- \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\, |x| \right]$$

$$+ \left[(1-\nu)(2+\nu)s^2 + \frac{\beta^2\lambda^2}{2} \right] \frac{\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}}{\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} (P_1 + P_2) \exp\left[- \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\, |x| \right]$$

$$\left. - \frac{\alpha\lambda}{2} \left[(1-\nu^2-2\nu)s^2 + \frac{\beta^2\lambda^2}{2} \right] \frac{(P_1 - P_2)}{\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \exp\left[- \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\, |x| \right] \right\} \cos ys\, ds = m_o \tag{13}$$

$$- \frac{\alpha^2 Ehc^2}{\lambda^2 R} \lim_{|x| \to 0} \int_0^\infty \left\{ (P_1 + P_2) \exp\left[- \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\, |x| \right] + \frac{\alpha\lambda}{2}(1+\nu) \frac{P_1 - P_2}{\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \right.$$

$$\exp\left[- \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\, |x| \right] - \nu \frac{\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}}{\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} (P_1 + P_2)$$

$$\left. \exp\left[- \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\, |x| \right] \right\} s^2 \cos ys\, ds = n_o \tag{14}$$

whereas continuity conditions on the functions and their derivatives for x = 0 and | y | > 1 may be satisfied if

$$\int_0^\infty \begin{pmatrix} \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\, (P_1 + P_2) \\ \frac{\alpha\lambda}{2} (P_1 - P_2) \end{pmatrix} \cos ys\, ds = 0; \quad |y| > 1 \tag{15, 16}$$

Therefore, we have reduced our problem to solving the dual integral equations 13-16 for the unknown functions $(P_1 - P_2)(s)$ and $(P_1 + P_2)(s)$. These may be transformed** to the following set of coupled singular integral equations

$$\int_{-1}^{1} \left\{ L_1 u_1 + L_2 u_2 \right\} d\xi = - \frac{n_o \pi \lambda^2 R y}{\alpha^2 Ehc^2} \quad ; \quad |y| < 1 \tag{17}$$

$$\int_{-1}^{1} \left\{ L_3 u_1 + L_4 u_2 \right\} d\xi = - m_o \pi y \quad ; \quad |y| < 1 \tag{18}$$

* P_3 and P_4 are given implicitly in terms of P_1, P_2 in the appendix, where we have used a shear free condition along the y-axis.

** See ref. 5.

where u_1 and u_2 are unknown functions defined by

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \int_0^\infty \begin{pmatrix} \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\ (P_1 + P_2) \\ \frac{\alpha\lambda}{2}\ (P_1 - P_2) \end{pmatrix} \cos ys\, ds \quad ; \quad |y| < 1 \qquad (19, 20)$$

and the kernels L_i $(i = 1, 2, 3, 4)$ are complicated modified Bessel functions (see appendix).

Following the same method of solution as in ref. 5, the singular integral equations may be solved for small values of the parameter λ. We do not show the details of this solution, but a list of the requisite steps can be found in the appendix. Finally, the displacement and stress functions are:

$$\begin{aligned} W(x^{\pm}, y) = \int_0^\infty \Bigg\{ & \left[\frac{\frac{\alpha\lambda}{2} A_1 + \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\ B_1}{\alpha\lambda\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}} \right] \frac{J_1(s)}{s} \exp\left[\left(\frac{\alpha\lambda}{2} - \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}} \right) |x| \right] \\ & + \left[\frac{\frac{\alpha\lambda}{2} A_1 - \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\ B_1}{\alpha\lambda\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}} \right] \frac{J_1(s)}{s} \exp\left[- \left(\frac{\alpha\lambda}{2} + \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}} \right) |x| \right] \\ & + \left[\left(\frac{\beta\lambda}{2} - \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}} \right) \frac{\nu(A_1 - B_1) J_1(s)}{\beta\lambda s\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \right. \\ & \left. - \left(\frac{\beta\lambda}{2} B_1 - \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\ A_1 \right) \frac{J_1(s)}{\beta\lambda s\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \right] \exp\left[- \left(\frac{\beta\lambda}{2} + \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}} \right) |x| \right] \\ & + \left[\left(\frac{\beta\lambda}{2} + \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}} \right) \frac{\nu(A_1 - B_1) J_1(s)}{\beta\lambda s\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} - \left(\frac{\beta\lambda}{2} B_1 + \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\ A_1 \right) \right. \\ & \left. \cdot \frac{J_1(s)}{\beta\lambda s\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \right] \exp\left[\left(\frac{\beta\lambda}{2} - \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}} \right) |x| \right] + \ldots \Bigg\} \cos ys\, ds \qquad (21) \end{aligned}$$

$$\begin{aligned} F(x^{\pm}, y) = \frac{\alpha^2 Ehc^2}{\lambda^2 R} \int_0^\infty \Bigg[& \frac{\frac{\alpha\lambda}{2} A_1 + \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\ B_1}{\alpha\lambda\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}} \frac{J_1(s)}{s} \exp\left[\left(\frac{\alpha\lambda}{2} - \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}} \right) |x| \right] \\ & + \frac{\frac{\alpha\lambda}{2} A_1 - \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\ B_1}{\alpha\lambda\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}} \frac{J_1(s)}{s} \exp\left[- \left(\frac{\alpha\lambda}{2} + \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}} \right) |x| \right] \\ & - \left(\frac{\beta\lambda}{2} - \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}} \right) \frac{\nu(A_1 - B_1) J_1(s)}{\beta\lambda s\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \exp\left[- \left(\frac{\beta\lambda}{2} + \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}} \right) |x| \right] \end{aligned}$$

$$+\left(\frac{\beta\lambda}{2} B_1 - \sqrt{s^2+\frac{\beta^2\lambda^2}{4}} A_1\right) \frac{J_1(s)}{\beta\lambda s\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}} \exp\left[-\left(\frac{\beta\lambda}{2}+\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}\right)|x|\right]$$

$$-\left(\frac{\beta\lambda}{2} + \sqrt{s^2+\frac{\beta^2\lambda^2}{4}}\right) \frac{\nu(A_1-B_1)J_1(s)}{\beta\lambda s\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}} \exp\left[\left(\frac{\beta\lambda}{2}-\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}\right)|x|\right]$$

$$\left.+\left(\frac{\beta\lambda}{2} B_1 + \sqrt{s^2+\frac{\beta^2\lambda^2}{4}} A_1\right) \frac{J_1(s)}{\beta\lambda s\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}} \exp\left[\left(\frac{\beta\lambda}{2}-\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}\right)|x|\right]+\ldots\right\}\cos ys\, ds \tag{22}$$

where

$$A_1 = -\frac{n_o\lambda^2 R}{2\alpha^2 Ehc^2}\left\{1+\frac{1-2\nu-\nu^2}{\nu_o(3+\nu)}\frac{\pi\lambda^2}{32}+\frac{(1+\nu)^2}{(1-\nu)(3+\nu)}\frac{\alpha^2\lambda^2}{32}\left(1+2\gamma+2\ell n\frac{\lambda\beta}{8}\right)\right\}$$

$$-\frac{m_o(1+\nu)}{2(1-\nu)(3+\nu)}\left\{1-\frac{(1+\nu)^2}{\nu_o(3+\nu)}-\frac{\pi\lambda^2}{32}-\frac{\alpha^2\lambda^2}{32}\left(1+2\gamma+2\ell n\frac{\beta\lambda}{8}\right)\right\}+0(\lambda^4\ell n\lambda). \tag{23}$$

$$B_1 = -\frac{n_o\lambda^2 R}{2\alpha^2 Ehc^2}\left\{1+\frac{\pi\lambda^2}{32}\frac{\alpha^2\lambda^2}{32(3+\nu)}\left[1+\nu-2\gamma-2\ell n\frac{\alpha\lambda}{8}+2(2+\nu)(\gamma+\ell n\frac{\beta\lambda}{8})\right]\right\}$$

$$+\frac{m_o}{2(3+\nu)}\left\{1-\frac{(1+\nu)^2}{(1-\nu)(3+\nu)}\frac{\pi\lambda^2}{32}+\frac{\alpha^2\lambda^2}{32(1-\nu)}\left(1+\nu+2\gamma+2\ell n\frac{\alpha\lambda}{8}+2\nu\gamma+2\nu\ell n\frac{\beta\lambda}{8}\right)\right\}$$

$$+\ 0(\lambda^4\ell n\lambda) \tag{24}$$

THE SINGULAR STRESSES

The bending and extensional stress components are defined in terms of the displacement function W and stress function F as:

$$\sigma_{x_b} = -\frac{Ez}{(1-\nu^2)c}\left[\frac{\partial^2 W}{\partial x^2}+\nu\frac{\partial^2 W}{\partial y^2}\right] \tag{25}$$

$$\sigma_{y_b} = -\frac{Ez}{(1-\nu^2)c}\left[\frac{\partial^2 W}{\partial y^2}+\nu\frac{\partial^2 W}{\partial x^2}\right] \tag{26}$$

$$\tau_{xy_b} = \frac{2Gz}{c}\frac{\partial^2 W}{\partial x\partial y} \tag{27}$$

$$\sigma_{x_e} = \frac{1}{hc^2}\frac{\partial^2 F}{\partial y^2} \tag{28}$$

$$\sigma_{y_e} = \frac{1}{hc^2}\frac{\partial^2 F}{\partial x^2} \tag{29}$$

$$\tau_{xy_e} = \frac{-1}{hc^2}\frac{\partial^2 F}{\partial x\partial y} \tag{30}$$

where z is the dimensionless distance through the thickness h of the shell, measured from the middle surface with positive direction inwards. The stresses may be expressed in an integral form from which, when evaluated, give for $\epsilon e^{i\theta} = y \pm ix$, $\epsilon \ll 1$

Bending Stresses: On the surface $z = -h/2c$ (upper fibers)

$$\sigma_{x_b} = -\frac{P_b}{\sqrt{2\epsilon}}\left(\frac{11+5\nu}{4}\cos\frac{\theta}{2} + \frac{1-\nu}{4}\cos\frac{5\theta}{2}\right) + 0(\epsilon^o) \tag{31}$$

$$\sigma_{y_b} = +\frac{P_b}{\sqrt{2\epsilon}}\left(-\frac{3-3\nu}{4}\cos\frac{\theta}{2} - \frac{1-\nu}{4}\cos\frac{5\theta}{2}\right) + 0(\epsilon^o) \tag{32}$$

$$\tau_{xy_b} = -\frac{P_b}{\sqrt{2\epsilon}}\left(-\frac{7+\nu}{4}\sin\frac{\theta}{2} - \frac{1-\nu}{4}\sin\frac{5\theta}{2}\right) + 0(\epsilon^o) \tag{33}$$

where

$$P_b = \frac{3n_o\lambda^2(1+\nu)}{16(3+\nu)\sqrt{12(1-\nu^2)hc^2}}\left\{1+2(\gamma+\ell n\frac{\lambda}{8})\right\} + \frac{6m_oD}{(3+\nu)c^2h}\left\{1-\frac{5+2\nu+\nu^2}{(1-\nu)(3+\nu)}\frac{\pi\lambda^2}{64}\right\} + 0(\lambda^4\ell n\lambda) \tag{34}$$

Similarly we find through the thickness

Extensional Stresses:

$$\sigma_{x_e} = \frac{P_e}{\sqrt{2\epsilon}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{5\theta}{2}\right) + 0(\epsilon^o) \tag{35}$$

$$\sigma_{y_e} = \frac{P_e}{\sqrt{2\epsilon}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{5\theta}{2}\right) + 0(\epsilon^o) \tag{36}$$

$$\tau_{xy_e} = \frac{P_e}{\sqrt{2\epsilon}}\left(\frac{1}{4}\sin\frac{\theta}{2} - \frac{1}{4}\sin\frac{5\theta}{2}\right) + 0(\epsilon^o) \tag{37}$$

where

$$P_e = \frac{n_o}{hc^2}\left\{1+\frac{\pi\lambda^2}{64}\right\} + \frac{\sqrt{12(1-\nu^2)}(1+\nu)}{32(1-\nu)(3+\nu)}\frac{m_o\lambda^2D}{h^2c^2}\left\{1+2(\gamma+\ell n\frac{\lambda}{8})\right\} + 0(\lambda^4\ell n\lambda) \tag{38}$$

CONCLUSIONS

As in the case of a spherical shell,

(i) the stresses near the crack tip are proportional to $1/\sqrt{\epsilon}$ and have the same angular distribution as that of a flat plate

(ii) an interaction occurs between bending and stretching

(iii) the stress intensity factors are function of R; in the limit as $R \to \infty$ we recover the intensity factors for bending (6) and extension(7) in a flat plate. Thus we may write

$$\frac{\sigma_{shell}}{\sigma_{plate}} \approx 1 + (a + b \ln \frac{c}{\sqrt{Rh}}) \frac{c^2}{Rh} + 0 (\frac{1}{R^2})$$

where the expression in parentheses for small values of the parameter $c/\sqrt{(Rh)}$ is a positive quantity.

From this and the corresponding result for a spherical cap, it would appear that the general effect of initial curvature is to increase the stress in the neighborhood of the crack point. It is also of some practical value to be able to correlate flat sheet behavior with that of initially curved specimens. In experimental work on brittle fracture for example, considerable effort might be saved since, by eqn 39, we would expect within elastic theory, to predict the behavior of curved sheets from flat sheet tests.

In conclusion it must be emphasized that the classical bending theory has been used in deducing the foregoing results. Hence, only the Kirchhoff shear condition is satisfied along the crack, and not the vanishing of both individual shearing stresses. While outside the local region the stress distribution should be accurate, one might expect the same type of discrepancy to exist near the crack point as that found by Knowles and Wang(8) in comparing Kirchhoff and Reissner bending results for the flat plate case. In this case the order of the stress singularity remained unchanged but the circumferential distribution around the crack changed so as to be precisely the same as that due to solely extensional loading. Pending further investigation of this effect for initially curved plates, one is tempted to conjecture that the bending amplitude and angular distribution would be the same as that of stretching.

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APPENDIX

The following is a list of the successive steps in the analysis.

$$P_3 + P_4 = -(1+\nu) \frac{\alpha\lambda}{2} \frac{(P_1 - P_2)}{\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} + \nu \frac{\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}}{\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} (P_1 + P_2)$$

$$P_3 - P_4 = \frac{\nu\alpha}{\beta} (P_1 - P_2) + \frac{2\nu_o}{\beta\lambda} \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}} (P_1 + P_2)$$

where $\nu_o \equiv 1-\nu$

$$L_1 \equiv \lim_{|x| \to 0} \int_0^\infty \left\{ \frac{s \exp\left[-\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\, |x|\right]}{\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}} - \nu \frac{s \exp\left[-\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\, |x|\right]}{\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \right\}$$

$$\sin \zeta s\, ds = \frac{\alpha\lambda}{2} K_1\left(\frac{\alpha\lambda|\zeta|}{2}\right) - \nu \frac{\beta\lambda}{2} K_1\left(\frac{\beta\lambda|\zeta|}{2}\right) \approx \frac{1-\nu}{\zeta} - (1+\nu)\frac{\alpha^2\lambda^2\zeta}{16}$$

$$+ \frac{\alpha^2\lambda^2\zeta}{8}\left(\gamma + \ell n \frac{\alpha\lambda|\zeta|}{4}\right) - \frac{\nu\beta^2\lambda^2\zeta}{8}\left(\gamma + \ell n \frac{\beta\lambda|\zeta|}{4}\right) + 0(\lambda^4\zeta^3 \ \ell n\ \lambda|\zeta|)$$

$$L_2 \equiv \lim_{|x|\to 0} \int_0^\infty (1+\nu)\frac{s \exp\left[-\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}|x|\right]}{\sqrt{s \ + \frac{\beta^2\lambda^2}{4}}} \sin \zeta s\, ds = (1+\nu)\frac{\beta\lambda}{2} K_1\left(\frac{\beta\lambda|\zeta|}{2}\right)$$

$$\approx \frac{1+\nu}{\zeta} + (1+\nu)\frac{\beta^2\lambda^2\zeta}{8}\left(\gamma + \ell n\frac{\beta\lambda|\zeta|}{4}\right) - (1+\nu)\frac{\beta^2\lambda^2\zeta}{16} + 0(\lambda^4\zeta^3\ \ell n\ \lambda|\zeta|)$$

$$L_3 \equiv \lim_{|x|\to 0} \int_0^\infty \left\{\frac{\nu_o s \exp\left[-\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}\ |x|\right]}{\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}} + (2+\nu)\nu_o \frac{s \exp\left[-\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}\ |x|\right]}{\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}}\right\}$$

$$\sin \zeta s\, ds + \frac{\alpha^2\lambda^2}{2}\int_0^\infty \left\{\frac{\exp\left[-\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}\ |x|\right]}{s\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}} - \frac{\exp\left[-\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}\ |x|\right]}{s\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}}\right\} \sin \zeta s\, ds$$

$$= \nu_o \frac{\alpha\lambda}{2} K_1\left(\frac{\alpha\lambda|\zeta|}{2}\right) + (2+\nu)\nu_o \frac{\beta\lambda}{2} K_1\left(\frac{\beta\lambda|\zeta|}{2}\right) + \frac{\alpha^2\lambda^2}{2}\int_0^\infty \left\{\frac{\exp\left[-\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}\ |x|\right]}{s\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}}\right.$$

$$\left. - \frac{\exp\left[-\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}\ |x|\right]}{s\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}}\right\} \sin \zeta s\, ds \approx \frac{\nu_o(3+\nu)}{\zeta} + (1+\nu)\ \nu_o \frac{\alpha^2\lambda^2\zeta}{16}$$

$$+ \nu_o \frac{\alpha^2\lambda^2\zeta}{8}\left(\gamma + \ell n\frac{\alpha\lambda|\zeta|}{4}\right) + \nu_o(2+\nu)\frac{\beta^2\lambda^2\zeta}{8}\left(\gamma + \ell n\frac{\beta\lambda|\zeta|}{4}\right) + \frac{\pi\lambda^2\zeta}{4} + 0(\lambda^4\zeta^3\ \ell n\ \lambda|\zeta|)$$

$$L_4 \equiv \lim_{|x|\to 0} \int_0^\infty \left\{\frac{2\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}\exp\left[-\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}\ |x|\right]}{s} + \frac{\beta^2\lambda^2}{2}\frac{\exp\left[-\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}\right]}{s\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}}\right\}$$

$$\sin \zeta s\, ds - (1-\nu^2-2\nu)\lim_{|x|\to 0}\int_0^\infty \frac{s \exp\left[-\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}|x|\right]}{\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}} \sin \zeta s\, ds$$

$$\approx -\frac{\nu_o(3+\nu)}{\zeta} + (1+\nu)^2\frac{\alpha^2\lambda^2\zeta}{16} - (1+\nu)^2\frac{\alpha^2\lambda^2\zeta}{8}\left(\gamma + \ell n\frac{\beta\lambda|\zeta|}{4}\right) - \frac{\pi\lambda^2\zeta}{8}$$

$$+ 0(\lambda^4\zeta^3\ \ell n\ \lambda|\zeta|)$$

where $\zeta \equiv y - \xi$

$$u_1(\xi) = \sqrt{1-\xi^2} \sum_{n=0}^{\infty} A_{n+1} \lambda^{2n} (1-\xi^2)^n$$

$$u_2(\xi) = \sqrt{1-\xi^2} \sum_{n=0}^{\infty} B_{n+1} \lambda^{2n} (1-\xi^2)^n$$

$$P_1 + P_2 = A_1 \frac{J_1(s)}{s\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}} + \dots$$

$$P_1 - P_2 = \frac{2B_1}{\alpha\lambda} \frac{J_1(s)}{s} + \dots$$

where A_1 and B_1 are given by 23, 24, and

$$(1-\nu)A_2 + (1+\nu)B_2 = (1+\nu) \frac{\alpha^2}{48} (A_1 - B_1)$$

$$(3+\nu)(1-\nu)(A_2 - B_2) = -\frac{\alpha^2(1+\nu)}{48} (1-\nu)A_1 + (1+\nu)B_1$$

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RÉSUMÉ - Faisant suite à une étude précédente qui traitait du cas de la fissure longitudinale dans une enveloppe cylindrique, le présent mémoire est relatif a l'analyse des états de leusiers autour d'une fissure circonferencielle.

La singularité de la fonction universe de la racine carrée des contreintes, qui est particulière aux problèmes de fission se retrouve pour les deux composantes de flexion et d'extension. En outre, il est possible de faire intervenir le rayen de courbrue initial et de se celier au cas de la tole plane grâce a une relation de la forme

$$\frac{\sigma_{enveloppe}}{\sigma_{tole}} \approx 1 + (a + b \,\ell n \frac{c}{\sqrt{Rh}}) \frac{c^2}{Rh} + \dots$$

dans laquelle le terme autre parentheses est de signe positif.

ZUSAMMENFASSUNG - Folgend einer früheren Analyse eines rechten Achsrisses in einer zylindrischen Schale, werden hier die Spannungen für einen endlichen Umkreis präsentiert. Die umgekehrte Wurzeleinzigartigkeit zu Rissproblemen besonderer Spannungen wird sowohl in Zugs- als Biegekomponenten gewonnen. Weiter, die Anfangsbiegung kann durch einen Formfaktor zu der einer am Anfang ebenen Platte verwandt werden:

$$\frac{\sigma_{shell}}{\sigma_{plate}} \approx 1 + (a + b \, \ell n \frac{c}{\sqrt{Rh}}) \frac{c^2}{Rh} + \ldots$$

wo die Grösse in Klammern positif ist.