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A CYLINDRICAL SHELL CONTAINING A PERIPHERAL CRACK

by

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INTRODUCTION

A problem in fracture mechanics relating to design of monocoque structures and pressure vessels deals with the stresses in the neighborhood of a crack in an initially curved sheet. The presence of curvature generates deviations from behavior of flat sheets in that imposed bending loads will induce extensional stresses and similarly, imposed stretching loads will lead to localized bending stresses. The imposed and induced stresses can combine so that the local stress level is higher than would be found in a flat plate similarly loaded. Thus initially curved panels have a reduced resistance to fracture initiation that is basically of geometric origins.

For the two simple geometries which come to mind, a spherical shell, and a cylindrical shell, the author has discussed the results in two recent reports.(1,2) The former is concerned with a line crack in a spherical cap, while the latter discusses a finite axial crack in a pressurized cylindrical shell. It is the intent of this report to extend the work of Ref. 2 by considering the conjugate problem, namely, that of a finite circumferential crack in a cylindrical shell.

FORMULATION OF THE PROBLEM

Consider a portion of a thin, shallow cylindrical shell of constant thickness h, subjected to an internal pressure q. This material of the shell is assumed to be homogeneous and isotropic; perpendicular to the axis there exists a cut of length 2c. Following Marguerre,(3) the coupled differential equations governing the displacement function w and the stress function F, with x and y as dimensionless rectangular coordinates of the base plane (see Figure 1) are given by:

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ABSTRACT

Investigations into the mechanics of fracture initiation with special emphasis on the effects of initial sheet curvature and the interaction of applied extensional and bending stresses have been pursued further at the University of Utah.

Using an integral formulation, the coupled Marguerre equations for a cylindrical shell with a peripheral crack of length 2c are solved for the in-plane and Kirchoff bending stresses. The inverse square root singularity of the stresses peculiar to crack problems was recovered in both the extensional and bending components. Furthermore, the initial curvature may be related to that found in an initially flat plate by a factor of the form

$$\frac{\sigma_{shell}}{\sigma_{plate}} \approx \frac{1 + f(c/\sqrt{Rh})}{1 + g(c/\sqrt{Rh})}$$

where the functions $f(c/\sqrt{Rh})$ and $g(c/\sqrt{Rh})$ are such that as the parameter increases the fraction also increases. This depicts that initially curved panels have a reduced resistance to fracture initiation that is basically of geometric origin.

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$$\frac{Ehc^2}{R}\frac{\partial^2 w}{\partial x^2} + \nabla^4 F = 0 \tag{1}$$

$$\nabla^4 w - \frac{c^2}{RD}\frac{\partial^2 F}{\partial x^2} = \frac{q}{D} c^4 \tag{2}$$

where R is the radius of the cylinder. As to boundary conditions, one must require that the normal moment, equivalent vertical shear, and normal and tangential in-plane stresses vanish along the crack. However, suppose that one has already found a particular solution* satisfying eqns 1 and 2, but that there is a residual normal moment M_x, equivalent vertical shear V_x, normal in-plane stress N_x, and in-plane tangential stress N_{xy}, along the crack $|y| < 1$ of the form:

$$M_x^{(P)} = -D\frac{m_o}{c^2}, \quad V_x^{(P)} = 0, \quad N_x^{(P)} = -\frac{n_o}{c^2}, \quad N_{xy}^{(P)} = 0 \tag{3-6}$$

where m_o and n_o will be considered constants for simplicity.

MATHEMATICAL STATEMENT OF THE PROBLEM

Assuming, therefore, that a particular solution has been found, we need to find two functions of the dimensionless coordinates (x,y), W(x,y) and F(x,y), such that they satisfy the homogeneous partial differential equations 1 and 2 and the following boundary conditions.

At x = 0 and $|y| < 1$:

$$M_x(o,y) = -\frac{D}{c^2}\left[\frac{\partial^2 W}{\partial x^2} + \nu\frac{\partial^2 W}{\partial y^2}\right] = \frac{Dm_o}{c^2} \tag{7}$$

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* As an illustration of how the local solution may be combined in a particular case see ref. 4.

$$V_x(0,y) = -\frac{D}{c^3}\left[\frac{\partial^3 W}{\partial x^3} + (2-\nu)\frac{\partial^3 W}{\partial y^2 \partial x}\right] = 0 \tag{8}$$

$$N_x(0,y) = \frac{1}{c^2}\frac{\partial^2 F}{\partial y^2} = \frac{n_o}{c^2} \tag{9}$$

$$N_{xy}(0,y) = -\frac{1}{c^2}\frac{\partial^2 F}{\partial x \partial y} = 0 \tag{10}$$

At $x = 0$ and $|y| > 1$ we must satisfy the continuity requirements, namely

$$\lim_{|x| \to 0}\left[\frac{\partial^n}{\partial x^n}(W^+) - \frac{\partial^n}{\partial x^n}(W^-)\right] = 0 \tag{11}$$

$$\lim_{|x| \to 0}\left[\frac{\partial^n}{\partial x^n}(F^+) - \frac{\partial^n}{\partial x^n}(F^-)\right] = 0 \tag{12}$$

for $n = 0,1,2,3$. Furthermore, we shall limit ourselves to large radii of curvature, i.e., small deviations from flat sheets; we thus require that the displacement function W and the stress function F together with their first derivaties be finite far away from the crack. In this manner, we avoid infinite stresses and infinite displacements in the region far away from the crack. These restrictions at infinity simplify the mathematical complexities of the problem considerably. Furthermore, the first corresponds to the usual expectations of St. Venant's principle.

METHOD OF SOLUTION

We construct the following integral representations which have the proper symmetrical behavior with respect to y, with $\lambda^4 \equiv Ehc^4/R^2D$:

$$W(x^{\pm},y) = \int_0^\infty P_1 \exp\left[\left(\frac{\alpha\lambda}{2} - \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\right)|x|\right] + P_2 \exp\left[-\left(\frac{\alpha\lambda}{2} + \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\right)|x|\right]$$

$$+ P_3 \exp\left[-\left(\frac{\beta\lambda}{2} + \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\right)|x|\right] + P_4 \exp\left[\left(\frac{\beta\lambda}{2} - \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\right)|x|\right]\Bigg\} \cos ys \, ds \tag{13}$$

$$F(x^{\pm},y) = \frac{\alpha^2 Ehc^2}{\lambda^2 R} \int_0^\infty \Bigg\{ P_1 \exp\left[\left(\frac{\alpha\lambda}{2} - \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\right)|x|\right] + P_2 \exp\left[-\left(\frac{\alpha\lambda}{2} + \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\right)|x|\right]$$

$$-P_3 \exp\left[-\left(\frac{\beta\lambda}{2} + \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\right)|x|\right] - P_4 \exp\left[\left(\frac{\beta\lambda}{2} - \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\right)|x|\right]\Bigg\} \cos ys \, ds \tag{14}$$

where the P_i $(i = 1,2,3,4)$ are arbitrary functions of s to be determined from the boundary conditions. The $\pm$ signs refer to $x>0$ and $x<0$, respectively, and $\alpha \equiv i^{\frac{1}{2}}$, $\beta \equiv (-i)^{\frac{1}{2}}$.

Imposition of the boundary condition requirements eqns 7-10, using eqns 8 and 10 to determine P_3 and P_4,* give for $x = 0$ and $|y| < 1$

$$-\lim_{|x| \to 0} \int_0^\infty \Bigg\{ \left[(1-\nu)s^2 + \frac{\alpha^2\lambda^2}{2}\right] (P_1+P_2) \exp\left[-\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\,|x|\right]$$

$$- \alpha\lambda\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\,(P_1-P_2) \exp\left[-\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\,|x|\right]$$

$$+ \left[(1-\nu)(2+\nu)s^2 + \frac{\beta^2\lambda^2}{2}\right] \frac{\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}}{\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}} (P_1+P_2) \exp\left[-\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\,|x|\right] \tag{13}$$

*P_3 and P_4 are given implicitly in terms of P_1, P_2 in the appendix, where we have used a shear free condition along the y-axis.

$$
-\frac{\alpha\lambda}{2}\left[(1-\nu^2-2\nu)s^2+\frac{\beta^2\lambda^2}{2}\right]\frac{(P_1-P_2)}{\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}}\exp\left[-\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}\,|x|\right]\Bigg\}\cos ys\,ds=m_o
$$

(13 contd.)

$$
-\frac{\alpha^2Ehc^2}{\lambda^2R}\lim_{|x|\to 0}\int_0^\infty\Bigg\{(P_1+P_2)\exp\left[-\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}\,|x|\right]+\frac{\alpha\lambda}{2}(1+\nu)\frac{P_1-P_2}{\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}}\exp\left[-\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}\,|x|\right]
$$

$$
-\nu\frac{\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}}{\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}}(P_1+P_2)\exp\left[-\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}\,|x|\right]\Bigg\}s^2\cos ys\,ds=n_o
$$

(14)

whereas continuity conditions on the functions and their derivatives for $x=0$ and $|y|>1$ may be satisfied if

$$
\int_0^\infty\begin{pmatrix}\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}\,(P_1+P_2)\\ \frac{\alpha\lambda}{2}(P_1-P_2)\end{pmatrix}\cos ys\,ds=0\ ;\ |y|>1 \qquad (15,16)
$$

Therefore, we have reduced our problem to solving the dual integral equations 13-16 for the unknown functions $(P_1-P_2)(s)$ and $(P_1+P_2)(s)$. These may be transformed* to the following set of coupled singular integral equations

$$
\int_{-1}^{1}\{L_1u_1+L_2u_2\}\,d\xi=-\frac{n_o\pi\lambda^2Ry}{\alpha^2Ehc^2}\quad ;\quad |y|<1 \qquad (17)
$$

$$
\int_{-1}^{1}\{L_3u_1+L_4u_2\}\,d\xi=-m_o\pi y\quad ;\quad |y|<1 \qquad (18)
$$

where u_1 and u_2 are unknown functions defined by

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* See ref. 5.

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \int_0^\infty \begin{pmatrix} \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}} \; (P_1+P_2) \\ \frac{\alpha\lambda}{2} \; (P_1-P_2) \end{pmatrix} \cos ys \, ds \qquad ; \; |y|<1 \tag{19,20}$$

and the kernels L_i ($i = 1,2,3,4$) are complicated modified Bessel functions (see appendix).

Following the same method of solution as in ref. 5, the singular integral equations may be solved for small values of the parameter λ. We do not show the details of this solution, but a list of the requisite steps can be found in the appendix. Finally, the displacement and stress functions are:

$$W(x^{\pm},y) = \int_0^\infty \Bigg\{ \left[\frac{\frac{\alpha\lambda}{2} A_1 + \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}} \, B_1}{\alpha\lambda\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}} \right] \frac{J_1(s)}{s} \exp\left[\left(\frac{\alpha\lambda}{2} - \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}} \right) |x| \right]$$

$$+ \left[\frac{\frac{\alpha\lambda}{2} A_1 - \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}} \, B_1}{\alpha\lambda\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}} \right] \frac{J_1(s)}{s} \exp\left[-\left(\frac{\alpha\lambda}{2} + \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}} \right) |x| \right]$$

$$+ \Bigg[\left(\frac{\beta\lambda}{2} - \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}} \right) \frac{\nu(A_1-B_1) \, J_1(s)}{\beta\lambda s \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}}$$

$$- \left(\frac{\beta\lambda}{2} B_1 - \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}} \, A_1 \right) \frac{J_1(s)}{\beta\lambda s \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \Bigg] \exp\left[-\left(\frac{\beta\lambda}{2} + \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}} \right) |x| \right]$$

$$+ \Bigg[\left(\frac{\beta\lambda}{2} + \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}} \right) \frac{\nu(A_1-B_1) \, J_1(s)}{\beta\lambda s \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} - \left(\frac{\beta\lambda}{2} B_1 + \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}} \, A_1 \right)$$

$$\cdot \frac{J_1(s)}{\beta\lambda s \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \Bigg] \exp\left[\left(\frac{\beta\lambda}{2} - \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}} \right) |x| \right] + \cdots \Bigg\} \cos ys \, ds \tag{21}$$

$$F(x^*,y) = \frac{\alpha^2 Ehc^2}{\lambda^2 R} \int_0^\infty \left\{ \frac{\frac{\alpha\lambda}{2} A_1 + \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\, B_1}{\alpha\lambda\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}} \frac{J_1(s)}{s} \exp\left[\left(\frac{\alpha\lambda}{2} - \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\right)|x|\right]\right.$$

$$+ \frac{\frac{\alpha\lambda}{2} A_1 - \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\, B_1}{\alpha\lambda\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}} \frac{J_1(s)}{s} \exp\left[-\left(\frac{\alpha\lambda}{2} + \sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\right)|x|\right]$$

$$- \left(\frac{\beta\lambda}{2} - \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\right) \frac{\nu(A_1 - B_1)\, J_1(s)}{\beta\lambda s\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \exp\left[-\left(\frac{\beta\lambda}{2} + \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\right)|x|\right]$$

$$+ \left(\frac{\beta\lambda}{2} B_1 - \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\, A_1\right) \frac{J_1(s)}{\beta\lambda s\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \exp\left[-\left(\frac{\beta\lambda}{2} + \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\right)|x|\right]$$

$$- \left(\frac{\beta\lambda}{2} + \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\right) \frac{\nu(A_1 - B_1)\, J_1(s)}{\beta\lambda s\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \exp\left[\left(\frac{\beta\lambda}{2} - \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\right)|x|\right]$$

$$\left. + \left(\frac{\beta\lambda}{2} B_1 + \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\, A_1\right) \frac{J_1(s)}{\beta\lambda s\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \exp\left[\left(\frac{\beta\lambda}{2} - \sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\right)|x|\right] + \ldots \right\} \cos ys\, ds \qquad (22)$$

where

$$A_1 - B_1 = -\frac{n_o \lambda^4 R(1+\nu)}{32Ehc^2\, \nu_o (3+\nu)\Delta^*} \left\{1 + 2(\gamma + \ln\frac{\lambda}{8}) - \left(\frac{17-22\nu-11\nu^2}{16\nu_o}\right) \pi(\lambda/4)^2\right\}$$

$$- \frac{m_o}{\nu_o(3+\nu)\Delta^*} \left\{1 - \frac{\pi\lambda^2}{64} + \frac{13+2\nu+\nu^2}{32(3+\nu)(1-\nu)} (\lambda/4)^4\right. \qquad (23)$$

$$\left. + \frac{-25+22\nu+11\nu^2}{8(3+\nu)\nu_o} (\gamma + \ln\frac{\lambda}{8}) (\lambda/4)^4\right\} + O(\lambda^6 \ln\lambda)$$

$$(1-\nu)A_1+(1+\nu)B_1 = -\frac{n_o \lambda^2 R}{\alpha^2 Ehc^2\Lambda^*}\left\{1+\frac{5+2\nu+\nu^2}{\nu_o(3+\nu)}\frac{\pi\lambda^2}{64}-\frac{19-2\nu-\nu^2}{32\nu_o(3+\nu)}(\lambda/4)^4\right.$$

$$\left.+\frac{23+22\nu+11\nu^2}{8\nu_o(3+\nu)}(\gamma+\ln\frac{\lambda}{8})(\lambda/4)^4\right\} \tag{24}$$

$$+\frac{m_o(1+\nu)}{\nu_o(3+\nu)\Lambda^*}\frac{\alpha^2\lambda^2}{32}\left\{1+2(\gamma+\ln\frac{\lambda}{8})-\frac{11\pi}{16}(\lambda/4)^2\right\}+O(\lambda^6\ln\lambda)$$

with

$$\Lambda^* = 1+\frac{(1+\nu)^2}{\nu_o(3+\nu)}\frac{\pi\lambda^2}{32}-\left[\frac{27+14\nu+7\nu^2}{32\nu_o(3+\nu)}+\frac{5+2\nu+\nu^2}{16\nu_o(3+\nu)}\pi^2\right](\lambda/4)^4 \tag{25}$$

$$-\frac{(1+\nu)^2}{\nu_o(3+\nu)}(\gamma+\ln\frac{\lambda}{8})^2(\lambda/4)^4+\frac{15+6\nu+3\nu^2}{8\nu_o(3+\nu)}(\gamma+\ln\frac{\lambda}{8})(\lambda/4)^4+(\lambda/4)^4\frac{\nu^2+2\nu+13}{32\nu_o(4-\nu_o)}$$

$$+(\lambda/4)^4\left[\frac{11\nu^2+22\nu-25}{4\nu_o(3+\nu)}\right](\gamma+\ln\frac{\lambda}{8})+O(\lambda^6\ln\lambda)$$

THE SINGULAR STRESSES

The bending and extensional stress components are defined in terms of the displacement function W and stress function F as:

$$\sigma_{x_b} = -\frac{Ez}{(1-\nu^2)c}\left[\frac{\partial^2 W}{\partial x^2}+\nu\frac{\partial^2 W}{\partial y^2}\right] \tag{26}$$

$$\sigma_{y_b} = -\frac{Ez}{(1-\nu^2)c}\left[\frac{\partial^2 W}{\partial y^2}+\nu\frac{\partial^2 W}{\partial x^2}\right] \tag{27}$$

$$\tau_{xy_b} = \frac{2Gz}{c}\frac{\partial^2 W}{\partial x\partial y} \tag{28}$$

$$\sigma_{x_e} = \frac{1}{hc^2}\frac{\partial^2 F}{\partial y^2} \tag{29}$$

$$\sigma_{y_e} = \frac{1}{hc^2}\frac{\partial^2 F}{\partial x^2} \tag{30}$$

$$\tau_{xy_e} = \frac{-1}{hc^2}\frac{\partial^2 F}{\partial x\partial y} \tag{31}$$

where z is the dimensionless distance through the thickness h of the shell, measured from the middle surface with positive direction inwards. The stresses may be expressed in an integral form from which, when evaluated, give for

$\varepsilon e^{i\theta} = y \pm ix,\ \varepsilon << 1$

<u>Bending Stresses:</u> On the surface $z = -h/2c$ (upper fibers)

$$\sigma_{x_b} = -\frac{P_b}{\sqrt{2\varepsilon}}\left(\frac{11+5\nu}{4}\cos\frac{\theta}{2} + \frac{1-\nu}{4}\cos\frac{5\theta}{2}\right) + O(\varepsilon^o) \tag{32}$$

$$\sigma_{y_b} = \frac{P_b}{\sqrt{2\varepsilon}}\left(\frac{3-3\nu}{4}\cos\frac{\theta}{2} + \frac{1-\nu}{4}\cos\frac{5\theta}{2}\right) + O(\varepsilon^o) \tag{33}$$

$$\tau_{xy_b} = \frac{P_b}{\sqrt{2\varepsilon}}\left(\frac{7+\nu}{4}\sin\frac{\theta}{2} + \frac{1-\nu}{4}\sin\frac{5\theta}{2}\right) + O(\varepsilon^o) \tag{34}$$

$$P_b = \frac{3n_o\lambda^2(1+\nu)}{16(3+\nu)\sqrt{12(1-\nu^2)}\ hc^2\Delta^*}\left\{1 + 2(\gamma+\ell n\,\frac{\lambda}{8}) - (\frac{17-22\nu-11\nu^2}{16\nu_o})\ \pi(\lambda/4)^2\right\}$$

$$+ \frac{6m_o D}{(3+\nu)c^2h\Delta^*}\left\{1 - \frac{\pi\lambda^2}{64} + \frac{13+2\nu+\nu^2}{32(3+\nu)(1-\nu)}(\lambda/4)^4 + \frac{-25+22\nu+11\nu^2}{8(3+\nu)\nu_o}(\gamma+\ell n\,\frac{\lambda}{8})\ (\lambda/4)^4\right\}$$

$$+ O(\lambda^6\ell n\lambda) \tag{35a}$$

which for small λ reduces to:

$$P_b = \frac{3n_o\lambda^2\ (1+\nu)}{16(3+\nu)\ \sqrt{12(1-\nu^2)}hc^2}\left\{1 + 2(\gamma+\ell n\,\frac{\lambda}{8})\right\} + \frac{6m_o D}{(3+\nu)c^2h}\left\{1 - \frac{5+2\nu+\nu^2}{(1-\nu)(3+\nu)}\ \frac{\pi\lambda^2}{64}\right\}$$

$$+ O(\lambda^4\ell n\lambda) \tag{35b}$$

($\gamma = 0.577$ Euler's constant)

Similarly we find through the thickness

<u>Extensional Stresses:</u>

$$\sigma_{x_e} = \frac{P_e}{\sqrt{2\varepsilon}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{5\theta}{2}\right) + O(\varepsilon^o) \tag{36}$$

$$\sigma_{y_e} = \frac{P_e}{\sqrt{2\varepsilon}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{5\theta}{2}\right) + O(\varepsilon^o) \tag{37}$$

$$\tau_{xy_e} = \frac{P_e}{\sqrt{2\varepsilon}}\left(\frac{1}{4}\sin\frac{\theta}{2} - \frac{1}{4}\sin\frac{5\theta}{2}\right) + O(\varepsilon^o) \tag{38}$$

$$P_e = \frac{n_o}{hc^2\Delta^*}\left\{1 + \frac{5+2\nu+\nu^2}{\nu_o(3+\nu)}\frac{\pi\lambda^2}{64} - \frac{19-2\nu-\nu^2}{32\nu_o(3+\nu)}(\lambda/4)^4 + \frac{23+22\nu+11\nu^2}{8\nu_o(3+\nu)}(\gamma+\ell n\,\frac{\lambda}{8})(\lambda/4)^4\right\}$$

$$+ \frac{\sqrt{12(1-\nu^2)}\,(1+\nu)}{32\nu_o(3+\nu)}\frac{m_o\lambda^2 D}{\Delta^* h^2 c^2}\left\{1 + 2(\gamma+\ell n\,\frac{\lambda}{8}) - \frac{11\pi}{16}(\lambda/4)^2\right\}$$

$$+ O(\lambda^6 \ell n\lambda) \qquad (39a)$$

which similarly for small λ reduces to:

$$P_e = \frac{n_o}{hc^2}\left\{1 + \frac{\pi\lambda^2}{64}\right\} + \frac{\sqrt{12(1-\nu^2)}(1+\nu)}{32(1-\nu)(3+\nu)}\frac{m_o\lambda^2 D}{h^2c^2}\left\{1 + 2(\gamma+\ell n\,\frac{\lambda}{8})\right\} + O(\lambda^4\ell n\lambda) \qquad (39b)$$

As a practical matter, consider a shell subjected to a uniform internal pressure q_o with an axial extension $N_x = \frac{q_o R}{2}$, $M_y = 0$ far away from the crack (see Fig. 2). The stress normal to the crack, along the line of crack prolongation, for $\nu = 0.32$ and in the upper fibers is

$$\underset{\substack{\text{circumferential}\\ \text{crack}}}{\sigma_n} \approx \frac{K}{\sqrt{2\varepsilon}}\left(\frac{q_o R}{2h}\right) \qquad (40)$$

where the stress intensity factor K is given by

$$K = \frac{1 + 0.81(\lambda/4)^2 - 2.46(\gamma+\ell n\,\frac{\lambda}{8})(\lambda/4)^2 + 2.80(\lambda/4)^4 + 1.78(\gamma+\ell n\,\frac{\lambda}{8})(\lambda/4)^4}{1 + 1.26(\lambda/4)^2 - 2.06(\lambda/4)^4 - 0.80(\gamma+\ell n\,\frac{\lambda}{8})^2(\lambda/4)^4 + 0.98(\gamma+\ell n\,\frac{\lambda}{8})(\lambda/4)^4}$$

(41a)

which for small values of the parameter λ reduces to

$$K = 1 + 0.20\lambda^2 - 0.15\lambda^2\,\ell n\,\lambda \qquad (41b)$$

A plot of the stress intensity factor for both equations (41a) and (41b) is given in figure 3. It should be emphasized at this point that the solution of the coupled singular integral equations was obtained in a series form, which for small values of the parameter λ depending on the number of terms used) one may show that it converges to the exact solution. However, by looking at figure 3 one will be somewhat surprised that eq. (41a), which is correct up to $O(\lambda^4)$, deviates considerably for large λ from eq. (41b), which is correct up to $O(\lambda^2)$, and might thus conclude that an asymptotic solution with higher order terms will not necessarily increase the radius of convergence. The author believes that when the $O(\lambda^6)$ term is included in the solution, its character corrects itself and resembles that of eq. (41b). This can be seen from the expansion of the kernels by noticing that the terms $O(\alpha^2\lambda^2)$ and $O(\alpha^6\lambda^6)$ have the same sign, however by definition α^6 is $-\alpha^2$, i.e. there exists a sign reversal which could alter the character of eq. (41a). Nevertheless, this is only a conjecture and as a result the matter is under further investigation.

CONCLUSIONS

As in the case of a spherical shell,

(i) the stresses near the crack tip are proportional to $1/\sqrt{\epsilon}$ and have the same angular distribution as that of a flat plate

(ii) an interaction occurs between bending and stretching

(iii) the stress intensity factors are function of R; in the limit as $R \to \infty$ we recover the intensity factors for bending[6] and extension[7] in a flat plate. Thus we may write

$$\frac{\sigma_{shell}}{\sigma_{plate}} \approx \frac{1 + f(c/\sqrt{Rh})}{1 + g(c/\sqrt{Rh})} \tag{42}$$

where the functions f and g are such that the fraction is positive and monotonically increasing.

From this and the corresponding result for a spherical cap, it would appear that the general effect of initial curvature is to increase the stress in the neighborhood of the crack point. It is also of some practical value to be able to correlate flat sheet behavior with that of initially curved specimens. In experimental work on brittle fracture for example, considerable effort might be saved since, by eqn 42, we would expect within elastic theory, to predict the behavior of curved sheets from flat sheet tests.

In conclusion it must be emphasized that the classical bending theory has been used in deducing the foregoing results. Hence, only the Kirchhoff shear condition as satisfied along the crack, and not the vanishing of both individual shearing stresses. While outside the local region the stress distribution should be accurate, one might expect the same type of discrepancy to exist near the crack joint as that found by Knowles and Wang[8] in comparing Kirchhoff and Reissner bending results for the flat plate case. In this case the order of the stress singularity remained unchanged but the circumferential distribution around the crack changed so as to be precisely the same as that due to solely extensional loading. Pending further investigation of this effect for initially curved plates, one is tempted to conjecture that the bending amplitude and angular distribution would be the same as that of stretching.

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APPENDIX

The following is a list of the successive steps in the analysis.

$$P_3+P_4 = -(1+\nu)\,\frac{\alpha\lambda}{2}\,\frac{(P_1-P_2)}{\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}} + \nu\,\frac{\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}}{\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}}\,(P_1+P_2)$$

$$P_3-P_4 = \frac{\nu\alpha}{\beta}\,(P_1-P_2) + \frac{2\nu_o}{\beta\lambda}\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}\,(P_1+P_2)$$

where $\nu_o \equiv 1-\nu$

$$L_1 \equiv \lim_{|x|\to o}\int_o^\infty \left\{ \frac{s\,\exp\left[-\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}\,|x|\right]}{\sqrt{s^2+\frac{\alpha^2\lambda^2}{4}}} - \nu\,\frac{s\,\exp\left[-\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}\,|x|\right]}{\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}}\right\}\sin\zeta s\,ds = \frac{\alpha\lambda}{2}\,K_1\!\left(\frac{\alpha\lambda|\zeta|}{2}\right)$$

$$-\nu\,\frac{\beta\lambda}{2}\,K_1\!\left(\frac{\beta\lambda|\zeta|}{2}\right) \approx \frac{1-\nu}{\zeta} - (1+\nu)\,\frac{\alpha^2\lambda^2\zeta}{16} + \frac{\alpha^2\lambda^2\zeta}{8}\left(\gamma+\ln\frac{\alpha\lambda|\zeta|}{4}\right) - \frac{\nu\beta^2\lambda^2\zeta}{8}\left(\gamma+\ln\frac{\beta\lambda|\zeta|}{4}\right)$$

$$-\frac{5\nu_o}{4}\,(\alpha\lambda/4)^4\,\zeta^3 + (\alpha\lambda/4)^4\,\zeta^3\left(\gamma+\ln\frac{\alpha\lambda|\zeta|}{4}\right) - \nu(\beta\lambda/4)^4\,\zeta^3\left(\gamma+\ln\frac{\beta\lambda|\zeta|}{4}\right)$$

$$+\,O(\lambda^6\zeta^5\ln\lambda|\zeta|)$$

$$L_2 \equiv \lim_{|x|\to o}\int_o^\infty (1+\nu)\,\frac{s\,\exp\left[-\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}\,|x|\right]}{\sqrt{s^2+\frac{\beta^2\lambda^2}{4}}}\,\sin\zeta s\,ds = (1+\nu)\,\frac{\beta\lambda}{2}\,K_1\!\left(\frac{\beta\lambda|\zeta|}{2}\right)$$

$$\approx \frac{1+\nu}{\zeta} + (1+\nu)\,\frac{\beta^2\lambda^2\zeta}{8}\left(\gamma+\ln\frac{\beta\lambda|\zeta|}{4}\right) - (1+\nu)\,\frac{\beta^2\lambda^2\zeta}{16}$$

$$-(1+\nu)\,\frac{5}{4}\,(\beta\lambda/4)^4\,\zeta^3 + (1+\nu)(\beta\lambda/4)^4\,\zeta^3\left(\gamma+\ln\frac{\beta\lambda|\zeta|}{4}\right) + O(\lambda^6\zeta^5\ln\lambda|\zeta|)$$

$$L_3 \equiv \underset{|x|\to 0}{} \int_0^\infty \left\{ \frac{\nu_0 s \exp\left[-\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\,|x|\right]}{\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}} + (2+\nu)\,\nu_0 \frac{s\,\exp\left[-\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\,|x|\right]}{\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \right\} \sin \zeta s\, ds$$

$$+ \frac{\alpha^2\lambda^2}{2} \int_0^\infty \left\{ \frac{\exp\left[-\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\,|x|\right]}{s\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}} - \frac{\exp\left[-\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\,|x|\right]}{s\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \right\} \sin \zeta s\, ds$$

$$= \nu_0 \frac{\alpha\lambda}{2} K_1\!\left(\frac{\alpha\lambda|\zeta|}{2}\right) + (2+\nu)\,\nu_0 \frac{\beta\lambda}{2} K_1\!\left(\frac{\beta\lambda|\zeta|}{2}\right) + \frac{\alpha^2\lambda^2}{2} \int_0^\infty \left\{ \frac{\exp\left[-\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\,|x|\right]}{s\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}} \right.$$

$$\left. - \frac{\exp\left[-\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\,|x|\right]}{s\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \right\} \sin \zeta s\, ds \approx \frac{\nu_0(3+\nu)}{\zeta} + (1+\nu)\,\nu_0 \frac{\alpha^2\lambda^2\zeta}{16}$$

$$+ \nu_0 \frac{\alpha^2\lambda^2\zeta}{8}\left(\gamma + \ln \frac{\alpha\lambda|\zeta|}{4}\right) + \nu_0(2+\nu)\frac{\beta^2\lambda^2\zeta}{8}\left(\gamma + \ln \frac{\beta\lambda|\zeta|}{4}\right) + \frac{\pi\lambda^2\zeta}{4}$$

$$- \frac{5\nu_0(3+\nu)}{4}(\alpha\lambda/4)^4\,\zeta^3 + \nu_0(\alpha\lambda/4)^4\,\zeta^3\left(\gamma + \ln \frac{\alpha\lambda|\zeta|}{4}\right)$$

$$+ (2+\nu)\nu_0\left(\gamma + \ln \frac{\beta\lambda|\zeta|}{4}\right)(\alpha\lambda/4)^4\,\zeta^3 - \frac{\lambda^4}{4.9}\,\zeta^3$$

$$+ \frac{\lambda^4\zeta^3}{4.12}\left(\gamma + \ln \frac{\lambda|\zeta|}{4}\right) + O(\lambda^6\zeta^5 \ln \lambda|\zeta|)$$

$$L_4 \equiv -\lim_{|x|\to 0} \int_0^\infty \left\{ \frac{2\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\,\exp\left[-\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}\,|x|\right]}{s} + \frac{\beta^2\lambda^2}{2}\,\frac{\exp\left[-\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\,|x|\right]}{s\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \right\} \sin \zeta s\, ds$$

$$- (1-\nu^2-2\nu) \lim_{|x|\to 0} \int_0^\infty \frac{s\,\exp\left[-\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}\,|x|\right]}{\sqrt{s^2 + \frac{\beta^2\lambda^2}{4}}} \sin \zeta s\, ds \approx$$

$$- \frac{\nu_o(3+\nu)}{\zeta} + (1+\nu)^2 \frac{\alpha^2\lambda^2\zeta}{16} - (1+\nu)^2 \frac{\alpha^2\lambda^2\zeta}{8} (\gamma+\ell n \frac{\beta\lambda|\zeta|}{4}) - \frac{\pi\lambda^2\zeta}{8}$$

$$- (1-\nu^2-2\nu) (\beta\lambda/4)^4 \zeta^3(\gamma+\ell n \frac{\beta\lambda|\zeta|}{4}) + (1-\nu^2-2\nu) \frac{5}{4} (\beta\lambda/4)^4 \zeta^3$$

$$- \frac{83}{2} (\alpha\lambda/4)^4 \frac{\zeta^3}{9} + \frac{2}{3} (\alpha\lambda/4)^4 \zeta^3 (\gamma+\ell n \frac{\alpha\lambda|\zeta|}{4}) + \frac{8}{3} (\alpha\lambda/4)^4 \zeta^3 (\gamma+\ell n \frac{\beta\lambda|\zeta|}{4})$$

$$+ O(\lambda^6\zeta^5 \ell n \lambda|\zeta|).$$

where $\zeta \equiv y - \xi$

$$u_1(\xi) = \sqrt{1-\xi^2} \sum_{n=0}^{\infty} A_{n+1} \lambda^{2n} (1-\xi^2)^n$$

$$u_2(\xi) = \sqrt{1-\xi^2} \sum_{n=0}^{\infty} B_{n+1} \lambda^{2n} (1-\xi^2)^n$$

$$P_1+P_2 = A_1 \frac{J_1(s)}{s\sqrt{s^2 + \frac{\alpha^2\lambda^2}{4}}} + \cdots$$

$$P_1-P_2 = \frac{2B_1}{\alpha\lambda} \frac{J_1(s)}{s} + \cdots$$

where A_1 and B_1 are given by 23, 24, and

$$(1-\nu)A_3 + (1+\nu)B_3 = + \frac{1}{4^4.20} \left\{(1-\nu)A_1 + (1+\nu)B_1\right\} + (1+\nu) \frac{\alpha^2}{160} \left\{A_2-B_2\right\}$$

$$-(1-\nu)(3+\nu)\pi\lambda^4\left(A_3-B_3\right) = A_1 \left\{(1-\nu)(3+\nu)\frac{\pi}{20} (\alpha\lambda/4)^4 + \frac{\pi\lambda^4}{4\cdot 12\cdot 20}\right\}$$

$$+ B_1 \left\{-(1-2\nu-\nu^2) (\alpha\lambda/4)^4 \frac{\pi}{20} + \frac{10}{3} (\alpha\lambda/4)^4 \frac{\pi}{20}\right\}$$

$$+ \frac{\alpha^2\lambda^4(1+\nu)\pi}{160} \left\{(1-\nu)A_2 + (1+\nu)B_2\right\}$$

$$\lambda^2\pi \left\{(1-\nu)A_2 + (1+\nu)B_2\right\} - \frac{\alpha^2\lambda^4\pi}{64} (1+\nu) \left\{A_2-B_2\right\}$$

$$= A_1\left\{ (1+\nu) \frac{\alpha^2\lambda^2\pi}{48} + (1-\nu) \frac{\pi}{2} (\alpha\lambda/4)^4 (\gamma + \frac{1}{3}) + \frac{\pi}{2}(\alpha\lambda/4)^4 (\ell n \frac{\alpha\lambda}{8} - \nu \ \ell n \frac{\beta\lambda}{8})\right\}$$

$$+ B_1 \left\{-(1+\nu) \frac{\alpha^2\lambda^2\pi}{48} + (1+\nu) \frac{\pi}{2} (\alpha\lambda/4)^4 (\gamma + \frac{1}{3}) + \frac{\pi}{2} (1+\nu)(\alpha\lambda/4)^4 \ \ell n \frac{\beta\lambda}{8} \right\}$$

$$\nu_o(3+\nu) \ \pi\lambda^2 \left\{A_2-B_2\right\} + \frac{\alpha^2\lambda^4}{64} \pi(1+\nu) \left\{(1-\nu)A_2 + (1+\nu)B_2 \right\}$$

$$= A_1 \left\{- (1-\nu)(1+\nu) \frac{\alpha^2\lambda^2\pi}{48} + (1-\nu)(3+\nu) (\alpha\lambda/4)^4 \frac{\pi}{2} (\gamma + \frac{1}{3})\right.$$

$$+ \frac{\pi\lambda^4}{48} (\frac{\gamma}{2} + \frac{1}{8}) + (1-\nu) (\alpha\lambda/4)^4 \frac{\pi}{2} \ell n \frac{\alpha\lambda}{8} + (2+\nu)(1-\nu) (\alpha\lambda/4)^4 \frac{\pi}{2} \ell n \frac{\beta\lambda}{8}$$

$$\left. + \frac{\lambda^4\pi}{96} \ell n \frac{\lambda}{8} \right\}$$

$$+ B_1 \left\{- (1+\nu)^2 \frac{\alpha^2\lambda^2\pi}{48} - (\alpha\lambda/4)^4 (1-\nu^2-2\nu)\frac{\pi}{2} (\gamma + \frac{1}{3}) + (\alpha\lambda/4)^4 \frac{\pi}{2}(\frac{10\gamma}{3} + \frac{2}{3})\right.$$

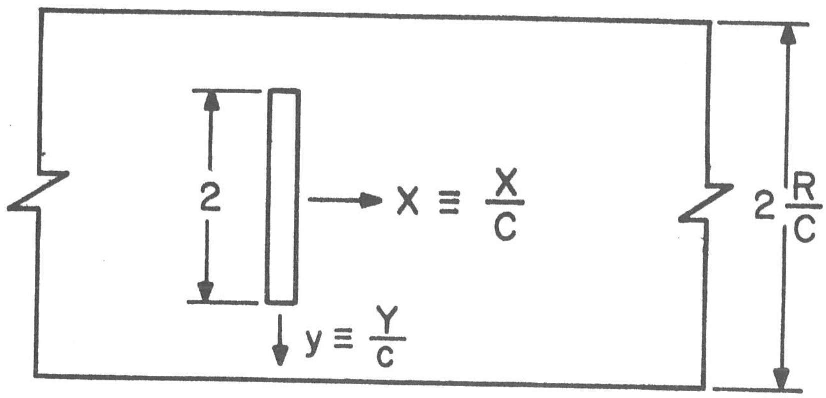
$$\left. - (1-\nu^2-2\nu) (\alpha\lambda/4)^4 \frac{\pi}{2} \ell n \frac{\beta\lambda}{8} + \frac{8}{3} (\alpha\lambda/4)^4 \frac{\pi}{2} \ell n \frac{\beta\lambda}{8} + \frac{\pi}{3} (\alpha\lambda/4)^4 \ \ell n \frac{\alpha\lambda}{8} \right\}$$

FIGURE CAPTIONS

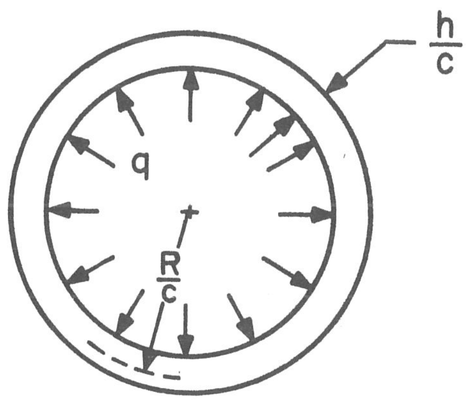
Figure 1. Geometry and coordinates.

Figure 2. Cracked shell under uniform axial extension N_x and internal pressure q_0.

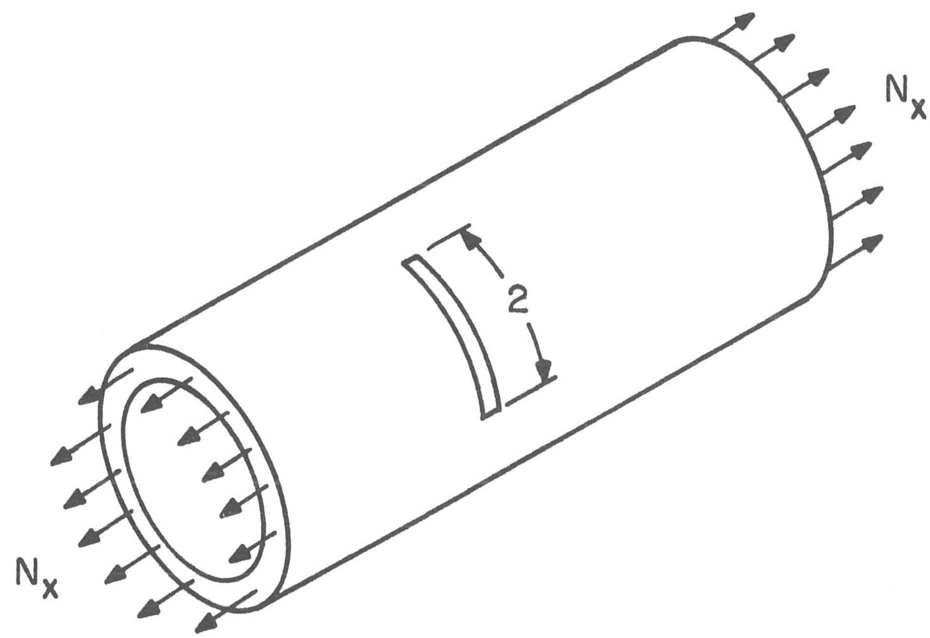
Figure 3. Stress intensity factor vs. λ .

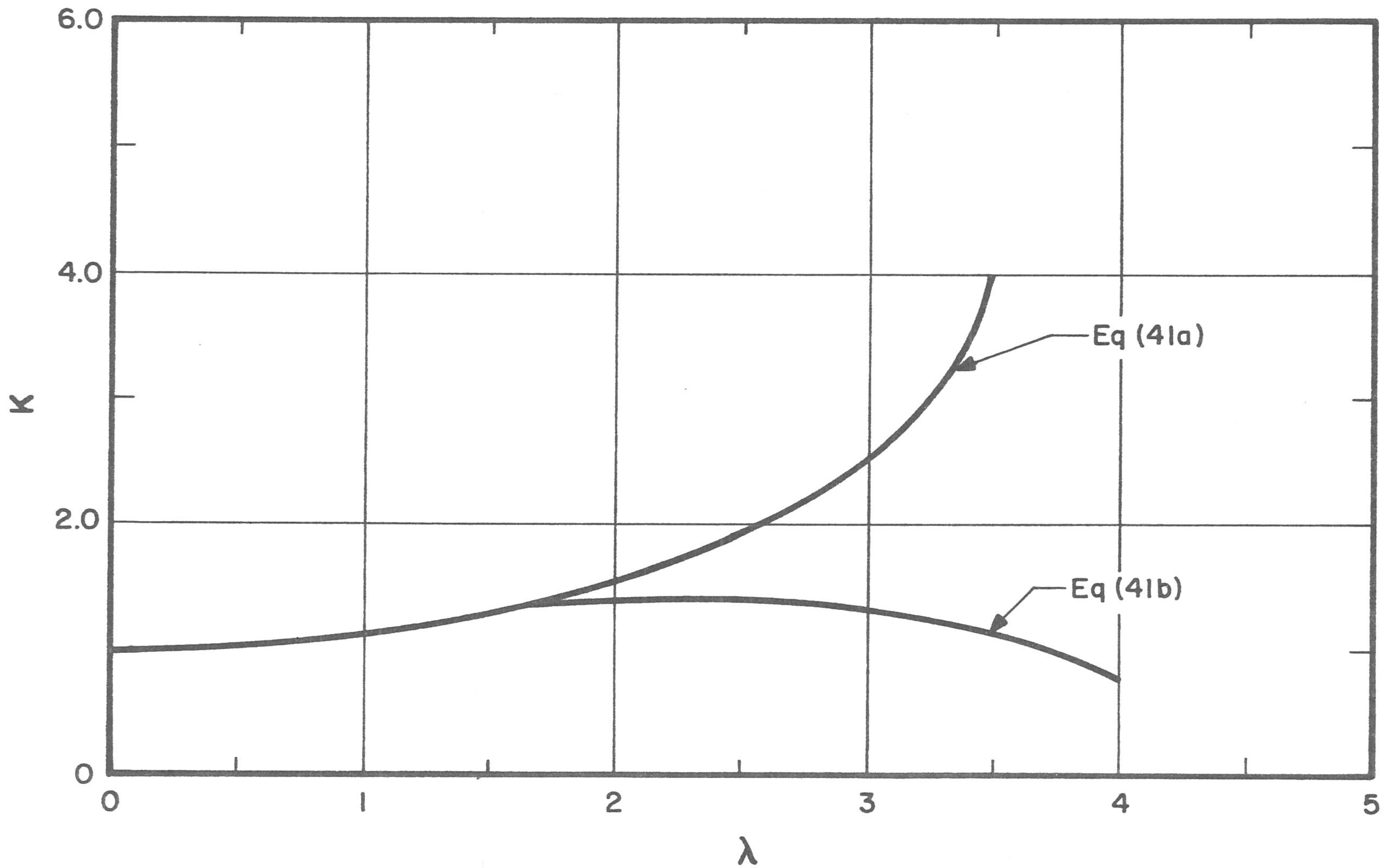


TOP VIEW



SIDE VIEW


N$_x$
2
N$_x$


6.0
4.0
2.0
0
K
0
1
2
3
4
5
λ
Eq (41a)
Eq (41b)