20

REPRINTED FROM THE

Proceedings of an international conference on

PROSPECTS OF FRACTURE MECHANICS

Held at Delft University of Technology, The Netherlands
June 24-28, 1974

Editors:

G. C. SIH
Lehigh University, Bethlehem, Pennsylvania, U.S.A.

H. C. van ELST
Metal Research Institute TNO, Apeldoorn, The Netherlands

D. BROEK
Delft University of Technology, Delft, The Netherlands

NOORDHOFF INTERNATIONAL PUBLISHING LEYDEN

ON THE PREDICTION OF CATASTROPHIC FAILURES IN PRESSURIZED VESSELS

E. S. Folias

University of Utah, Salt Lake City, Utah 84112, USA

ABSTRACT

Using a Dugdale-type model, the size of the plastic zone ahead of the crack tip is estimated, and a fracture criterion incorporating geometry and plasticity corrections is suggested. This criterion predicts failures in pressurized vessels of arbitrary shape by knowing only shell geometry, material properties and crack size. A comparison with some of the experimental data in existing literature substantiates the validity of the fracture criterion and its potential use.

INTRODUCTION

Experience has taught us that when a sharp object penetrates a pressurized balloon two things may occur: (i) if the hole is small the balloon loses pressure; (ii) if the hole is large the balloon explodes. One conjectures, therefore, that for a given pressure there exist a critical flow size beyond which the balloon fails catastrophically and vice versa. In fact, one may further conjecture that the critical pressure is a function of the material properties, the geometry of the structure and the flaw shape and size.

Thin-walled pressure vessels do resemble balloons and like balloons are subject to puncture and explosive loss. For a given material, under a specified stress field due to internal pressure, there will be a crack length in the material which will be self-propagating. Crack lengths less than the critical value will cause leakage but not destruction. However, if the critical length is ever reached, either by penetration or by the growth of a small fatigue crack, an explosion and complete loss of the structure may occur. The subject of eventual concern therefore is to assess analytically the relation between critical pressure and critical crack lengths in sheets which are initially curved.

GENERAL THEORY

Let us consider a portion of a thin, shallow shell*, of constant thickness h, subjected to an internal pressure $q(x,y)$ and containing a through the thickness line crack of length $2c$. In addition, we shall limit our considerations to elastic, isotropic and homogeneous segments of shells which are subjected to small deformations.

The basic variables in the theory of shallow shells are the displacement function $w(x,y)$ in the direction of the z axis and a stress function $F(x,y)$ which represents the stress resultants tangent to the middle surface of the shell. Following Marguerre [2], the coupled differential equations governing w and F, with x and y as rectangular Cartesian coordinates of the base plane, are given by

$$\nabla^4 F = Eh\left[2\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w_0}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right] \tag{1}$$

$$D\nabla^4 w = -q - 2\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2 w_0}{\partial x\partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w_0}{\partial x^2} \tag{2}$$

where $w_0(x,y)$ describes the initial shape of the shell in reference to that of a flat plate and D and E the flexural rigidity and Young's modules respectively.

A theoretical attach of the general problem for an arbitrary initial curvature presents formidable mathematical complexities. However, for spherical and cylindrical shells, exact solutions have been obtained in an asymptotic form and the results can be found in references [3] - [11]. For other more complicated shell geometries, the results can be obtained by a proper superposition of the above two solutions [10].

Without going into the mathematical details, the stress distribution around the crack tip for a symmetrical loading is found to be:

---

*According to Ogibalov [1], a shell will be called shallow if the least radius of curvature is greater by one order of magnitude than the linear dimensions, i.e. $L/R \leq 0.1$; and thin if $h/R \leq 0.01$.

Extensional stresses--through the thickness:

$$\sigma_x^{(e)} = P^{(e)}\left(\frac{c}{2r}\right)^{1/2}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{5\theta}{2}\right) + O(r^0) \tag{3}$$

$$\sigma_y^{(e)} = P^{(e)}\left(\frac{c}{2r}\right)^{1/2}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{5\theta}{2}\right) + O(r^0) \tag{4}$$

$$\tau_{xy}^{(e)} = P^{(e)}\left(\frac{c}{2r}\right)^{1/2}\left(-\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{5\theta}{2}\right) + O(r^0) \tag{5}$$

Bending stresses--on the "tension side" of the shell:

$$\sigma_x^{(b)} = P^{(b)}\left(\frac{c}{2r}\right)^{1/2}\left(-\frac{3-3\nu}{4}\cos\frac{\theta}{2} - \frac{1-\nu}{4}\cos\frac{5\theta}{2}\right) + O(r^0) \tag{6}$$

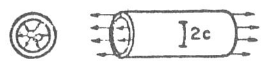
$$\sigma_y^{(b)} = P^{(b)}\left(\frac{c}{2r}\right)^{1/2}\left(\frac{11+5\nu}{4}\cos\frac{\theta}{2} + \frac{1-\nu}{4}\cos\frac{5\theta}{2}\right) + O(r^0) \tag{7}$$

$$\tau_{xy}^{(b)} = P^{(b)}\left(\frac{c}{2r}\right)^{1/2}\left(-\frac{7+\nu}{4}\sin\frac{\theta}{2} - \frac{1-\nu}{4}\sin\frac{5\theta}{2}\right) + O(r^0) \tag{8}$$

where ν is Poisson's ratio and (r,θ) are the polar coordinates around the crack tip. The stress coefficients $P^{(e)}$ and $P^{(b)}$ are functions of the crack size, geometry of the shell, material properties and loading characteristics. While their exact behavior can be found in reference [10], for certain simple loadings one may approximate them within a 7% error by the relations* of Table 1,

*The expressions in this table are valid for all λ.

TABLE 1

Long cylinder axial crack 	$P^{(e)} \approx \sqrt{1 + 0.317\lambda^2}\ (q_0 R/h)$ $P^{(b)} \approx 0$
Long cylinder peripheral crack 	$P^{(e)} \approx \sqrt{1 + 0.05\lambda^2}\ (q_0 R/2h)$ $P^{(b)} \approx 0$
Spherical cap 	$P^{(e)} \approx \sqrt{1 + 0.466\lambda^2}\ (q_0 R/2h)$ $P^{(b)} \approx 0$

where for convenience we have defined R to be the radius of curvature and

$$\lambda \equiv \left(\frac{Ehc^4}{R^2 D}\right)^{1/4} \equiv \{12(1 - \nu^2)\}^{1/4}\left\{\frac{R}{h}\right\}^{1/2}\frac{c}{R} \equiv \{12(1 - \nu^2)\}^{1/4}\frac{c}{(Rh)^{1/2}} \quad (9)$$

It must be emphasized that classical bending theory has been used in deducing the foregoing results. Hence it is inherent that only the Kirchhoff equivalent shear free condition is satisfied along the crack, and not the vanishing of both individual shearing stresses. While outside the local region the stress distribution should be accurate, one might expect the same type of discrepancy to exist near the crack point as that found by Knowles and Wang [12] in comparing Kirchhoff and Reissner bending results for a flat plate. In this case the order of the stress singularity remained unchanged but the angular distribution around the crack changed so as to precisely be the same as that due to solely extensional loading.

Recently, Sih and Hagendorf [13] investigated this matter further by deriving an improved theory of shallow shells which incorporates the effect of a transverse shear deformation. As expected, their results showed that classic theory cannot adequately predict the exact angular dependence of the bending stresses in the vicinity of a crack. However, in general, these bending stresses are so small when compared to the extensional stresses that can be neglected. On the other hand, for very long cracks such contributions become significant and consequently may no longer be neglected. Unfortunately, in such cases bulging effects become extremely important and any theory, whether classic or shear, is inadequate.

ELASTIC BRITTLE FRACTURE

From the foregoing discussion it becomes apparent that initially curved sheets containing through cracks present a reduced resistance to fracture initiation. Consequently, the presence of a crack in the walls of a pressure vessel can severely reduce the strength of the structure and can cause sudden failure at nominal tensile stresses less than the material yield strength. To ensure, therefore, the integrity of a structure, the designer must be cognizant of the relation that exists between fracture load, material properties, flow shape and size and structural geometry.

The principal task, however, of Fracture Mechanics is precisely the prediction of failure due to the presence of sharp discontinuities. Specifically, the approach is based on a corollary of the first law of thermodynamics which was first applied to the phenomenon of fracture by Griffith [14]. His hypothesis was that the total energy of a cracked system subjected to external loading remains constant as the crack extends an infinitesimal distance. That is

$$\frac{\partial U_{system}}{\partial c} = 0. \tag{10}$$

Following the work of Griffith, the total energy of the system is given by

$$U_{system} = U_{loading} + U_{surface} + U_{strain}, \tag{11}$$

where the increase in strain energy due to the presence of the crack may be calculated by considering the discontinuity in the v-displacement across the crack. In particular

$$U_{strain} = -\frac{1}{2}\int_{-c}^{c} \sigma_y(x,0)(v^+ - v^-)dx, \tag{12}$$

which in view of the results of ref. [3] now becomes

$$U_{strain} = -\frac{\pi}{E} c^2 \left\{\left[P^{(e)}\right]^2 + \frac{2}{3}(1+\nu)^2\left[P^{(b)}\right]^2\right\}. \tag{13}$$

Thus for crack instability

$$4\gamma c = -\frac{\partial U_{strain}}{\partial c}, \tag{14}$$

where γ is the surface energy per unit area, or

$$4\gamma c = \frac{2\pi c}{E}\left\{\left[P^{(e)}\right]^2 + \frac{2}{3}(1+\nu)^2\left[P^{(b)}\right]^2\right\} + + 2\frac{\pi}{E}c^2\left\{P^{(e)}\frac{dP^{(e)}}{dc} + \frac{2}{3}(1+\nu)^2 P^{(b)}\frac{dP^{(b)}}{dc}\right\} \tag{15}$$

The reader should notice that as $R \to \infty$ the elastic fracture criterion for flat plates is recovered.

PLASTICITY CORRECTION

Due to the presence of high stresses in the vicinity of the tip of a crack, when the appropriate yield criterion is satisfied then localized plastic deformation occurs and a plastic zone size is created. This phenomenon effectively increases the crack length and therefore must be accounted for. Following Dugdale [15], the size of the plastic zone (see fig. 1) is determined by the relation

$$\frac{c}{c_e} = \cos\left(\frac{\pi\sigma}{2\sigma_y}\right) \tag{16}$$

where $2c_e$ is the effective crack length and σ_y the yield stress of the material. This relation, however, applies only to a perfect elasticplastic non-strain-hardening material. Subsequently, McClintock [16] has suggested that a strain-hardening material may be approximated by an ideally, plastic one if a stress higher than the yield stress and lower than the ultimate stress is chosen. In particular, the author suggests the value

$$\sigma^* = \frac{1}{2}\left\{\sigma_y + \frac{\sigma_y + \sigma_u}{2}\right\}. \tag{17}$$

Correcting, therefore, the Griffith-Irwin equation so as to include plasticity and geometry effects one has

$$P^{(e)} = \frac{2\sigma^*}{\pi}\cos^{-1}\left[\exp\left(-\frac{\pi K^2}{8\sigma^{*2}c}\right)\right]. \tag{18}$$

where K represents the fracture toughness of the material. It is interesting to note that one may also treat σ^* and K as floating constants to be determined from scaled-down models.

EXPERIMENTAL VERIFICATION

In general, a theory is inadequate unless there exists experimental evidence to substantiate its validity and its potential use. Therefore, in the following we compare our results with some of the experimental data existing in literature.

1. R. P. Sopher, A. L. Lowe, D. C. Martin, and P. J. Rieppel, Evaluation of weld joint flaws on initiating points of brittle fracture, Welding J. (Nov. 1959), p. 4415. The results of tests on 9-ft-diameter, 3/4-in.-thick spheres with full thickness cracks are given.

   Material: ABS-B Steel

   σ_y: 30.7 ksi, σ^* = 37.9 ksi

   σ_u: 59.4 ksi, K = 102.0 ksi in

   The results are plotted in fig 2. The agreement is very good.

2. R. B. Anderson and T. L. Sullivan, Fracture mechanics of through-cracked cylindrical pressure vessels. NASA TN D-3252. This paper gives results of tests on 6-in.-diameter, 0.060-in.-thick cylinders with full thickness cracks.

   Material: 2014-T6Al

   σ_y: 90.5 ksi, σ^* = 91.9 ksi

   σ_u: 93.9 ksi, K = 51.6 ksi in

   The results are plotted in fig. 3. The agreement is good.

3. W. H. Irvine, A. Quirk and E. Bevitt, Fast fracture of pressure vessels: An appraisal of theoretical and experimental aspects and application to operational safety. J. Brit. Nucl. Energy Soc. (Jan. 1964). The results of test on 5-ft-diameter, 1-in.-thick, cylindrical vessels with through-cracks are given.

   Material: 0.36% C Steel

   σ_y: 33 ksi, σ^* = 40 ksi

   σ_u: 61 ksi, K = 179 ksi in

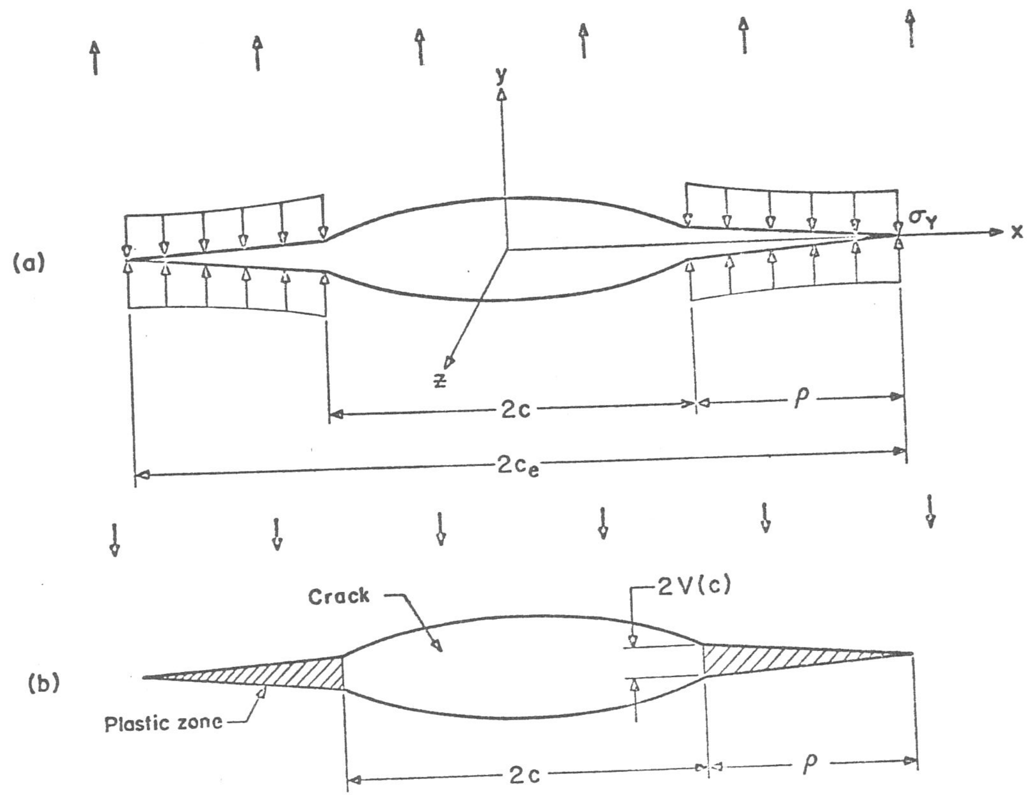


Fig. 1. (a) Internal stress distribution used in the Dugdale model of elastic-plastic deformation near a crack of length 2c under plane-stress tensile loading. (b) Displacements 2V associated with crack opening.

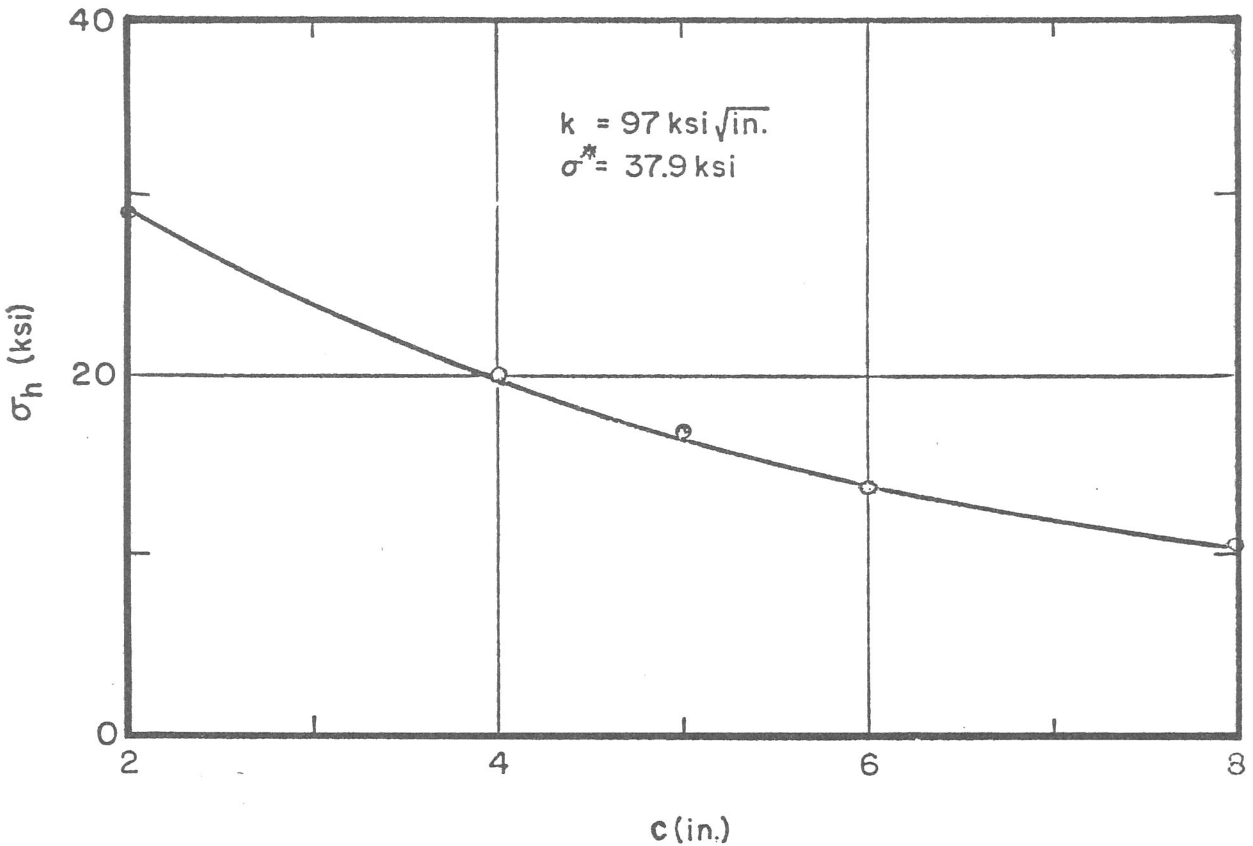


Fig. 2. Comparison between theory and experiment for ABS-B steel spherical vessels.

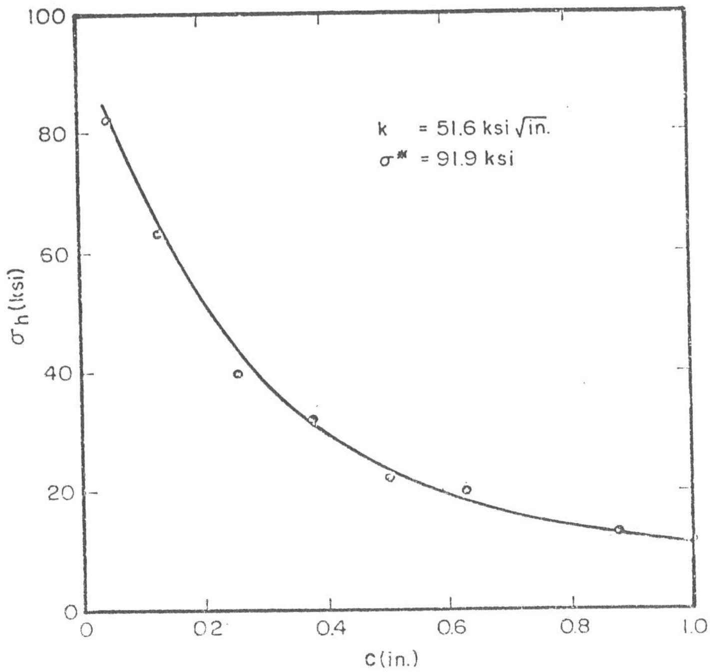


Fig. 3. Comparison between theory and experiment for 2014-T6Al cylindrical vessels at -423°F.

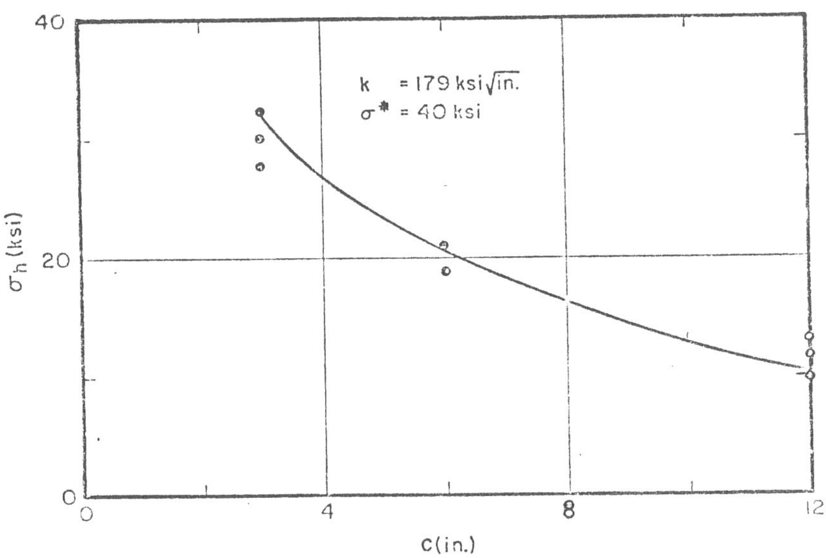


Fig. 4. Comparison between theory and experiment for 0.36% C steel cylindrical vessels.

The results are plotted in fig. 4. The agreement is fairly good.

Comparison with other experimental data available in the literature leads to equally good agreements.

CONCLUSIONS

The close agreement between the theoretically predicted fracture strengths and the experimental data suggests that eq. (18) may be used to predict failures in pressurized vessels knowing only the structural geometry, the crack length, the ultimate and yield stresses, and the fracture toughness of the material.

An interesting point, however, should be stressed. It seems from the above examples that for very small crack lengths the cosine angle is greater than $\pi/2$ and the criterion, therefore, fails. This failure is due to the fact that the material, at least local to the crack, undergoes considerable strain-hardening and as a result the fracture hoop stress is higher than σ^*. At the present time there are no adequate criteria to handle this problem.

Recently, using a slightly different model than that of Dugdale's, the author was able to derive* a more accurate fracture criterion which for very small crack lengths shows better agreement with experimental observations. On the other hand, for larger crack lengths the predicted results are very close to those of eq. (18).

As an example, for a long cylindrical pressure vessel containing an axial crack the more accurate fracture criterion gives the hoop stress σ_h as

$$\sigma_h = \sigma^* \left\{ \frac{1 + 0.317\lambda^2 \left(\frac{1-\alpha}{\alpha}\right)^2}{1 + 0.317\lambda^2 \left(\frac{1}{\alpha}\right)^2} (1 - \alpha) \right\}^{1/2} \tag{19}$$

where α is the ratio of the actual to the effective crack length, ie

$$\alpha = \frac{c}{c_e} = e^{-\frac{\pi K^2}{8\sigma^{*2} c}} \tag{20}$$

Although eq. (19) will yield more accurate results, it does require some rather lengthy calculations.

---

*For details see ref. [17].

In addition to the external applied loads, pressure vessels are also subjected to vibrations. Consequently, an investigation was carried out ref. [18] in order to assist analytically what effect, if any, do vibrations have on the mechanism of fracture. The analysis has shown that in general transverse vibrations reduce the stress intensity factor. However, when the forcing frequency approaches the natural frequency of the uncracked shell, the stress intensity factor increases without bound. This phenomenon coupled with the usual $1/\sqrt{r}$ singular behavior, causes the pressure vessel to fail at nominal values even lower than the yield stress.

REFERENCES

[1] Ogibalov, Dynamics and strength of shells, translated from Russian, published for NASA and NSF by the Israel Program for Scientific Translations.

[2] Marguerre, K., Zur Theorie der Gekrummten Platte Grosser Formanderung, Proceedings of the 5th International Congress of Applied Mechanics, 93-101, (1938).

[3] Folias, E. S., Journal of Mathematics and Physics, Vol. 44, No. 2, June 1965, pp. 164-176.

[4] Erdogan, F. and Kibler, International Journal of Fracture Mechanics, 5, 3 (September 1969).

[5] Folias, E. S., International Journal of Fracture Mechanics, 1, 2, 104-113 (1965); ARL 64-174, Aerospace Research Laboratories, United States Air Force, Dayton, Ohio, October 1964.

[6] Folias, E. S., International Journal of Fracture Mechanics, 3, 1, 1-11 (1967).

[7] Copley, L. G. and Sanders, J. L., International Journal of Fracture Mechanics, 5, 2 (June 1969).

[8] Duncan, M. E. and Sanders., J. L., International Journal of Fracture Mechanics, 5, 2 (June 1969).

[9] Folias, E. S., International Journal of Fracture Mechanics, 1, 1, 20-46 (1965).

[10] Folias, E. S., On the effect of initial curvature on cracked flat sheets, International Journal of Fracture Mechanics, Vol. 5, No. 4, pp. 327-346, December 1969.

[11] Fung, Y. C. and Sechler, E. E., Thin Shell Structures, Prentice-Hall, Chapter 21, January 1974.

[12] Knowles, J. K. and Wang, N. M., On the bending of an elastic plate containing a crack, Journal of Mathematics and Physics, Vol. 39, pp. 223-236, 1960.

[13] Sih, G. C. and Hagendorf, H. C., A new theory of spherical shells with cracks, Thin Shell Structures, edited by Y. C. Fung and E. E. Sechler, Prentice-Hall, pp. 519-545, 1974.

[14] Griffith, A. A., The theory of rupture, Proceedings of the 1st International Congress of Applied Mechanics, pp. 55-63, Delft, 1924.

[15] Dugdale, D. S., Journal of Mechanics and Physics of Solids, Vol. 8, p. 100, 1960.

[16] McClintock, F. A., On notch sensitivity, Welding Journal of Research Supplement, Vol. 26, pp. 202-208.

[17] Folias, E. S., Estimating plastic zone sizes ahead of a crack, to appear in International Journal of Fracture Mechanics, 1974.

[18] Do, S. H. and Folias, E. S., On the steady-state transfers vibrations of a cracked spherical shell, International Journal of Fracture Mechanics, Vol. 7, No. 1, March 1971.

DISCUSSION by D. Radenkovic, Ecole Polytechnique, France

The solutions of Marguerre's equations presented in this communication will prove certainly very useful in the applications where membrane forces are predominant - the experimental results quoted by the author show this clearly. However, I would like to point out that on the vicinity of the crack tip the work of shearing forces cannot be neglected as a rule; in order to take it into account, on the calculations concerning through-cracks on plates and shells, it is necessary to use a theory of Reissner's type. In this way, by a local analysis, Radenkovic and Bergez [1,2] were able to define the five intensity factors which, in the general case, resume at the crack tip, the shell geometry and the loading. Once these factors are clearly defined, the solution of the global problem, which permits to find their values, can be obtained by numerical methods.

REFERENCES

[1] Bergez, D. and Radenkovic, D., "La caracterization des", etc.

[2] Bergez, D. and Radenkovic, D., "On the definition of stress-intensity factors in cracked plates and shells", Proc. 2nd Int. Conf. on Pressure Vessel Technology, San Antonio, Texas, pp. 1089-1096, 1973.

AUTHOR'S CLOSURE

Yes, this inadequacy was also pointed out by the author in the paper. I would furthermore like to add that in the case of spherical shells, Sih and Hagendorf upon deriving a higher order theory, they were able to explain precisely this very point. Their results were reported at the Symposium on Thin Shell Structures, 1972. (See author's paper for reference).