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On the equilibrium of a linear elastic layer

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Abstract. A general method for the construction of a 3D solution applicable to the equilibrium of a linear elastic layer which is subjected to a general load of bending or stretching is discussed. In the special case of a layer with faces free of stress, the general solution is derived explicitly. The general solution has a sufficient number of arbitrary functions to allow it to be used to solve a whole class of practical 3D problems, e.g. an inclusion, a partial through-the-thickness crack, a cylindrical hole etc.

1 Introduction

Engineering fracture mechanics techniques are able to compensate for the inadequacies of conventional design consepts based on tensile strength or yield strength. While such criteria are adequate for many engineering structures, they are insufficient when the likelihood of cracks exists. Now, after approximately three decades of development, fracture mechanics has become a useful design tool, particularly for high strength materials. Interestingly enough, this technological advancement was based on the 2D elasticity solution of the crack problem. Once the classical stress field was determined, it was then integrated to form a physically meaningful concept, the energy, release rate (G). Unstable crack propagation occurs when G attains a critical value, which is a material constant.

While this approach is well understood in two dimensions, it is less so in three. Moreover, configurations occur quite often in practice that are inherently three-dimensional in nature, e.g. the intersection of a partially, through-the-thickness crack with a free surface. The objective, therefore, is to further our understanding of the role of such three-dimensional features such as specimen thickness, re-entrant angle etc., within the context of LEFM and beyond.

The mathematical difficulties, however, posed by three-dimensional elasticity problems are substantially greater than those associated with plane stress or plane strain. Previous work on the subject has been reported by Sokolnikoff (1983), Marguerre (1955), Sternberg (1960) and others. A most comprehensive recent survey of the various methods of solution is given by Panasyuk et al. (1980). To the best of the author's knowledge, such a general solution to the equilibrium of linear elastic layers is not available. However, based on work carried out by Folias (1975, 1988a), it is now possible to construct such a solution, which in turn may be used to determine the stress field in a plate with a partial through-the-thickness crack, or an inclusion, or even a void of an arbitrary shape.

2 Formulation of the problem

We shall restrict our attention to the realm of classical static elasticity in which it is assumed that the strains are small and that the physical constants of the solid under consideration are independent of position and stress. We consider the elastic solid to be a layer bounded by two parallel planes, $x_3 = 0$ and $x_3 = h$, and containing, e.g. in the plane $x_2 = 0$ a crack[1] which may or may not be all the

1 Although our objective is to derive a general solution for the 3D equilibrium of linear elastic plates, for the sake of clarity and rigour we will specialize the discussion to the case of a plate that has been weakened by a plane crack. Two such plane cracks which are of great practical importance are (i) a through-the-thickness line crack and (ii) a partial-through-the-thickness elliptical crack. For simplicity, we assume that the crack is symmetric with respect to the x_3-axis

way through-the-thickness. The surface of the crack will be denoted by S_c and the total volume of the body V. A typical point of the solid will be denoted by the three rectangular Cartesian coordinates x_i ($i = 1, 2, 3$), and the displacement vector by $\boldsymbol{u}_i$.

In the absence of body forces, the displacement functions $\boldsymbol{u}_i$ are governed by the well-known Navier's equations, i.e.

$$\nabla^2 \boldsymbol{u}_i + \frac{1}{1-2\nu} e_{,i} = 0 \quad \text{for all} \quad x_i \in V, \tag{1}$$

where the quantity e represents the divergence of the displacement vector $\boldsymbol{u}_i$, ∇^2 the Laplacian operator, and ν Poisson's ratio. Furthermore, the stress-displacement relations are given by Hook's law as

$$\boldsymbol{\sigma}_{ij} = \frac{2\nu}{1-2\nu} G e \delta_{ij} + G(u_{i,j} + u_{j,i}) \tag{2}$$

where G now stands for the shear modulus, $\boldsymbol{\sigma}_{ij}$ for the stress tensor and δ_{ij} for the Kronecker delta.

As to boundary conditions, one must require that (i) on the planes $z = 0$ and $z = h$, the appropriate loading conditions are to be satisfied; (ii) on the faces of the crack all stresses must vanish; and (iii) away from the crack, the appropriate loading and support conditions are to be satisfied.

In treating this type of problem, it is convenient to seek the solution in two parts, the undisturbed or 'particular' solution, $\boldsymbol{u}_i^{(p)}$, which satisfies equation (1) and the loading and support conditions but leaves residual forces along the crack, and the 'complementary' solution, $\boldsymbol{u}_i^{(c)}$, which precisely nullifies these residuals and offers no contribution far away from the crack. Particular solutions of this type can easily be constructed and consequently will be assumed as known.

3 Mathematical statement of the complimentary problem

Assuming, therefore, that a particular solution has been found, we need to construct a displacment vector field $\boldsymbol{u}_i^{(c)}$ such that it satisfies Navier's equations and the following boundary conditions

$$\sigma_{2i}^{(c)} = -\sigma_{2i}^{(p)} \quad \text{for all} \quad x_i \in S_c \tag{3}$$

$$\sigma_{3i}^{(c)} + \sigma_{3i}^{(p)} = f_i \quad \text{for all} \quad x_i \in \{\text{plane}\, x_3 = 0\}, \qquad \sigma_{3i}^{(c)} + \sigma_{3i}^{(p)} = g_i \quad \text{for all} \quad x_i \in \{\text{plane}\, x_3 = h\} \tag{4a, b}$$

where f_i and g_i are given functions of x_1 and x_2, and far away from the crack the displacements $\boldsymbol{u}_i^{(c)}$ are to be bounded.

4 Method of solution

For the construction of a solution to Eq. (1), we use a method that was first introduced by Lur'e (1964) and was subsequently extended by Folias (1975). It should be noted that the above symbolic method leads to the same form of a solution as that obtained by the use of a double Fourier integral transform with the subsequent use of a contour integration[2]. This matter has been discussed extensively by Wilcox (1978).

This object of the method of Lur'e is to reduce the partial differential equations to ordinary differential equations that in turn may be solved by elementary methods. This can be accomplished if one introduces the following operators

$$\partial_1 \equiv \frac{\partial}{\partial x_1}, \qquad \partial_2 \equiv \frac{\partial}{\partial x_2}, \qquad D^2 \equiv \partial_1^2 + \partial_2^2,$$

which are to be interpreted as numbers. Thus Eq. (1) may now be thought of as a system of ordinary differential equations of the independent variable x_3

$$l_{ij} u_j^{(c)} = 0 \tag{5}$$

2 The matter was investigated by Folias in 1977. As further verification, however, Folias posed the question of completeness to Professor Wilcox who independently arrived at the same eigenfunctions as those reported by Folias (1975)

where the differential operator l_{ij} is symmetric and is defined by

$$l_{11}=\frac{d^2}{dz^2}+D^2+\frac{1}{1-2\nu}\partial_1^2 \qquad l_{22}=\frac{d^2}{dz^2}+D^2+\frac{1}{1-2\nu}\partial_2^2$$

$$l_{12}=\frac{1}{1-2\nu}\partial_1\partial_2 \qquad l_{33}=2\left(\frac{1-\nu}{1-2\nu}\right)\frac{d^2}{dz^2}+D^2 \tag{6}$$

$$l_{13}=\frac{1}{1-2\nu}\partial_1\frac{d}{dz} \qquad l_{23}=\frac{1}{1-2\nu}\partial_2\frac{d}{dz}.$$

Upon integrating the above system subject to the following initial conditions

$$u_i^{(c)}={}_ou_i,\ \frac{\partial u_i^{(c)}}{\partial x_3}={}_ou_i' \quad at \quad x_3=0 \tag{7}$$

one has, after a few simple calculations[3],

$$u^{(c)}=\cos(zD)\,{}_ou-\frac{1}{2(1-2\nu)}\frac{z\sin(zD)}{D}\partial_1\theta_o+\frac{\sin(zD)}{D}{}_ou'$$
$$-\frac{1}{4(1-\nu)}\left[\frac{\sin(zD)}{D^3}-\frac{z\cos(zD)}{D^2}\right]\partial_1\theta_o' \tag{8}$$

$$v^{(c)}=\cos(zD)\,{}_ov-\frac{1}{2(1-2\nu)}\frac{z\sin(zD)}{D}\partial_2\theta_o+\frac{\sin(zD)}{D}{}_ov'$$
$$-\frac{1}{4(1-\nu)}\left[\frac{\sin(zD)}{D^3}-\frac{z\cos(zD)}{D^2}\right]\partial_2\theta_o' \tag{9}$$

$$w^{(c)}=\frac{\sin(zD)}{D}{}_ow'+\frac{1}{2(1-2\nu)}\left[\frac{\sin(zD)}{D}-z\cos(zD)\right]\theta_o$$
$$+\cos(zD)\,{}_ow-\frac{1}{4(1-\nu)}\frac{z\sin(zD)}{D}\theta_o', \tag{10}$$

where, in order to avoid confucion, we have used the longhand notation for the displacement functions and

$$\theta_o=\partial_1\,{}_ou+\partial_2\,{}_ov+{}_ow';\qquad \theta_o'=\partial_1\,{}_ou'+\partial_2\,{}_ov'-D^2\,{}_ow. \tag{11),(12}$$

The reader should note that ${}_ou_i$ and ${}_ou_i'$ are still arbitrary functions of x_1 and x_2 and are to be determined from the boundary conditions on the two plate faces $x_3=0$ and $x_3=h$. In effect, we are perturbing the solution from the plane $x_3=0$ to the plane $x_3=h$.

It is worthwhile at this point to pose and examine a few special cases in which the displacement functions can be reduced considerably.

Problem (i): a plate with a through-the-thickness line crack and under the action of a stretching load. This may be solved if one lets $z=0$ be the middle plane and ${}_ou'={}_ov'={}_ow=0$.

Problem (ii): a plate with a through-the-thickness line crack and under the action of a bending load. This can be solved if one lets $z=0$ be the middle plane and ${}_ou={}_ov={}_ow'=0$.

Problem (iii): a plate with vanishing stresses on the boundary $z=0$ and under the action of either a bending or a stretching load (Fig. 1). This can be solved if one lets ${}_ou'=-\partial_1\,{}_ow$, ${}_ov'=-\partial_2\,{}_ow$, ${}_ow'=-\frac{\nu}{1-2\nu}\theta_o$.

3 x_1, x_2, x_3 corresponds to x, y, z in the longhand notation and u_1, u_2, u_3 to u, v, w

In general, however, the unknown functions ${}_0u$, ${}_0v$, ${}_0w$, ${}_0u'$, ${}_0v'$, ${}_0w'$ are to be determined from the boundary conditions (Eq. 4), i.e.

$$\sigma_{3i}^{(c)} = -\sigma_{3i}^{(p)} + f_i \equiv F_i \quad \text{on the plane} \quad x_3 = z = 0 \tag{13}$$

$$\sigma_{3i}^{(c)} = -\sigma_{3i}^{(p)} + g_i \equiv G_i \quad \text{on the plane} \quad x_3 = z = h \tag{14}$$

where F_i and G_i are known functions of x_1 and x_2 or x and y. Or, in view of Eqs. (2) and (8)–(10), they may also be written in the form[4]

$$d_{mn}\xi_n = H_m \qquad (m, n, = 1, 2, \ldots, 6) \tag{15}$$

where the ξ_n and H_n now stand for the vectors (${}_0u$, ${}_0v$, ${}_0w$, ${}_0u'$, ${}_0v'$, ${}_0w'$) and (F_1, F_2, F_3, G_1, G_2, G_3) respectively, and the operators d_{mn} are defined by

$$d_{11} = d_{12} = d_{15} = d_{16} = d_{21} = d_{22} = d_{24} = d_{26} = d_{33} = d_{34} = d_{35} = 0$$

$$d_{13} = \partial_1, \quad d_{23} = \partial_2, \quad d_{14} = d_{25} = 1$$

$$d_{31} = \frac{2\nu}{1-2\nu}\partial_1, \quad d_{32} = \frac{2\nu}{1-2\nu}\partial_2, \quad d_{36} = \frac{2(1-\nu)}{1-2\nu}$$

$$d_{41} = -\frac{\sin(hD)}{D}D^2 - \frac{1}{1-2\nu}h\cos(hD)\partial_1^2$$

$$d_{52} = -\frac{\sin(hD)}{D}D^2 - \frac{1}{1-2\nu}h\cos(hD)\partial_2^2$$

$$d_{42} = d_{51} = -\frac{1}{1-2\nu}h\cos(hD)\partial_1\partial_2$$

$$d_{43} = \partial_1 d^*, \quad d_{53} = \partial_2 d^*, \quad d^* = \cos(hD) + \frac{1}{2(1-\nu)}(hD)\sin(hD)$$

$$d_{44} = \cos(hD) - \frac{1}{2(1-\nu)}\frac{h\sin(hD)}{D}\partial_1^2 \tag{16}$$

$$d_{55} = \cos(hD) - \frac{1}{2(1-\nu)}\frac{h\sin(hD)}{D}\partial_2^2$$

$$d_{45} = d_{54} = -\frac{1}{2(1-\nu)}\frac{h\sin(hD)}{D}\partial_1\partial_2$$

$$d_{46} = \partial_1 d^{**}, \quad d_{56} = \partial_2 d^{**}, \quad d^{**} = \frac{\sin(hD)}{D} - \frac{1}{1-2\nu}h\cos(hD)$$

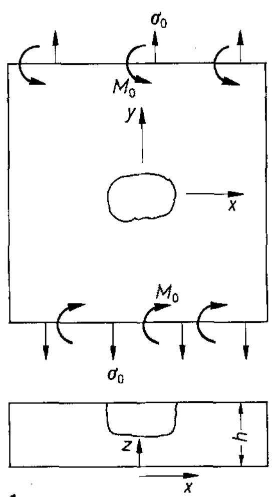
$$d_{61} = \partial_1 d_*^*, \quad d_{63} = \partial_2 d_*^*, \quad d_*^* = \frac{2\nu}{1-2\nu}\cos(hD) + \frac{1}{1-2\nu}(hD)\sin(hD)$$

$$d_{64} = -\partial_1 d_{**}^{**}, \quad d_{65} = -\partial_2 d_{**}^{**}, \quad d_{**}^{**} = \frac{1-2\nu}{2(1-\nu)}\frac{\sin(hD)}{D} + \frac{1}{2(1-\nu)}h\cos(hD)$$

$$d_{63} = -2D\sin(hD) + \left[\frac{1-2\nu}{2(1-\nu)}\frac{\sin(hD)}{D} + \frac{1}{2(1-\nu)}h\cos(hD)\right]D^2$$

$$d_{66} = 2\cos(hD) + \left[\frac{2\nu}{1-2\nu}\cos(hD) + \frac{1}{1-2\nu}(hD)\sin(hD)\right].$$

4 In some cases, the ${}_0u_i'$ can by inspection be expressed in terms of the ${}_0u_i$. If that is the case, the reader should take advantage of this in order to reduce the dimension of the system (Eq. 15) and as a result avoid a lot of unnecessary algebra



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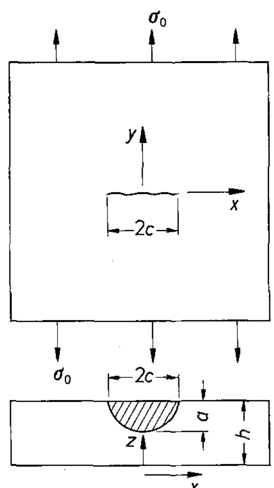

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Figs. 1 and 2. Geometrical and loading configura tion 1 of a plate with a flaw of an arbitrary shape; 2 of a partial through-the-thickness crack

Next, we will construct a solution to the system (Eq. 15) in the form

$$\xi_n = \xi_n^{(h)} + \xi_n^{(p)} \qquad (n = 1, 2, \ldots, 6). \tag{17}$$

Thus keeping in mind that the differential operators ∂_1, ∂_2, D^2 obey the same formal rules of addition and multiplication as numbers, the homogeneous solution of the system is

$$\xi_n^{(h)} = \chi_n(x, y), \qquad (n = 1, 2, 3, \ldots 6) \tag{18}$$

where the functions χ_n satisfy the differential relations

$$Q\chi_n = 0, \qquad (n = 1, 2, \ldots 6) \tag{19}$$

with

$$Q = \det|d_{mn}|. \tag{20}$$

Moreover, the operator Q is an even function (hD), for example:

$$\text{in problem (i):} \quad Q = \frac{2}{1-2\nu}\,\frac{(hD)^4}{h^2}\,\frac{\sin(hD)}{(hD)}\left[1 + \frac{\sin(2hD)}{(2hD)}\right] \tag{20a}$$

$$\text{in problem (ii):} \quad Q = \frac{1}{1-\nu}\,\frac{(hD)^2}{h^2}\cos(hD)\left[1 - \frac{\sin(2hD)}{(2hD)}\right] \tag{20b}$$

$$\text{in problem (iii):} \quad Q = \frac{1-\nu}{\nu^3}\,\frac{(hD)^4}{h^3}\,\frac{\sin(hD)}{(hD)}\left[(hD)^2 - \sin^2(hD)\right]. \tag{20c}$$

It remains, therefore, for us to construct a solution of the differential Eq. (19). Using a Fourier integral formulation, we write[5]

$$\xi_n^{(h)} = \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} R_n^{(k)} e^{\pm\sqrt{s^2+\beta_k^2}\,y} e^{isx}\, \mathrm{d}s, \tag{21}$$

where the β_k's are the roots of the equation

$$Q|_{D=\beta_k} = 0. \tag{22}$$

5 Note that any multiple roots must be handled in the usual way. Also, for a plate with finite dimensions in the x-direcction, a Fourier Series formulation must be used. Thus, three dimensional beam problems may also be solved by the same method

This equation, in general, has an infinite number of complex roots which appear in groups of four, one in each quadrant of the complex plane and only two of each group of four roots are relevant to the present work. These are the complex conjugate pairs with $Re(\beta_k) \geq 0$.

Returning to Eq. (21) we see that not all cofficients $R_n^{(k)}$ are linearly independent. Upon substituting Eq. (21) into the homogeneous part of equation (15) we rescover the corresponding relationships.

On the other hand, the particular solution can be expressed in terms of the Green's function $G_{mn}(s, y; \eta, \zeta)$ of the system (Eq. 15), in particular

$$\xi_n^{(p)} = \int_0^x \int_0^y \sum_{m=1}^{6} G_{mn}(x, y; \eta, \zeta) H_m(\eta, \zeta) \, d\eta \, d\zeta \qquad (m, n = 1, 2, \ldots, 6). \tag{23}$$

Finally, in view of Eqs. (23), (21), (17), (8), (9), (10) and the definition of ξ_n, the complementary displacement functions can be written in the form

$$u_i^{(c)} = \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \sum_{n=1}^{6} M_i(R_n^{(k)}, \beta_k z) e^{\pm\sqrt{s^2+\beta_k^2}\,y} e^{isx} \, ds + \int_0^x \int_0^y N_i(x, y, z; \eta, \zeta) \, d\eta \, d\zeta \qquad (i = 1, 2, 3 \quad \text{and} \quad n = 1, 2, \ldots, 6), \tag{24}$$

where the functions N_i are known and the remaining coefficients $R_n^{(k)}$ are to be determined from the boundary conditions (Eq. 3) and the appropriate continuity conditions. Moreover, if the stress σ_{3i} vanish on the two planes $z = 0$ and $z = h$, $\xi_n^{(p)} = 0$ and consequently $N_i = 0$.

Once the coefficients $R_n^{(k)}$ have been chosen so that the remaining boundary conditions (Eq. 3) are satisfied, the displacements $u_i^{(c)}$ and the stresses $\sigma_{ij}^{(c)}$ can be determined. For example, without going into the mathematical details, the general solution for the case of a uniform elastic layer with zero stresses on faces $z = 0$ and $z = h$ becomes:

(i) the displacement field[6]:

$$\begin{aligned} u^{(c)} = & -\frac{\partial I_1}{\partial y} + z\frac{\partial^2 I_3}{\partial y^2} + \frac{3m-2}{2(m-1)} z^2 \frac{\partial^2 I_4}{\partial y^2} - \frac{m}{2(m-1)} z^2 \left(\frac{\partial^3 I_1}{\partial y^3} + \frac{\partial^3 I_2}{\partial y^3}\right) - |y| \frac{\partial I_4}{\partial y} \\ & + \sum_{v=1}^{\infty} \frac{\partial H_v}{\partial x} \left\{ \frac{1 - \cos\beta_v h}{\beta_v^2 h} \left[\cos\beta_v z - \frac{m}{2(m-1)} \beta_v z \sin\beta_v z\right] - \frac{m-2}{2(m-1)} \frac{1}{\beta_v} \left[\sin(\beta_v z) \right.\right. \\ & \left.\left. + \frac{m}{m-2} \beta_v z \cos(\beta_v z)\right]\right\} + \sum_{v=1}^{\infty} \frac{\partial \tilde{H}_v}{\partial x} \left\{ \frac{1 + \cos\gamma_v h}{\gamma_v^2 h} \left[\cos(\gamma_v z) - \frac{m}{2(m-1)} \gamma_v z \sin(\gamma_v z)\right] \right. \\ & \left. - \frac{m-2}{2(m-1)} \frac{1}{\gamma_v} \left[\sin(\gamma_v z) + \frac{m}{m-2} \gamma_v z \cos(\gamma_v z)\right]\right\} + \sum_{n=1}^{\infty} \frac{\partial H_n}{\partial y} \cos(\alpha_n z) \end{aligned} \tag{25}$$

$$\begin{aligned} v^{(c)} = & -\frac{\partial I_2}{\partial x} - z\frac{\partial^2 I_3}{\partial x \partial y} - \frac{3m-2}{2(m-1)} z^2 \frac{\partial^3 I_4}{\partial x^2 \partial y} + \frac{m}{2(m-1)} z^2 \left(\frac{\partial^3 I_1}{\partial x \partial y^2} + \frac{\partial^3 I_2}{\partial x \partial y^2}\right) + |y| \frac{\partial I_4}{\partial x} \\ & - \sum_{n=1}^{\infty} \frac{\partial H_n}{\partial x} \cos(\alpha_n z) + \sum_{v=1}^{\infty} \frac{\partial H_v}{\partial y} \left\{ \frac{1 - \cos(\beta_v h)}{\beta_v^2 h} \left[\cos(\beta_v z) - \frac{m}{2(m-1)} \beta_v z \sin(\beta_v z)\right] \right. \\ & \left. - \frac{m-2}{2(m-1)} \frac{1}{\beta_v} \left[\sin(\beta_v z) + \frac{m}{m-2} \beta_v z \cos(\beta_v z)\right]\right\} + \sum_{v=1}^{\infty} \frac{\partial \tilde{H}_v}{\partial y} \left\{ \frac{1 + \cos(\gamma_v h)}{\gamma_v^2 h} \left[\cos(\gamma_v z) \right.\right. \\ & \left.\left. - \frac{m}{2(m-1)} \gamma_v z \sin(\gamma_v z)\right] - \frac{m-2}{2(m-1)} \frac{1}{\gamma_v} \left[\sin(\gamma_v z) + \frac{m}{m-2} \gamma_v z \cos(\gamma_v z)\right]\right\} \end{aligned} \tag{26}$$

6 In writing Eqs. (25)–(33), we assumed symmetry with respect to the x-axis

$$w^{(c)} = \frac{\partial I_3}{\partial x} + \frac{1}{m-1} z \left(\frac{\partial^2 I_1}{\partial x \partial y} + \frac{\partial^2 I_2}{\partial x \partial y} \right) - \frac{1}{m-1} z \frac{\partial I_4}{\partial x} + \sum_{v=1}^{\infty} H_v \left\{ \frac{1-\cos(\beta_v h)}{\beta_v h} \left[\frac{m-2}{2(m-1)} \sin(\beta_v z) - \frac{m}{2(m-1)} \beta_v z \cos(\beta_v z) \right] + \cos(\beta_v z) + \frac{m}{2(m-1)} \beta_v z \sin(\beta_v z) \right\} + \sum_{v=1}^{\infty} \tilde{H}_v \left\{ \frac{1+\cos(\gamma_v h)}{\gamma_v h} \times \left[\frac{m-2}{2(m-1)} \sin(\gamma_v z) - \frac{m}{2(m-1)} \gamma_v z \cos(\gamma_v z) \right] + \cos(\gamma_v z) + \frac{m}{2(m-1)} \gamma_v z \sin(\gamma_v z) \right\} \tag{27}$$

(ii) the stress field:

$$\frac{\sigma_{xx}^{(c)}}{2G} = -\frac{\partial^2 I_1}{\partial x \partial y} + z \frac{\partial^3 I_3}{\partial x \partial y^2} + \frac{1}{m+1} z^2 \frac{\partial^3 I_4}{\partial x \partial y^2} - |y| \frac{\partial^2 I_4}{\partial x \partial y} + \sum_{v=1}^{\infty} \frac{\partial^2 H_v}{\partial x^2} \left[\frac{1-\cos\beta_v h}{\beta_v^2 h} \left[\cos(\beta_v z) - \frac{m}{2(m-1)} \beta_v z \sin(\beta_v z) \right] - \frac{m-2}{2(m-1)} \frac{1}{\beta_v} \left[\sin(\beta_v z) + \frac{m}{m-2} \beta_v z \cos(\beta_v z) \right] \right]$$
$$+ \frac{1}{m-1} \sum_{v=1}^{\infty} \frac{1}{h} H_v [(1-\cos(\beta_v h)) \cos(\beta_v z) - \beta_v h \sin(\beta_v z) + \sum_{v=1}^{\infty} \frac{\partial^2 \tilde{H}_v}{\partial x^2} \left[\frac{1+\cos(\gamma_v h)}{\gamma_v^2 h} \times \left[\cos(\gamma_v z) - \frac{m}{2(m-1)} \gamma_v z \sin(\gamma_v z) \right] - \frac{m-2}{2(m-1)} \frac{1}{\gamma_v} \left[\sin(\gamma_v z) + \frac{m}{m-2} \gamma_v z \cos(\gamma_v z) \right] \right]$$
$$+ \frac{1}{m-1} \sum_{v=1}^{\infty} \frac{1}{h} \tilde{H}_v [(1+\cos(\gamma_v h)) \cos(\gamma_v z) - \gamma_v h \sin(\gamma_v z)] + \sum_{n=1}^{\infty} \frac{\partial^2 H_n}{\partial x \partial y} \cos(\alpha_n z) \tag{28}$$

$$\frac{\sigma_{yy}^{(c)}}{2G} = -\frac{\partial^2 I_2}{\partial x \partial y} - z \frac{\partial^3 I_3}{\partial z \partial y^2} - \frac{z^2}{m+1} \frac{\partial^3 I_4}{\partial x \partial y^2} + |y| \frac{\partial^2 I_4}{\partial x \partial y} + \left(\frac{m-1}{m+1} \right) \frac{\partial I_4}{\partial x}$$
$$+ \frac{1}{m-1} \sum_{v=1}^{\infty} \frac{1}{h} H_v [(1-\cos(\beta_v h)) \cos(\beta_v z) - \beta_v h \sin(\beta_v z)] + \sum_{v=1}^{\infty} \frac{\partial^2 H_v}{\partial y^2} \left[\frac{1-\cos(\beta_v h)}{h} \times \left[\cos(\beta_v z) - \frac{m}{2(m-1)} \beta_v z \sin(\beta_v z) \right] - \frac{m-2}{2(m-1)} \frac{1}{\beta_v} \left[\sin(\beta_v z) + \frac{m}{m-2} \beta_v z \cos(\beta_v z) \right] \right]$$
$$- \sum_{n=1}^{\infty} \frac{\partial^2 H_n}{\partial x \partial y} \cos(\alpha_n z) + \frac{1}{m-1} \sum_{v=1}^{\infty} \frac{1}{h} \tilde{H}_v [(1+\cos(\gamma_v h)) \cos(\gamma_v z) - \gamma_v h \sin(\gamma_v z)]$$
$$+ \sum_{v=1}^{\infty} \frac{\partial^2 \tilde{H}}{\partial y^2} \left[\frac{1+\cos(\gamma_v h)}{\gamma_v^2 h} \left[\cos(\gamma_v z) - \frac{m}{2(m-1)} \gamma_v \sin(\gamma_v z) \right] - \frac{m-2}{2(m-1)} \frac{1}{\gamma_v} \left[\sin(\gamma_v z) + \frac{m}{m-2} \gamma_v z \cos(\gamma_v z) \right] \right] \tag{29}$$

$$\frac{\sigma_{zz}^{(c)}}{2G} = \frac{m}{2(m-1)} \sum_{v=1}^{\infty} \frac{1}{h} H_v [(1-\cos(\beta_v h)) \beta_v z \sin(\beta_v z) - \beta_v h \sin(\beta_v z) + \beta_v h \beta_v z \cos(\beta_v z)]$$
$$+ \frac{m}{2(m-1)} \sum_{v=1}^{\infty} \frac{1}{h} \tilde{H}_v [(1+\cos(\gamma_v h)) \gamma_v z \sin(\gamma_v z) - \gamma_v h \sin(\gamma_v z) + \gamma_v h \gamma_v z \cos(\gamma_v z)] \tag{30}$$

$$\frac{\tau_{xy}^{(c)}}{G} = -\left(\frac{\partial^2 I_1}{\partial y^2} - \frac{\partial^2 I_2}{\partial y^2}\right) + 2z\frac{\partial^3 I_3}{\partial y^3} + \frac{2}{m+1} z^2 \frac{\partial^3 I_4}{\partial y^3} - 2|y|\frac{\partial^2 I_4}{\partial y^2} - \frac{\partial I_4}{\partial y}$$

$$+ \sum_{v=1}^{\infty} \frac{\partial^2 H_v}{\partial x \partial y}\left[\frac{1-\cos\beta_v h}{\beta_v^2 h}\left(2\cos(\beta_v z) - \frac{m}{m-1}\beta_v z \sin(\beta_v z)\right) - \frac{m-2}{m-1}\frac{1}{\beta_v}\left(\sin(\beta_v z)\right.\right.$$

$$\left.\left. + \frac{m}{m-2}\beta_v z\cos(\beta_v z)\right)\right] + \sum_{v=1}^{\infty}\frac{\partial^2 \tilde{H}_v}{\partial x \partial y}\left[\frac{1+\cos\gamma_v h}{\gamma_v^2 h}\left(2\cos(\gamma_v z) - \frac{m}{m-1}\gamma_v z\sin(\gamma_v z)\right)\right.$$

$$\left. - \frac{m-2}{m-1}\frac{1}{\gamma_v}\left(\sin(\gamma_v z) + \frac{m}{m-2}\gamma_v z\cos(\gamma_v z)\right)\right] + \sum_{n=1}^{\infty}\left(-\frac{\partial^2 H_n}{\partial x^2} + \frac{\partial^2 H_n}{\partial y^2}\right)\cos(\alpha_n z) \tag{31}$$

$$\frac{\tau_{yz}^{(c)}}{G} = \frac{m}{m-1}\sum_{v=1}^{\infty}\frac{\partial H_v}{\partial y}\left[\frac{1-\cos(\beta_v h)}{\beta_v h}(\sin(\beta_v z) + \beta_v z\cos(\beta_v z)) - \beta_v z\sin(\beta_v z)\right]$$

$$+ \frac{m}{m-1}\sum_{v=1}^{\infty}\frac{\partial \tilde{H}_v}{\partial y}\left[\frac{1+\cos\gamma_v h}{\gamma_v h}(\sin(\gamma_v z) + \gamma_v z\cos(\gamma_v z)) - \gamma_v z\sin(\gamma_v z)\right]$$

$$- \sum_{n=1}^{\infty}\alpha_n\frac{\partial H_n}{\partial x}\sin(\alpha_n z) \tag{32}$$

$$\frac{\tau_{zx}^{(c)}}{G} = -\frac{m}{m-1}\sum_{v=1}^{\infty}\frac{\partial H_v}{\partial x}\left[\frac{1-\cos\beta_v h}{\beta_v h}(\sin(\beta_v z) + \beta_v z\cos(\beta_v z)) - \beta_v z\sin(\beta_v z)\right]$$

$$- \frac{m}{m-1}\sum_{v=1}^{\infty}\frac{\partial \tilde{H}_v}{\partial x}\left[\frac{1+\cos\gamma_v h}{\gamma_v h}(\sin(\gamma_v z) + \gamma_v z\cos(\gamma_v z)) - \gamma_v z\sin(\gamma_v z)\right]$$

$$- \sum_{n=1}^{\infty}\frac{\partial H_n}{\partial y}\alpha_n\sin(\alpha_n z) \tag{33}$$

where $m \equiv 1/v$, v is Poisson's ration, $\alpha_n = n\frac{\pi}{h}$ $(n = 0, 1, 2, \ldots)$ and β_v, γ_v are the roots of the transcendental equations:

$$\sin(\beta_v h) = (\beta_v h); \quad \sin(\gamma_v h) = -(\gamma_v h), \tag{34), (35}$$

and the functions I_1, I_2, I_3, I_4 are 2D harmonic functions[7] and H_v, $\tilde{H}_v$, H_n satisfy the equations:

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} - \begin{Bmatrix}\beta^2\\ \alpha^2\\ \gamma^2\end{Bmatrix} H = 0. \tag{36}$$

It may be noted that the above complementary displacement field is very general and can be used to solve a whole class of 3D elastic problems, including the partial through-the-thickness crack (Fig. 2). By virtue of its construction, the solution satisfies the governing equations as well as zero stress boundary conditions at the top and bottom surfaces of the layer. The remaining arbitrary functions H_v, $\tilde{H}_v$, H_n and I_i $(i = 1, 2, 3, 4)$ are to be used in order to satisfy the remaining boundary conditions on the surface of the discontinuity.

The general 3D solution has been used successfully to construct an analytical solution for a plate, of an arbitrary thickness 2h, which has been weakened by a cylindrical hole. For the case of an extensional load, the details have been worked out by Folias et al. (1986). The work has also been submitted as a follow up paper.

The general solution has also been used to solve the problem of a cylindrical inclusion embedded in a plate of an arbitrary thickness (Folias et al. 1987a). The results of this analysis are of great

7 For a definition in terms of the Fourier integral transform see Appendix

pracitcal importance to the field of composites and are presently extended to also include a compressive load along the axis of the inclusion.

Other papers which are being completed and will be forthcoming are:

(i) a plate weakened by a hole and subjected to a uniform bending load

(ii) a periodic distribution of anisotropic inclusions embedded in a plate of an arbitrary thickness

(iii) a laminated plate with a cylindrical hole (laminates are isotropic but of different material properties)

(iv) a plate of finite thickness containing a partial-through-the-thickness crack.

Perhaps it is noteworthy to point out another important feature of the general 3D solution.

It reveals the inherent form of the solution, particularly in regions where a geometrical discontinuity intersects with the free surface of a plate, and permits a simplistic approach, a la Williams (1952), for the construction of such stress fields. For example, Folias (1987b) utilizes the form of the general solution in order to extract the explicit, 3D, analytical stress field in the neighborhood of the intersection of a hole and a free surface. The results were subsequently extended (Folias 1987c) to the case of a cylindrical inclusion. The analysis shows that the stress field is singular and that the order of the prevailing singularity depends on the material properties. The limiting cases of a soft, as well as a rigid inclusion, are also discussed. In a recent paper, Folias (1988b) derived the explicit 3D stress field in a laminated composite plate particularly in the vicinity of the intersection of a hole and a material interface. The analysis shows the stress field to be singular and provides a better understanding of the dependance of the interlaminar stresses on the material constants of the two adjacent laminates.

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Appendix

Let:

$$I_1 = \int_0^\infty \frac{P_1}{s} e^{-s|y|} \sin(xs)\,ds \qquad H_v = \int_0^\infty R_v\, e^{-\sqrt{s^2+\beta_v^2}|y|} \cos(xs)\,ds$$

$$I_2 = \int_0^\infty \frac{P_2}{s} e^{-s|y|} \sin(xs)\,ds \qquad \tilde{H}_v = \int_0^\infty \tilde{R}_v\, e^{-\sqrt{s^2+\gamma_v^2}|y|} \cos(xs)\,ds$$

$$I_3 = \int_0^\infty \frac{P_3}{s} e^{-s|y|} \sin(xs)\,ds \qquad H_n = \int_0^\infty s_v\, e^{-\sqrt{s^2+\alpha_n^2}|y|} \sin(xs)\,ds$$

$$I_4 = \int_0^\infty \frac{Q_1}{s} e^{-s|y|} \sin(xs)\,ds$$

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