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Bounds for Effective Parameters of Multicomponent Media by Analytic Continuation

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Recently D. Bergman introduced a method for obtaining bounds for the effective dielectric constant (or conductivity) of a two-component medium. This method does not rely on a variational principle but instead exploits the properties of the effective parameter as an analytic function of the ratio of the component parameters. We extend the method to multicomponent media using techniques of several complex variables.

KEY WORDS: Multicomponent stationary random media; bounds for the effective dielectric constant; integral representations; several complex variables.

1. INTRODUCTION

Because of the difficulty of calculating the effective parameters (e.g. dielectric constant, or thermal or electrical conductivity) of a heterogeneous medium, there has been much interest in obtaining bounds for these parameters. Wiener[1] gave optimal bounds for the effective parameters of a multicomponent material with fixed volume fractions and real component parameters. For isotropic materials, Hashin and Shtrikman[2] improved Wiener's bounds using variational principles. Recently Bergman[3-10] introduced a method for obtaining bounds on complex effective parameters which does not rely on variational principles. Instead it exploits the properties of the effective parameters as analytic functions of the component parameters. The method of Bergman has been elaborated upon in detail and applied to several problems by Milton.[11-16] A mathematical formulation of it was given by the authors in Ref. 17. However, aside from Bergman's trajectory approach which is discussed below, the method has

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been restricted to two-component materials, where the effective parameters are functions of a single complex variable, the ratio of the component parameters. In this paper the analytic continuation method is extended to multicomponent media in a direct way for the first time. In particular, we obtain the analog for several complex variables of the single variable integral representation for effective parameters given in Ref. 17. The extension is illustrated by rederiving the Wiener bounds for multicomponent media and by proposing new complex versions of them which are based on a hypothesis that cannot be verified at present. In another paper, Golden[18] rederives the Hashin–Shtrikman bounds for multicomponent media and gives new complex versions of them, based again on an unproven hypothesis.

In Ref. 17 the integral representation involves a complex kernel containing the component parameter information and a positive measure containing information about the geometry of the composite. For three-component materials, one of the two complex variables can be fixed as a multiple of the other, so that the effective parameters are treated as analytic functions of a single complex variable. Bergman[3,9] has applied the single variable analytic method in this case to obtain the real Hashin–Shtrikman bounds. However, this approach makes the above-mentioned measure depend upon the component parameters as well as the geometry of the composite. The problem is to give a direct extension of the analytic continuation method. That is, to find a representation for the effective parameters from information about the geometry of the composite. We have done this by treating the effective parameters explicitly as analytic functions of several complex variables.

The multicomponent representation formula is significant for the following reason. As in the two-component case, the effective parameters can be expanded about a homogeneous medium where the component parameters are equal. The information in this perturbation expansion can then be used along with the representation formula to continue the effective parameters beyond nearly homogeneous materials to their full domain of analyticity.

2. ANALYTICITY OF THE EFFECTIVE PARAMETER

We assume that the medium under study is an N-component microscopically isotropic dielectric. Our formulation of the multicomponent problem is the same as in Refs. 17 and 19. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\varepsilon_{ij}(x, \omega)$ be strictly stationary random fields taking N complex values $\varepsilon_1, \ldots, \varepsilon_N$ on $x \in \mathbb{R}^d$, $\omega \in \Omega$. Assume that there is a translation group τ_x, $x \in \mathbb{R}^d$, which is one to one on Ω and preserves P.

Then with each stationary random field $f(x, \omega)$ we associate a measurable function $f(\omega)$ via $f(\tau_{-x}\omega) = f(x, \omega)$.

Let $E^k(x, \omega)$ and $D^k(x, \omega)$ be two stationary random vector fields satisfying

$$D_i^k(x, \omega) = \sum_{j=1}^{d} \varepsilon_{ij}(x, \omega)\, E_j^k(x, \omega), \qquad i = 1,..., d \tag{2.1}$$

$$\nabla \cdot D^k(x, \omega) = 0 \tag{2.2}$$

$$\nabla \times E^k(x, \omega) = 0 \tag{2.3}$$

$$\int_\Omega P(d\omega)\, E^k(x, \omega) = e_k \tag{2.4}$$

where e_k is a unit vector in the kth direction for some $k = 1, 2,..., d$. The effective dielectric constant ε_{ik}^* may now be defined as

$$\varepsilon_{ik}^* = \int_\Omega P(d\omega)\, D_i^k(\omega) \tag{2.5}$$

In Ref. 17 we showed that this ensemble average coincides with the more standard definition involving a volume average.

We focus attention on media of the form $\varepsilon_{ij}(\omega) = \varepsilon(\omega)\, \delta_{ij}$, where $\varepsilon(\omega) = \varepsilon_1 \chi_1(\omega) + \cdots + \varepsilon_N \chi_N(\omega)$. The indicator function $\chi_l(\omega)$ of medium l equals one for all realizations $\omega \in \Omega$ which have medium l at, say $x = 0$, and equals zero otherwise. Since (2.1)–(2.5) are linear in $\varepsilon(\omega)$, ε_{ik}^* depends only on the ratios $h_i = \varepsilon_i/\varepsilon_N$, $i = 1,..., N-1$. We write

$$m_{ik}(h_1,..., h_{N-1}) = \frac{\varepsilon_{ik}^*}{\varepsilon_N} = \int_\Omega P(d\omega) \left[\sum_{j=1}^{N-1} h_j \chi_j(\omega) + \chi_N \right] E_i^k(\omega) \tag{2.6}$$

Clearly m_{ik} has the same domain of analyticity in $\mathbb{C}^{N-1}$ as does E_i^k. From analysis of (2.2) it can be shown[20] that if $(h_1,..., h_{N-1})$ is finite and such that there exists a unique solution E_i^k to (2.2)–(2.4), then E_i^k is analytic at $(h_1,..., h_{N-1})$. Determining the existence and uniqueness of E_i^k is facilitated by reformulating (2.2)–(2.4) in the Hilbert space $\mathscr{H} = \{f_i(\omega) \in L^2(\Omega, \mathscr{F}, P)$, $i = 1,..., d \mid L_i f_j = L_j f_i$ weakly and $\int_\Omega P(d\omega)\, f_i(\omega) = 0\}$. Here the L_i, $i = 1,..., d$, are the infinitesimal generators of the unitary group T_x induced by τ_x through $(T_x f)(\omega) = f(\tau_{-x}\omega)$. Now the problem becomes to find $G_i^k(\omega) \in \mathscr{H}$ such that

$$\int_\Omega P(d\omega) \sum_{i,j=1}^{d} \varepsilon_{ij}(\omega) [G_j^k(\omega) + \delta_{jk}]\, \overline{f_i(\omega)} = 0 \tag{2.7}$$

which is a weak version of (2.2) under (2.3) and (2.4), where the bar denotes complex conjugation. By applying the Lax–Milgram lemma[21] to the bilinear form associated with (2.7), one finds[20] that there exists a unique E_i^k when the convex hull of $\{1, h_1, ..., h_{N-1}\}$ does not contain the origin in $\mathbb{C}$. Thus m_{ik} is analytic when $(h_1, ..., h_{N-1})$ satisfies this condition. Furthermore, from the symmetric form of the definition

$$\varepsilon_{ik}^* = \int_\Omega P(d\omega) \sum_{j=1}^{d} \varepsilon(\omega)\, E_j^k(\omega)\, \overline{E_j^i(\omega)} \tag{2.8}$$

it is apparent that the diagonals m_{kk} map $\{\operatorname{Im} h_1 > 0\} \times \cdots \times \{\operatorname{Im} h_{N-1} > 0\}$ into the upper half-plane with $m_{kk}(\bar{h}_1, ..., \bar{h}_{N-1}) = \overline{m_{kk}}(h_1, ..., h_{N-1})$.

3. BOUNDS FOR TWO-COMPONENT MEDIA

Let $h = \varepsilon_1/\varepsilon_2$, $s = 1/(1-h)$, and $F_{ik}(s) = \delta_{ik} - m_{ik}(h)$. From Section 2, $m_{ik}(h)$ is analytic off the negative real axis $(-\infty, 0]$, or $F_{ik}(s)$ is analytic off $[0, 1]$. In Ref. 17 we proved that there exist finite Borel measures $\mu_{ik}(dz)$ on $[0, 1)$ such that the diagonals $\mu_{kk}(dz)$ are positive and

$$F_{ik}(s) = \int_0^1 \frac{\mu_{ik}(dz)}{s - z}, \qquad i, k = 1, ..., d, \quad s \notin [0, 1] \tag{3.1}$$

One proof of (3.1) depends on the operator representation arising from (2.2),

$$F_{ik}(s) = \int_\Omega P(d\omega)\, \chi_1 [(s + \Gamma\chi_1)^{-1} e_k] \cdot e_i \tag{3.2}$$

where $\Gamma = \nabla(-\Delta)^{-1} \nabla\cdot$ with L_i replacing $\partial/\partial x_i$ in the differential operators. In the Hilbert space $L^2(\Omega, \mathscr{F}, P)$ with weight $\chi_1(\omega)$ in the inner product, $\Gamma\chi_1$ is a bounded, self-adjoint operator of norm less than or equal to 1. The formula (3.1) is the spectral representation of the resolvent $(s + \Gamma\chi_1)^{-1}$ where $\mu_{ik}(dz)$ is the spectral measure of the family of projections of $\Gamma\chi_1$. Another related proof exploits the fact that $-F_{kk}(s)$ has positive imaginary part when $\operatorname{Im} s > 0$, and is analytic at $s = \infty$. Then a general representation theorem in function theory[22] gives (3.1) for the diagonals $i = k$. It is this function theory approach that we will use to extend the analytic method to multicomponent media.

For $|s|>1$, (3.1) can be expanded about a homogeneous medium ($s=\infty$ or $h=1$),

$$F_{ik}(s)=\frac{\mu_{ik}^{(0)}}{s}+\frac{\mu_{ik}^{(1)}}{s^2}+\frac{\mu_{ik}^{(2)}}{s^3}+\cdots,\qquad \mu_{ik}^{(n)}=\int_0^1 z^n\mu_{ik}(dz) \tag{3.3}$$

Equating (3.3) to the same expansion of (3.2) yields

$$\mu_{ik}^{(n)}=(-1)^n\int_\Omega P(d\omega)\,[\chi_1(\Gamma\chi_1)^n e_k]\cdot e_i \tag{3.4}$$

When $i=k$ the moments $\mu_{kk}^{(n)}$ uniquely determine the positive μ_{kk}.[23] Thus (3.1) provides the analytic continuation of (3.3) to the full complex s plane excluding $[0, 1]$. When $i\neq k$, μ_{ik} is a signed measure of mass 0.

We now focus on one diagonal coefficient ε_{kk}^* and call it ε^*, with $F=1-m=1-\varepsilon^*/\varepsilon_2$ and representative μ. Bounds on ε^* are obtained as follows. If only the volume fractions p_1 and $p_2=1-p_1$ of the two media are known, then (3.4) fixes only the mass of μ, with $\mu^{(0)}=p_1$. For fixed $s\notin[0, 1]$, extremal values of the linear functional $F(s, \mu)$ are attained by one-point measures $\mu^{(0)}\delta_z(dz)$, $0\leqslant z<1$, since they are the extreme points of the set of positive Borel measures of mass $\mu^{(0)}$ on $[0, 1)$. Applying the same considerations to $E(s)=1-\varepsilon_1/\varepsilon^*$ restricts ε^* to a region in the complex plane bounded by two circular arcs. When $s>1$ this region collapses to an interval $(p_1/\varepsilon_1+p_2/\varepsilon_2)^{-1}\leqslant\varepsilon^*\leqslant p_1\varepsilon_1+p_2\varepsilon_2$, the classical Wiener bounds, which are attained by parallel plane configurations of the materials.

If the material is further assumed to be statistically isotropic, then $\mu^{(1)}$ can be calculated as well,[17] the result being $\mu^{(1)}=p_1p_2/d$, where d is dimension. Now F in (3.3) is known to second order, but can be transformed to a function F_1 of the same type (3.1)[9,10,20] known only to first order via

$$F_1(s)=\frac{1}{p_1}-\frac{1}{sF(s)} \tag{3.5}$$

The measure $\mu_1(dz)$ that represents $F_1(s)$ has mass $\mu_1^{(0)}=p_2/p_1\,d$. Applying the above considerations to F_1 and $E_1=1/p_2-1/s\,E(s)$ restricts ε^* to a region again bounded by circular arcs which lie inside the first order bounds above. When $s\geqslant 1$ the region collapses to the Hashin–Shtrikman bounds

$$\varepsilon_1+\frac{p_2}{1/(\varepsilon_2-\varepsilon_1)+p_1/d\varepsilon_1}\leqslant\varepsilon^*\leqslant\varepsilon_2+\frac{p_1}{1/(\varepsilon_1-\varepsilon_2)+p_2/d\varepsilon_2},\qquad \varepsilon_1\leqslant\varepsilon_2 \tag{3.6}$$

which are attained by coated sphere geometries. The complex bounds can be improved[15,10] by incorporating the interchange inequality[24,25] $m(h)\,m(1/h) \geqslant 1$, which becomes an equality for $d=2$.

The higher moments $\mu^{(n)}$, $n \geqslant 2$, depend on $(n+1)$-point correlation functions and cannot be calculated in general, although the interchange inequality forces relations among them.[11,20] If $\mu^{(0)},..., \mu^{(n-1)}$ were known, then the transformation (3.5) can be iterated to produce a function of type (3.1) known only to first order, so that its extremization is the same as above. Baker[26,27] was the first to use such an iteration procedure in a slightly different form to obtain nth-order complex bounds on upper half-plane functions like $F(s)$. His work was done in the context of Padé approximants to Stieltjes series. Independently, in the context of heterogeneous media Milton[11] used another method to obtain nth-order complex bounds on ε^*, which reduce for real component parameters to those obtained from variational principles. Felderhof[28] reformulates Milton's bounds, and Golden[18,20] discusses the iterative approach based on the transformation (3.5), while Baker's approach is the most general. The relationship of Milton's bounds to bounds on Stieltjes series is discussed by Milton and Golden,[29] where a slightly different formulation of the iteration is given. Bergman[10,9] introduced the transformation (3.5). It is in fact equivalent to the transformation which underlies Baker's work. Bergman used it and a variant to derive second-order bounds on ε^*. Subsequently, Kantor and Bergman[30] suggested iteration of (3.5).

4. THE POLYDISK REPRESENTATION FORMULA

For simplicity we consider three-component media so that $m(h_1, h_2) = \varepsilon^*/\varepsilon_3$ and $F(s_1, s_2) = 1 - m(h_1, h_2)$ are functions of two complex variables with $h_1 = \varepsilon_1/\varepsilon_3$, $h_2 = \varepsilon_2/\varepsilon_3$, $s_1 = 1/(1-h_1)$ and $s_2 = 1/(1-h_2)$. Let $U^2 = \{\operatorname{Im} s_1 > 0\} \times \{\operatorname{Im} s_2 > 0\}$. As a counterpart of $F(s_1, s_2)$: $U^2 \to \{\operatorname{Im} F < 0\}$ we consider $f(\zeta_1, \zeta_2)$: $D^2 \to \{\operatorname{Re} f > 0\}$ where $D^2 = \{|\zeta_1| < 1\} \times \{|\zeta_2| < 1\}$ is the polydisk, which is conformally equivalent to U^2. We derive the analog of (3.1) with $i = k$ for f in D^2, by first giving a polydisk Schwartz formula which expresses f restricted to $D_R^2 = \{|\zeta_1| < R\} \times \{|\zeta_2| < R\}$, $R < 1$, in terms of an integral of its real part over $T_R^2 = \{|w_1| = R\} \times \{|w_2| = R\}$.

Cauchy's formula for $(\zeta_1, \zeta_2) \in D_R^2$ is

$$f(\zeta_1, \zeta_2) = \left(\frac{1}{2\pi i}\right)^2 \iint_{T_R^2} \frac{f(w_1, w_2)}{(w_1 - \zeta_1)(w_2 - \zeta_2)}\, dw_1\, dw_2 \tag{4.1}$$

Let $\zeta_j^* = (R^2/r_j)\, e^{i\theta_j}$ be the reflection of $\zeta_j = r_j e^{i\theta_j}$ in the circle $\{|w_j| = R\}$, $j = 1, 2$. From the one variable Cauchy formula one can see that if ζ_1 or ζ_2

or both are replaced by their reflections in the integral in (4.1), then the integral vanishes. Therefore $f(\zeta_1, \zeta_2)$ may be written as

$$f(\zeta_1, \zeta_2) = \left(\frac{1}{2\pi i}\right)^2 \iint_{T_R^2} f(w_1, w_2) \times \left(\frac{1}{w_1 - \zeta_1} \pm \frac{1}{w_1 - \zeta_1^*}\right)\left(\frac{1}{w_2 - \zeta_2} \pm \frac{1}{w_2 - \zeta_2^*}\right) dw_1\, dw_2 \quad (4.2)$$

for any of the four combinations of +'s and −'s. With $dw_j = i\,\mathrm{Re}^{it_j}\,dt_j$, $f(\zeta_1, \zeta_2)$ has the following equivalent representations:

$$f(\zeta_1, \zeta_2) = \left(\frac{1}{2\pi i}\right)^2 \int_0^{2\pi} \int_0^{2\pi} f(t_1, t_2) \begin{cases} (1 + iQ_1)\, P_2 \\ (1 + iQ_2)\, P_1 \\ (1 + iQ_1)(1 + iQ_2) \\ P_1 P_2 \end{cases} dt_1\, dt_2 \quad (4.3)$$

where $P_j = \mathrm{Re}[H_j]$, $Q_j = \mathrm{Im}[H_j]$, $H_j = (w_j + \zeta_j)/(w_j - \zeta_j)$ and $w_j = Re^{it_j}$. Manipulation of these four forms[20] yields

$$f(\zeta_1, \zeta_2) = iv(0, 0) + \frac{1}{2}\left(\frac{1}{2\pi i}\right)^2 \int_0^{2\pi} \int_0^{2\pi} (H_1 H_2 + H_1 + H_2 - 1) \times u(Re^{it_1}, Re^{it_2})\, dt_1\, dt_2 \quad (4.4)$$

where $f = u + iv$. The representation (4.4) can be verified by expanding f in a power series $f(\zeta_1, \zeta_2) = \sum_{n,m=0}^{\infty} A_{nm} \zeta_1^n \zeta_2^m$ and observing that for $n, m \geqslant 1$,

$$2A_{nm}\zeta_1^n \zeta_2^m = \left(\frac{1}{2\pi i}\right)^2 \int_0^{2\pi} \int_0^{2\pi} H_1 H_2\, \mathrm{Re}[A_{nm} w_1^n w_2^m]\, dt_1\, dt_2 \quad (4.5)$$

By taking the weak * limit of $u(Re^{it_1}, Re^{it_2})\, dt_1\, dt_2$ as $R \to 1$ and calling it $\mu(dt_1, dt_2)$ we extend (4.4) to the full polydisk D^2. One should note, however, that this μ is special, i.e., not all positive measures on T^2 arise from the boundary values of holomorphic functions in D^2.[31] Indeed, for $R < 1$, $u(Re^{it_1}, Re^{it_2})$ has nonzero Fourier coefficients only in $\mathbb{Z}_+^2 \cup \mathbb{Z}_-^2$, where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and $\mathbb{Z}_- = -\mathbb{Z}_+$. This follows from the fact that $u = \frac{1}{2}(f + \bar{f})$ and on T_R^2, f has a Fourier series with only nonnegative powers of $Re^{i\theta_1}$ and $Re^{i\theta_2}$. Thus the Fourier transform of μ vanishes in the interiors of the second and fourth quadrants of $\mathbb{Z}^2$.

We have now proven necessity in the following:

Theorem. For $f(\zeta_1, \zeta_2)$ to be holomorphic with nonnegative real part in D^2 it is necessary and sufficient that f may be represented as

$$f(\zeta_1, \zeta_2) = iv(0, 0) + \frac{1}{2}\iint_{T^2}\left(\frac{e^{it_1}+\zeta_1}{e^{it_1}-\zeta_1}\frac{e^{it_2}+\zeta_2}{e^{it_2}-\zeta_2} + \frac{e^{it_1}+\zeta_1}{e^{it_1}-\zeta_1} + \frac{e^{it_2}+\zeta_2}{e^{it_2}-\zeta_2} - 1\right)$$
$$\times \mu(dt_1, dt_2) \tag{4.6}$$

where μ is a positive Borel measure satisfying

$$\iint_{T^2} e^{i(nt_1+mt_2)}\mu(dt_1, dt_2) = 0 \qquad \text{when} \quad nm < 0, \quad n, m \in \mathbb{Z} \tag{4.7}$$

Sufficiency is proved by first noting that the series generated by (4.6) converges uniformly on compact subsets of D^2, so that the right side of (4.6) is holomorphic in D^2. That the real part of (4.6) is positive follows from the fourth form in (4.3) since $P_1 P_2$ is positive. Representations of $f(\zeta_1, \zeta_2)$ equivalent to (4.6) have been given by Korányi and Pukánzky in Ref. 32 and by Vladimirov and Drozhzhinov in Ref. 33.

Because $F(s_1, s_2)$ is analytic for real (s_1, s_2) when both are off $[0, 1]$, the measure $\mu(dt_1, dt_2)$ must vanish on a subset E of T^2 corresponding to those real (s_1, s_2). In Ref. 20 it is shown that as $(\zeta_1, \zeta_2) \in D^2$ is sent radially to E, the real part of (4.6) vanishes identically on E, and that (4.6) gives rise to a purely imaginary analytic function on E. A formula for this imaginary function is given in Ref. 20, as well as some interesting consequences of the Fourier condition (4.7) on the smoothness of the measure $\mu(dt_1, dt_2)$.

5. BOUNDS FOR MULTICOMPONENT MEDIA

For two-component media the bounds were obtained by examining the images of extreme points of the set of positive measures of mass $\leqslant 1$ under the mapping (3.1). Denote $M_1 = \{$positive Borel measures μ on T^2 that satisfy the Fourier condition (4.7) and have total mass $\leqslant 1\}$. The simplest extreme points of M_1 have the form

$$\mu_1^* = \alpha\delta_{t_1^*}(dt_1) \times \frac{dt_2}{2\pi}, \qquad \mu_2^* = \beta\delta_{t_2^*}(dt_2) \times \frac{dt_1}{2\pi} \tag{5.1}$$

where $0 \leqslant t_1^*$, $t_2^* < 2\pi$ and α, $\beta \leqslant 1$. However, the full set of extreme points of M_1 has not been completely characterized.[34-36] Nevertheless, we have been able to recover the Wiener and Hashin–Shtrikman bounds for three-component media with real ε_1, ε_2, and ε_3 by using sums of the measures in

(5.1) with appropriate weights. It appears then that such measures give extremal values of functions represented by (4.6) for fixed ζ_1 and ζ_2. Below we will give the forms of this conjecture which yield Wiener, or first order bounds. In Ref. 18, stronger forms are given, which yield Hashin–Shtrikman, or second-order bounds.

To derive the Wiener bounds we first give the operator representation analogous to (3.2) for three-component materials,

$$F_{ik}(s_1, s_2) = \int_\Omega P(d\omega) \left(\frac{1}{s_1} \chi_1 + \frac{1}{s_2} \chi_2 \right) \times \left[\left(I + \frac{1}{s_1} \Gamma \chi_1 + \frac{1}{s_2} \Gamma \chi_2 \right)^{-1} e_k \right] \cdot e_i \tag{5.2}$$

where I is the identity. Note that the operators $\Gamma\chi_1$ and $\Gamma\chi_2$ in (5.2) do not commute, so that it is not immediately clear how to extend the spectral analysis given in Section 3 to multicomponent media. For $|s_1| > 1$ and $|s_2| > 1$, (5.2) can be expanded about a homogeneous medium ($s_1 = s_2 = \infty$) for $i = k$,

$$F(s_1, s_2) = \int_\Omega P(d\omega) \left[\left(\frac{\chi_1}{s_1} + \frac{\chi_2}{s_2} - \frac{\chi_1 \Gamma \chi_1}{s_1^2} - \frac{\chi_2 \Gamma \chi_2}{s_2^2} - \frac{(\chi_1 \Gamma \chi_2 + \chi_2 \Gamma \chi_1)}{s_1 s_2} + \cdots \right) e_k \right] \cdot e_k \tag{5.3}$$

If only the volume fractions p_1, p_2, and $p_3 = 1 - p_1 - p_2$ of the three materials are known, then $F(s_1, s_2)$ in (5.3) is known only to first order

$$F(s_1, s_2) = \frac{p_1}{s_1} + \frac{p_2}{s_2} + \cdots \tag{5.4}$$

To get the bounds assuming (5.4) for fixed, real s_1, $s_2 > 1$, we state the following:

Hypothesis 1. Let $K(s_1, s_2)$ be analytic with negative imaginary part in $U^2 = \{\operatorname{Im} s_1 > 0\} \times \{\operatorname{Im} s_2 > 0\}$ such that $K(\bar{s}_1, \bar{s}_2) = \bar{K}(s_1, s_2)$, $K(\infty, \infty) = 0$, and $K(s_1, s_2)$ is analytic for real s_1 and s_2 when both s_1 and s_2 are off $[0, 1]$. If for fixed $\alpha_1, \alpha_2 > 0$

$$K(s_1, s_2) = \frac{\alpha_1}{s_1} + \frac{\alpha_2}{s_2} + \cdots \tag{5.5}$$

then for fixed $s_1, s_2 > 1$, K is minimized by

$$K(s_1, s_2) = \frac{\alpha_1}{s_1} + \frac{\alpha_2}{s_2} \tag{5.6}$$

Back on the torus K has a counterpart $f(\zeta_1, \zeta_2) = iK(s_1, s_2)$ with $\zeta_j = (s_j - i)/(s_j + i)$, $j = 1, 2$. The minimizer (5.6) corresponds to a sum of the measures in (5.1) with (t_1^*, t_2^*) in ζ variables mapping to $(0, 0)$ in s variables. Note that the functions included under Hypothesis 1 form a much broader class than those arising from sums of product measures on T^2. The point of the conjecture is that K attains its minimum within the special class $\mu_1^* + \mu_2^*$.

Applying Hypothesis 1 to $F(s_1, s_2)$ under (5.4) gives

$$F(s_1, s_2) \geqslant \frac{p_1}{s_1} + \frac{p_2}{s_2} \tag{5.7}$$

Doing the same for $H(t_1, t_2) = 1 - \varepsilon_3/\varepsilon^*$ where $t_1 = 1 - s_1$, and $t_2 = 1 - s_2$, yields along with (5.7) for ε^*

$$\frac{1}{p_1/\varepsilon_1 + p_2/\varepsilon_2 + p_3/\varepsilon_3} \leqslant \varepsilon^* \leqslant p_1\varepsilon_1 + p_2\varepsilon_2 + p_3\varepsilon_3 \tag{5.8}$$

which are the classical Wiener bounds. These bounds are again attained by parallel plane configurations of the materials.

For complex $(s_1, s_2) \in U^2$, we employ the following:

Hypothesis 2. If $K(s_1, s_2)$ is as in Hypothesis 1 with (5.5), then for fixed $(s_1, s_2) \in U^2$, the values of $K(s_1, s_2)$ lie inside the region generated by

$$K(s_1, s_2) = \frac{\alpha_1}{s_1 - z_1} + \frac{\alpha_2}{s_2 - z_2} \tag{5.9}$$

as both z_1 and z_2 vary in $[-\infty, \infty]$.

The measures on T^2 that give rise to (5.9) are again of the form $\mu_1^* + \mu_2^*$, where t_1^* and t_2^* vary throughout $[0, 2\pi)$. The region R_3 obtained by applying Hypothesis 2 to F under (5.4) can be constructed as follows. First note that $p_1/(s_1 - z_1)$ and $p_2/(s_2 - z_2)$ with $-\infty \leqslant z_1, z_2 \leqslant \infty$ are both circles in the lower half-plane that contain the origin and are thus tangential to the real axis. Now add to each point $p_1/(s_1 - z_1)$ the circle

$p_2/(s_2 - z_2)$, $-\infty \leqslant z_2 \leqslant \infty$. The outside boundary of R_3 is a circle characterized by

$$\arg\left(\frac{\partial F}{\partial z_1}\right) = \arg\left(\frac{\partial F}{\partial z_2}\right) \tag{5.10}$$

where "arg" dentoes argument and $F = K$ is as in (5.9) with $\alpha_1 = p_1$ and $\alpha_2 = p_2$. This condition is equivalent to $\arg(s_1 - z_1) = \arg(s_2 - z_2)$, or $z_2 = (b_2/b_1) z_1 + (a_2 - b_2 a_1/b_1)$, where $s_1 = a_1 + ib_1$ and $s_2 = a_2 + ib_2$. Thus the outside boundary $C_3(z_1)$ of R_3 is the image of this line in (z_1, z_2) space under (5.9) and can be parameterized in the F plane by

$$C_3(z_1) = \frac{p_1 + p_2 b_1/b_2}{s_1 - z_1}, \qquad -\infty \leqslant z_1 \leqslant \infty \tag{5.11}$$

With $\arg \varepsilon_3 < \arg \varepsilon_2 < \arg \varepsilon_1$ we can apply the same considerations as above to $1 - \varepsilon^*/\varepsilon_1$ as a function of $q_2 = 1/(1 - \varepsilon_2/\varepsilon_1)$ and $q_3 = 1/(1 - \varepsilon_3/\varepsilon_1)$ to obtain a circular region bounded by

$$C_1(z_1) = \frac{p_2 + p_3 \operatorname{Im} q_2/\operatorname{Im} q_3}{q_2 - z_1}, \qquad -\infty \leqslant z_1 \leqslant \infty \tag{5.12}$$

In the ε^* plane we obtain two circular regions R_1^* and R_3^*. The bounds obtained by intersecting R_1^* and R_3^* are sometimes quite crude. It is then necessary to intersect these first-order bounds with zeroth-order bounds[37,18,20] which incorporate knowledge of only ε_1, ε_2, and ε_3 without regard to volume fraction. These zeroth-order bounds restrict ε^* to a region bounded by three arcs, each of which may be circular or straight depending on the configuration of ε_1, ε_2, and ε_3, are optimal, and can be obtained from measures of the form μ_1^* and μ_2^*.

The complex first-order bounds we obtain in this way do not reduce to the classical Wiener bounds when the parameters become real. In addition, we have completely ignored the support restrictions on μ in (4.6) imposed by the analyticity of F for certain real (s_1, s_2). However, under certain circumstances parts of these complex bounds are attainable,[37,18,20] and Milton[37] has proven them using an extended version of Bergman's trajectory method. Nevertheless, we expect that the nonattainable sections of the bounds can be improved.

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