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NASH ESTIMATES AND THE ASYMPTOTIC BEHAVIOR OF DIFFUSIONS[1]

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In order to analyze the asymptotic behavior of a particle diffusing in a drift field derived from a smooth bounded potential, we develop Nash-type a priori estimates on the transition density of the process. As an immediate consequence of the estimates, we find that for a rapidly decaying potential in $\mathbb{R}^d$, the mean squared displacement behaves like $td + C(t)$, where $\dot{C}(t)$ (the time integral of the "velocity autocorrelation function") decays like $t^{-d/2}$. We also prove, using the estimates, that for a potential in $\mathbb{R}^d$ of the form $V + B$, where V is stationary random ergodic and B has compact support, the diffusion converges under space and time scaling to the same Brownian motion as does the diffusion with $B = 0$.

1. Introduction. We consider the diffusion in $\mathbb{R}^d$ of a particle at $\mathbf{X}_t$ at time t in a drift field which is the gradient of a smooth bounded potential V, described by $d\mathbf{X}_t = -\nabla V(\mathbf{X}_t)\,dt + d\mathbf{W}_t$, where $\mathbf{W}_t$ is standard Brownian motion. It has been established [5, 14, 17] that for stationary random ergodic potentials (a class which includes periodic and quasiperiodic potentials), the diffusion on a macroscopic scale behaves like Brownian motion with some effective diffusion tensor $\mathsf{D}(V)$. Since the limiting motion is Brownian, the mean squared displacement (MSD) $E(t) \equiv E|\mathbf{X}_t - \mathbf{X}_0|^2$ behaves like Dt, $D = \mathrm{tr}(\mathsf{D})$ for large t. This result, however, gives no information on the correction $C(t)$ to the dominant behavior, with $E(t) = Dt + C(t)$. The correction is directly related to the "velocity" autocorrelation function (VAF), with $\ddot{E}(t) \sim$ VAF, which is of general interest in a variety of physical systems [1, 6, 19, 20]. For diffusion in a potential (for which the actual particle velocity does not exist), $\langle \nabla V(\mathbf{X}_0) \cdot \nabla V(\mathbf{X}_t) \rangle$ plays the role of the VAF and its Fourier transform is the frequency (ν) dependent diffusivity $\mathsf{D}(\nu)$.

The present paper is one of several [9–12] which contain results on the detailed structure of the MSD. As a first step in understanding this structure for general V, here and in [12] we investigate the effect of adding a "local" perturbation B to some V whose MSD is easily analyzed, such as $V = 0$ or V periodic. The principal tool that we develop here and use extensively to facilitate this is Nash-type a priori estimates on the transition density $u(\mathbf{x}, t)$ of the process, i.e., upper and lower Gaussian bounds on u. They hold in the present situation because the generator L of $\mathbf{X}_t$ can be written in essentially divergence

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form, $L = \frac{1}{2}e^{2V}\nabla \cdot (e^{-2V}\nabla)$, and then the arguments of Fabes and Stroock [7] for divergence form generators $\nabla \cdot (\mathbf{a}\nabla)$ with uniformly elliptic $\mathbf{a}$ are easily seen to apply.

An immediate consequence of the Nash estimates is that for smooth bounded "local" B with $V = 0$, i.e., $\int|\mathbf{x}|\,|\nabla B(\mathbf{x})|\,d\mathbf{x} < \infty$, $E(t) = t + O(\sqrt{t})$ for $d = 1$, $E(t) = 2t + O(\log t)$ for $d = 2$ and $E(t) = dt + C + O(1/t^{d/2-1})$ for $d \geq 3$. We also prove using the estimates that for any smooth bounded V, the Laplace transform $\tilde{E}(s)$ of $E(t)$ is analytic off the negative real axis $(-\infty, 0]$ and approaches zero as $|s| \to \infty$ away from $(-\infty, 0]$. This technical result is useful for deducing $t \to \infty$ asymptotics about $E(t)$ from $\tilde{E}(s)$.

As a preliminary in analyzing the effect of a perturbation, we prove that adding to a stationary random potential V a smooth bump $B(\mathbf{x})$ of compact support leaves the dominant behavior of $E(t)$ for V unchanged. Furthermore, the diffusion in $V + B$ converges under scaling to the same Brownian motion as diffusion in V. The proof of these statements is more difficult than might be expected and requires the Nash estimates at each stage in the proof. These estimates give a very useful characterization of diffusivity, and allow one to rigorously use such heuristic statements as, "a diffusing particle can only be a distance of order $\sqrt{t}$ away from its starting point after time t."

In [12] we use the Nash estimates to study $E(t)$ for $V + B$ in one dimension when V is periodic. In [9] we find in any dimension quasiperiodic potentials of the simplest kind such that $C(t)/t$, roughly speaking, approaches zero slower than any function that can be explicitly written down, be it power law, logarithmic or even slower decay, so that there is *no* law.

It is interesting to note that the second time derivatives of our results here for $E(t)$ for rapidly decaying potentials in $\mathbb{R}^d$ have the same power laws as one expects for the VAF for "truly" random media [6, 20].

2. Formulation. Let $V(\mathbf{x})$, $\mathbf{x} \in R^d$, be uniformly bounded and smooth, i.e., having uniformly bounded first, second and third derivatives. Given $V(\mathbf{x})$, we consider on a probability space (S, G, P), the $\mathbb{R}^d$-valued process $\mathbf{X}_t$, $t \in \mathbb{R}$, governed by the stochastic differential equation

$$d\mathbf{X}_t = -\nabla V(\mathbf{X}_t)\,dt + d\mathbf{W}_t \tag{2.1}$$

with $\mathbf{X}_0 = \mathbf{x}_0$, where $\mathbf{W}_t$ is standard d-dimensional Brownian motion with mean 0 and covariance matrix tI, where I is the identity. Associated with (2.1) is the transition probability $p[A, t, \mathbf{y}, t'] = P[\mathbf{X}_t \in A | \mathbf{X}_{t'} = \mathbf{y}]$, where $t, t' \in R$, $t' < t$, $\mathbf{y} \in \mathbb{R}^d$ and A is a Borel subset of $\mathbb{R}^d$. Under the preceding smoothness conditions, $p[A, t, \mathbf{y}, t']$ has a density $u(\mathbf{x}, t, \mathbf{y}, t')$ which is a fundamental solution of both the backward equation

$$\frac{\partial u}{\partial t'} + Lu = 0, \qquad \lim_{t' \uparrow t} u(\mathbf{x}, t, \mathbf{y}, t') = \delta_{\mathbf{x}}(\mathbf{y}) \tag{2.2}$$

and the forward equation

$$\frac{\partial u}{\partial t} - L^* u = 0, \qquad \lim_{t \downarrow t'} u(\mathbf{x}, t, \mathbf{y}, t') = \delta_{\mathbf{y}}(\mathbf{x}), \tag{2.3}$$

where L is the backward generator

$$L = \tfrac{1}{2}\Delta - \nabla V \cdot \nabla \tag{2.4}$$

which acts in the $\mathbf{y}$-variable and L^* is the forward generator

$$L^* = \tfrac{1}{2}\Delta + \nabla \cdot \nabla V \tag{2.5}$$

which acts in the $\mathbf{x}$-variable and is the (formal) adjoint of L.

We shall be interested in the MSD of the diffusing particle

$$E\left[|\mathbf{X}_t - \mathbf{x}_0|^2\right] = \int_S |\mathbf{X}_t - \mathbf{x}_0|^2 \, dP \tag{2.6}$$

$$= \int_{\mathbb{R}^d} |\mathbf{x} - \mathbf{x}_0|^2 u(x, t, \mathbf{x}_0, 0)\, dx. \tag{2.7}$$

A representation of the MSD which will be useful in the next section is

$$E\left[|\mathbf{X}_t - \mathbf{x}_0|^2\right] = td - 2\int_0^t E\left[(\mathbf{X}_s - \mathbf{x}_0) \cdot \nabla V(\mathbf{X}_s)\right] ds, \tag{2.8}$$

which may be obtained by using Itô's formula or by multiplying (2.3) with $\mathbf{y} = \mathbf{x}_0$ and $t' = 0$ by $(\mathbf{x} - \mathbf{x}_0)^2$, integrating by parts over $\mathbb{R}^d$ and then time integrating up to t.

We shall also be considering the case when V is a sample of a stationary random potential $V(\mathbf{x}, \omega)$, defined on a probability space $(\Omega, \mathscr{F}, \rho)$, $\omega \in \Omega$, which is smooth and uniformly bounded in $\mathbf{x}$ and ω. We may assume that Ω is the space of potential fields, so that the translations $\tau_{\mathbf{x}}$, $\mathbf{x} \in \mathbb{R}^d$, are naturally defined on Ω. Given $\omega \in \Omega$, (2.1) defines a process $\mathbf{X}_t = \mathbf{X}_t^\omega$ starting at $\mathbf{x}_0$. For a function f on path space let

$$E_e\big(f(\mathbf{X}_t)_{t\geq 0}\big) = \int_\Omega \mu(d\omega) E\big(f(\mathbf{X}_t^{\tau_{\mathbf{x}_0}\omega})_{t\geq 0}\big),$$

where

$$\mu(d\omega) = e^{-2V(0,\omega)}\, d\rho \Big/ \int_\Omega e^{-2V(0,\omega)}\, d\rho,$$

the "equilibrium" measure on the space of potentials, is reversible for the environment process $\omega_t = \tau_{-\mathbf{X}_t}\omega$, the potential field seen by the particle at time t.

The equilibrium averaged MSD has the "velocity" autocorrelation representation [4, 5, 18]

$$E_e|\mathbf{X}_t - \mathbf{x}_0|^2 = td - 2\int_0^t (t-s) E_e\left[\nabla V(\mathbf{x}_0) \cdot \nabla V(\mathbf{X}_s)\right] ds. \tag{2.9}$$

Using the semigroup $e^{\hat{L}t}$, where $\hat{L}$ is the backward generator of the environment process, and then carrying out the integration in (2.9) via the spectral theorem in $L^2(\Omega, d\mu)$ (with inner product $\langle\ \rangle$) where $\hat{L}$ is negative and self-adjoint, yields

$$E_e|\mathbf{X}_t - \mathbf{x}_0|^2 = Dt - 2\langle \nabla V \cdot \hat{L}^{-2}(e^{\hat{L}t} - 1)\nabla V\rangle. \tag{2.10}$$

In (2.10),

$$D = d + 2\langle \nabla V \cdot \hat{L}^{-1} \nabla V \rangle \tag{2.11}$$

is the trace of the diffusion matrix

$$D_{ij} = \delta_{ij} + 2\left\langle \frac{\partial V}{\partial x_i} \hat{L}^{-1} \frac{\partial V}{\partial x_j} \right\rangle, \qquad i, j = 1, \dots, d.$$

[We remark that ∇V is orthogonal to the constants in $L^2(\Omega, d\mu)$.] In one dimension [8]

$$D = \left(\int_\Omega e^{2V}\, d\rho \int_\Omega e^{-2V}\, d\rho \right)^{-1}, \tag{2.12}$$

provided the random environment V is ergodic under translations.

3. Nash estimates on the transition density in $\mathbb{R}^d$. It appears that the sharpest estimates for solutions of parabolic pde that hold for all time apply to the fundamental solutions of divergence form equations

$$\frac{\partial u}{\partial t} = \nabla \cdot (\mathsf{a} \nabla u), \qquad u(\mathbf{x}, 0) = \delta(\mathbf{x} - \mathbf{x}_0), \tag{3.1}$$

where a is symmetric, smooth and uniformly elliptic, i.e., there is a $\lambda > 0$ such that

$$\lambda |\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}(\mathbf{x}) \xi_i \xi_j \le \lambda^{-1} |\xi|^2, \qquad \mathbf{x}, \xi \in \mathbb{R}^d. \tag{3.2}$$

Nash [15] proved estimates on u which are slightly weaker than Gaussian upper and lower bounds. Then Aronson [2] obtained the Gaussian bounds

$$\frac{1}{Ct^{d/2}} e^{-C(|\mathbf{x}-\mathbf{x}_0|^2/t)} \le u(\mathbf{x}, t) \le \frac{C}{t^{d/2}} e^{-(|\mathbf{x}-\mathbf{x}_0|^2/Ct)}, \tag{3.3}$$

where C depends only on λ and d. Fabes and Stroock [7] have expanded upon the original ideas of Nash to reprove (3.3) without using Moser's parabolic Harnack inequality, as did Aronson. In fact, Fabes and Stroock obtain this Harnack inequality from (3.3). It is interesting to note that for a general parabolic operator the bounds of Aronson [2] only hold for $t \in [0, T]$ and then C depends in general on T.

We wish to obtain bounds like (3.3) for the fundamental solution of

$$\frac{\partial u}{\partial t} = L^* u, \qquad u(\mathbf{x}, 0) = \delta(\mathbf{x} - \mathbf{x}_0), \tag{3.4}$$

$$L^* = \tfrac{1}{2}\Delta + \nabla \cdot (\nabla V \cdot). \tag{3.5}$$

These bounds will follow by writing (3.5) in essentially divergence form,

$$L^* = \tfrac{1}{2} \nabla \cdot (e^{-2V} \nabla e^{2V}) \tag{3.6}$$

so that $y = e^{2V}u$ satisfies

$$\frac{\partial y}{\partial t} = Ly, \qquad y(\mathbf{x},0) = e^{2V}\delta(\mathbf{x} - \mathbf{x}_0), \tag{3.7}$$

$$L = \tfrac{1}{2}e^{2V}\nabla \cdot (e^{-2V}\nabla). \tag{3.8}$$

The Nash–Fabes–Stroock argument will then go through easily for L in (3.8) as well as for strictly divergence form operators. In fact, the estimates (3.3) can be easily shown to hold for a simple generalization of (3.6) and (3.8), as we see in

THEOREM 3.1. *Let $a(\mathbf{x})$ and $b(\mathbf{x})$ be positive functions on $\mathbb{R}^d$ which are smooth and uniformly elliptic, i.e., bounded away from infinity and zero. Also, let $u(\mathbf{x}, t)$ be the fundamental solution to*

$$\frac{\partial u}{\partial t} = \nabla \cdot (a\nabla bu) \tag{3.9}$$

or

$$\frac{\partial u}{\partial t} = b\nabla \cdot (a\nabla u), \tag{3.10}$$

$$u(\mathbf{x},0) = \delta(\mathbf{x} - \mathbf{x}_0).$$

Then the estimates (3.3) *hold for u, where C depends only on d and the maxima and minima of a and b.*

PROOF. It suffices to consider only (3.10) since $u \mapsto bu$ converts (3.9) to (3.10).

The proof in the paper of Fabes and Stroock of the estimates (3.3) for solution of (3.1) begins through analysis of the form

$$\left(A_\psi f, f^{2p-1}\right) \equiv \int_{\mathbb{R}^d}\left(A_\psi f(\mathbf{x})\right) f^{2p-1}(\mathbf{x})\, d\mathbf{x}, \tag{3.11}$$

$$A_\psi f = e^{-\psi}\nabla \cdot \left(a\nabla\left(e^{\psi}f\right)\right), \tag{3.12}$$

with $\psi(\mathbf{x}) = \boldsymbol{\alpha} \cdot \mathbf{x}$, $\boldsymbol{\alpha} \in \mathbb{R}^d$ and f a positive function from the Schwartz test function space. For equation (3.11) we define, analogously to (3.13),

$$B_\psi f = bA_\psi f \tag{3.13}$$

and then

$$\left(B_\psi f, f^{2p-1}\right)_b = \int_{\mathbb{R}^d}\left(B_\psi f(\mathbf{x})\right) f^{2p-1}(\mathbf{x})\left(b^{-1}\, d\mathbf{x}\right) \tag{3.14}$$

$$= \left(A_\psi f, f^{2p-1}\right). \tag{3.15}$$

The proof of (3.3) proceeding from (3.11) and (3.12) involves sequences of integral inequalities and relations among $L^p(\mathbb{R}^d, d\mathbf{x})$ norms of relevant functions. That this same proof holds for (3.13) and (3.14) is easy to check using the fact that there are positive constants C_1 and C_2 such that (for $f \neq 0$)

$$C_1 \leq \|f\|_{p,b}/\|f\|_p \leq C_2, \tag{3.16}$$

where

$$\|f\|_p = \left(\int_{\mathbb{R}^d} |f|^p \, d\mathbf{x}\right)^{1/p}, \qquad \|f\|_{p,b} = \left(\int_{\mathbb{R}^d} |f|^p b^{-1} \, d\mathbf{x}\right)^{1/p}. \tag{3.17}$$

□

COROLLARY 3.1. *Let $V(x)$ be smooth and bounded in $\mathbb{R}^d$ and let $u(\mathbf{x}, t)$ be the fundamental solution to*

$$\frac{\partial u}{\partial t} = L^* u = \frac{1}{2}\Delta u + \nabla \cdot (\nabla V u), \qquad u(\mathbf{x}, 0) = \delta(\mathbf{x} - \mathbf{x}_0). \tag{3.18}$$

Then (3.3) *holds for u in* (3.18) *with C depending only on d and the maximum and minimum of V.*

(Presumably the upper bound in this corollary can be proven with the general methods developed in [3].)

It is interesting to see why (3.3) breaks down for

$$L^* = \tfrac{1}{2}\Delta - \nabla \cdot (\mathbf{b}\, \cdot) \tag{3.19}$$

for general bounded $\mathbf{b}$. Consider, for example, the case of uniform drift $b = 1$ in $\mathbb{R}^1$. The potential associated with $b = 1$, namely, $V = -x$ ($b = -dV/dx$), is unbounded and we lose the uniform ellipticity of e^{2V} and e^{-2V} which is required for the bounds (3.3), which clearly do not hold in this case.

Another approach which must therefore break down is the trick of Oleinik and Kruzhkov [16]. They write $L = \nabla \cdot \mathsf{a}\nabla + \mathbf{b} \cdot \nabla$ in $\mathbb{R}^d$ as a divergence form operator in $\mathbb{R}^{d+1}$:

$$\frac{\partial u}{\partial t} = \nabla \cdot (\mathsf{a}\nabla u) + \frac{\partial}{\partial y}(\mathbf{b}y) \cdot \nabla u + \nabla \cdot \left(\mathbf{b}y \frac{\partial u}{\partial y}\right) + \frac{\partial^2 u}{\partial y^2} \tag{3.20}$$

$$= \nabla' \cdot \left(\begin{array}{c|c} \mathsf{a} & \mathbf{b}y \\ \hline \mathbf{b}y & 1 \end{array}\right) \nabla' u = \nabla' \cdot \mathsf{a}' \nabla' u, \qquad \nabla' = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial y}\right). \tag{3.21}$$

In order for bounds like (3.3) to apply to (3.21), a' in (3.21) must be uniformly elliptic in $\mathbb{R}^{d+1}$. However, in, say $d = 1$, the two eigenvalues λ_1 and λ_2 of this matrix are

$$\begin{aligned} 2\lambda_1 &= (1 + a) + \sqrt{4b^2y^2 + (1 - a)^2}, \\ 2\lambda_2 &= (1 + a) - \sqrt{4b^2y^2 + (1 - a)^2}. \end{aligned} \tag{3.22}$$

For $|y| \leq \alpha < \sqrt{a}/b$, a' is uniformly elliptic, $0 < M^{-1} < \lambda_1 \leq \lambda_2 < M$, for some M. However, as y is allowed to vary all over $\mathbb{R}$, as must be the case to apply (3.3) in $\mathbb{R}^2$, λ_2 in (3.22) can become negative as well as unbounded or zero, so that we lose uniform ellipticity.

As an immediate consequence of Corollary 3.1 and (2.8), we obtain the behavior of the MSD for diffusion in a rapidly decaying potential.

COROLLARY 3.2. *For* $\mathbf{X}_t$ *in* (2.1) *with* $\mathbf{X}_0 = \mathbf{x}_0$ *and smooth bounded* V *satisfying* $\int_{\mathbb{R}^d} |\mathbf{x}| \cdot |\nabla V(\mathbf{x})| \, d\mathbf{x} < \infty$,

$$E\left[|\mathbf{X}_t - \mathbf{x}_0|^2\right] = \begin{cases} t + O(\sqrt{t}), & d = 1, \\ t2 + O(\log t), & d = 2, \\ td + C + O\left(\dfrac{1}{t^{d/2-1}}\right), & d \geq 3, \end{cases} \tag{3.23}$$

where

$$C = -2\int_0^\infty dt \int_{\mathbb{R}^d} d\mathbf{x}(\mathbf{x} - \mathbf{x}_0) \cdot \nabla V(\mathbf{x}) u(\mathbf{x}, t), \qquad |C| < \infty. \tag{3.24}$$

The lower bound in Corollary 3.1 indicates that the t-dependent corrections in (3.23) will generically "attain" the upper bounds indicated. For example, in one dimension, with $x_0 = 0$, if $dV/dx < 0$ for $x < 0$ and $dV/dx > 0$ for $x > 0$, then the coefficient of t in (3.23) is nonzero (and negative). One can also construct similar attractive wells in higher dimensions.

Upper bounds on the coefficients of the time-dependent corrections in (3.23) can be easily obtained by repeating Nash's proof [15] of $u(\mathbf{x}, t) \leq k/t^{d/2}$ for $\nabla \cdot \mathsf{a} \nabla$ in the case of $L^* = \frac{1}{2}\nabla \cdot (e^{-2V} \nabla e^{2V})$, keeping track of the constants involved, to obtain the upper bound

$$u(\mathbf{x}, t) \leq \left(\left(\frac{d}{c_d} \right)^{d/2} \exp(4 + d)(V_{\max} - V_{\min}) \right) \frac{1}{t^{d/2}}, \tag{3.25}$$

where $c_d = (4\pi d/(d+2))[(d/2)!/(1 + d/2)]^{2/d}$. Then with $E|\mathbf{X}_t - \mathbf{x}_0|^2 = td + C + C(t)$ as in Corollary 3.2,

$$|C(t)| \leq \begin{cases} 4\beta_{1,V}\sqrt{t}, & d = 1, \\ 2\beta_{2,V}\log t + M & d = 2, \\ (4\beta_{d,V}/(d-2))/t^{d/2-1}, & d \geq 3, \end{cases} \tag{3.26}$$

$$\beta_{d,V} = \int_{\mathbb{R}^d} |(\mathbf{x} - \mathbf{x}_0) \cdot \nabla V(\mathbf{x})| \, d\mathbf{x} \left(\left(\frac{d}{c_d} \right)^{d/2} \exp(4 + d)(V_{\max} - V_{\min}) \right), \tag{3.27}$$

where $M > 0$ and C is in Corollary 4.2 for $d \geq 3$ and is zero otherwise.

Laplace transform. We now give some consequences of the a priori estimates for the Laplace transform in time of the transition density in Corollary 3.1,

$$\tilde{u}(\mathbf{x}, s) = \int_0^\infty e^{-st} u(\mathbf{x}, t) \, dt, \qquad \operatorname{Re} s > 0, \tag{3.28}$$

which satisfies

$$L^*\tilde{u} - s\tilde{u} = -\delta(\mathbf{x} - \mathbf{x}_0), \tag{3.29}$$

as well as for the transform of the MSD. The large t behavior of these functions is reflected in the small s behavior of the transforms. We first give an immediate consequence of Corollary 3.1 which, however, holds only for $d = 1$.

COROLLARY 3.3. *Let $\tilde{u}(x, s)$ be the Laplace transform of $u(x, t)$ in Corollary 3.1 for $d = 1$. Then for each x, there exist positive constants a_1 and a_2 such that for sufficiently small $s > 0$,*

$$\frac{a_1}{\sqrt{s}} \le \tilde{u}(x, s) \le \frac{a_2}{\sqrt{s}}. \tag{3.30}$$

We now consider implications of the a priori estimates and the additional fact that $L^* = \frac{1}{2}\nabla \cdot (e^{-2V}\nabla e^{2V})$ in (3.29) is self-adjoint in $L^2(\mathbb{R}^d, e^{2V}\,d\mathbf{x})$ with spectrum in the negative real axis. Note first that it follows from the Nash estimates that for $d \le 3$, $\tilde{u}(\cdot, s)$ is an $L^2(\mathbb{R}^d, e^{2V}\,d\mathbf{x})$-valued solution to (3.29). The resolvent structure of (3.29),

$$\tilde{u} = -(L^* - s)^{-1}\delta(\mathbf{x} - \mathbf{x}_0), \tag{3.31}$$

and the self-adjointness of L^* imply, as we shall see, that $\tilde{u}$, as an $L^2(\mathbb{R}^d, e^{2V}\,d\mathbf{x})$-valued function on the complex s-plane, is holomorphic off the negative real axis $(-\infty, 0]$ for $d \le 3$. [This would be trivial if $\delta(\mathbf{x} - \mathbf{x}_0)$ were in $L^2(\mathbb{R}^d, e^{2V}\,d\mathbf{x})$, but, of course, it is not.] Thus it is natural to ask if the asymptotic information in Corollary 3.3 for small positive real s can be continued into the complex s-plane away from the negative real axis. It turns out that for $d \le 3$ we can easily obtain upper bounds on the $L^2(\mathbb{R}^d, e^{2V}\,dx)$ norm of $\tilde{u}(\mathbf{x}, s)$ in the cut plane (and not just for small s).

COROLLARY 3.4. *Let L^* be as in Corollary 4.1 with $d \le 3$ and let $\mathbf{x}_0 \in \mathbb{R}^d$. For each $s \notin (-\infty, 0]$, the equation*

$$(L^* - s)\tilde{u} = -\delta(\mathbf{x} - \mathbf{x}_0) \tag{3.32}$$

has a unique $L^2(\mathbb{R}^d, e^{2V}\,d\mathbf{x})$ solution $\tilde{u}(\cdot, s)$. As an $L^2(\mathbb{R}^d, e^{2V}\,d\mathbf{x})$-valued function on $\mathbb{C} - (-\infty, 0]$, $\tilde{u}$ is holomorphic. Moreover, for any $\varepsilon > 0$, there exists a $C > 0$ such that for any s in the region $|\arg s| \le \pi - \varepsilon$,

$$\|\tilde{u}(\cdot, s)\|_{L^2(\mathbb{R}^d, e^{2V}\,d\mathbf{x})} \le \frac{C}{|s|^{1-d/4}}. \tag{3.33}$$

PROOF. Let $R_s = (L^* - s)^{-1}$ and $R_\lambda = (L^* - \lambda)^{-1}$, where λ and s are complex. The resolvent identity

$$R_s - R_\lambda = (s - \lambda)R_\lambda R_s$$

can be written as

$$R_s = \big(I(s - \lambda)R_\lambda\big)^{-1}R_\lambda, \tag{3.34}$$

where I is the identity. They key observation is that for $\lambda = |s|$,

$$\left\|(I - (s - \lambda)R_\lambda)^{-1}\right\| \leq 1/\sin(\varepsilon/2),$$

where $\|\cdot\|$ here denotes operator norm in $L^2(\mathbb{R}^d, e^{2V}\,d\mathbf{x})$, so that $\|\tilde{u}(\cdot, s)\|_{L^2} \leq \|\tilde{u}(\cdot, |s|)\|_{L^2}/\sin(\varepsilon/2)$. Corollary 3.1 yields

$$\|\tilde{u}(\mathbf{x}, |s|)\|_{L^2(\mathbb{R}^d, d\mathbf{x})} \leq C\int_0^\infty \frac{e^{-|s|t}}{t^{d/2}}\left\|\exp\left(\frac{-|\mathbf{x} - \mathbf{x}_0|^2}{Ct}\right)\right\|_{L^2(\mathbb{R}^d, d\mathbf{x})} \tag{3.35}$$

$$= C\left(\frac{\pi C}{2}\right)^{d/4}\int_0^\infty e^{-|s|t}/t^{d/4}\,dt \tag{3.36}$$

$$= \frac{B}{|s|^{1-d/4}}, \tag{3.37}$$

where $B > 0$, which yields (3.33). Moreover, since by (3.34),

$$\begin{aligned}\tilde{u}(\cdot, s) &= (I - (s - \lambda)R_\lambda)^{-1}\tilde{u}(\cdot, \lambda)\\ &= \left(\sum_{n=0} [(s - \lambda)R_\lambda]^n\right)\tilde{u}(\cdot, \lambda),\end{aligned} \tag{3.38}$$

where the series is operator norm convergent for $|s - \lambda| \leq |\lambda|$, analyticity follows easily, as usual. Finally uniqueness follows from the fact that the spectrum of L^* on $L^2(\mathbb{R}^d, e^{2V}\,d\mathbf{x})$ is contained in $(-\infty, 0]$. □

The preceding argument can presumably be refined for $d = 1$ to give a pointwise bound on $\tilde{u}$ with asymptotic behavior $O(|s|^{-1/2})$ in the given region. Note that for large s, $\|\tilde{u}(\cdot, s)\|_{L^2}$ decays more slowly than would be allowed if $\delta(\mathbf{x} - \mathbf{x}_0)$ were in L^2, i.e., $O(1/|s|)$, while for small s it grows more slowly than could be expected a priori, which is a reflection of the Nash estimates (Corollary 3.1). We remark that for $d \geq 4$, $\tilde{u}(\cdot, s) \notin L^2$ even for $s > 0$ since it diverges at $\mathbf{x}_0$ too rapidly.

We close this section with a useful fact about $\tilde{E}(s)$, the Laplace transform of the MSD in $\mathbb{R}^d$, $d \leq 3$, namely, that $\tilde{E}(s) \to 0$ as $s \to \infty$ along any path outside of a wedge containing $(-\infty, 0]$. This fact will play an essential role in the inversion of $\tilde{E}(s)$ to obtain an asymptotic series in [12] for the MSD in a periodic potential with a bump.

It follows from Corollary (3.1) that for $\operatorname{Re} s > 0$,

$$\tilde{E}(s) = \int_{\mathbb{R}^d} |\mathbf{x} - \mathbf{x}_0|^2 \tilde{u}(\mathbf{x}, s)\,d\mathbf{x}. \tag{3.39}$$

Note that it is not a priori clear that $\tilde{E}(s)$ can be analytically continued into the left half plane. It is easy to see, though, that $\tilde{E}(s) \to 0$ as $s \to \infty$ in the region $|\arg s| \leq \pi/2 - \varepsilon$, but extending this information, as well as (3.39), to the left half plane where (3.28) is no longer valid, requires some work. As with Corollary 3.4 we employ the resolvent identity to accomplish the extension.

THEOREM 3.2. *The Laplace transform of the MSD, for a fixed starting point* $\mathbf{x}_0$ *with* $u(\mathbf{x}, t)$ *as in Corollary* 3.1 *for* $d \le 3$, *can be analytically continued into* $\mathbb{C} - (-\infty, 0]$, *where it is given by* (3.39). *Furthermore, for any* $\varepsilon \to 0$, *as* $s \to \infty$ *in the region* $|\arg s| \le \pi - \varepsilon$,

$$\tilde{E}(s) \to 0. \tag{3.40}$$

PROOF. We give the details for $d = 1$. Without loss of generality we take $x_0 = 0$. We first obtain a bound on $|\tilde{u}(x, \lambda)|$ in the right half plane, where (3.28) holds. With $\lambda = a + ib$,

$$\begin{aligned} |\tilde{u}(x, \lambda)| &= \left| \int_0^\infty e^{-\lambda t} u(x, t)\, dt \right| \\ &\le \int_0^\infty e^{-at} u(x, t)\, dt, \end{aligned} \tag{3.41}$$

where $u(x, t)$ is the fundamental solution of (3.18). From the upper estimate in (3.4) applied to $u(x, t)$ and explicit computation of the Laplace transform of the Gaussian, there is a constant $C > 0$ such that

$$|\tilde{u}(x, \lambda)| \le \frac{C}{\sqrt{a}} e^{-\sqrt{a}|x|/C}. \tag{3.42}$$

Note that (3.42) by itself implies that, for any $\delta > 0$, $\tilde{E}(s) \to 0$ as $s \to \infty$ in the region $|\arg s| \le \pi/2 - \delta$. The idea of the proof is to use the resolvent identity to relate $\tilde{u}(x, s)$ for s in the left half plane to $\tilde{u}(x, \lambda)$ for $|\arg \lambda| \le \pi/2 - \delta$. We will then let s and λ both go to ∞. With $R_s = (L^* - s)^{-1}$ and $R_\lambda = (L^* - \lambda)^{-1}$, the resolvent identity gives

$$\tilde{u}(x, s) = -R_s \delta(x) = -(I - (s - \lambda) R_\lambda)^{-1} R_\lambda \delta(x). \tag{3.43}$$

It will be useful to expand $(I - (s - \lambda)R_\lambda)^{-1}$ in a Neumann series. Since $\|R_\lambda\| \le 1/|\lambda|$, it will be necessary to have $|s - \lambda| < |\lambda|$. For $\pi/3 < \varepsilon < \pi/2$, we may choose λ such that $\arg \lambda = \pi/3$ and $\operatorname{Im} \lambda = \operatorname{Im} s$, which satisfies $|s - \lambda| < |\lambda|$. For $0 < \varepsilon \le \pi/3$, λ can be chosen so that $|s - \lambda| < |\lambda|$ and the proof still goes through, but for simplicity we give the detailed proof only for $\pi/3 < \varepsilon < \pi/2$.

We will use the Neumann series to establish the following estimate, from which the theorem follows easily.

LEMMA 3.1. *Let* $I_r(x)$ *be the indicator function of* $|x| > r$, *where* $r \in \mathbb{Z}^+$. *Then for* $|s| > \varepsilon$ *and* $|\arg s| \le \pi - \varepsilon$,

$$\|I_r \tilde{u}(x, s)\|_{L^2(\mathbb{R}, dx)} \le \frac{g(r)}{|s|^{3/4}}, \tag{3.44}$$

where

$$\sum_{r=0}^{\infty} (r + 1)^2 g(r) < \infty. \tag{3.45}$$

Before proving this lemma, we indicate how it implies the corollary. Note first that

$$\int_{-\infty}^{\infty} x^2|\tilde{u}(x,s)|\,dx = \sum_{r=0}^{\infty}\int_{r<|x|<r+1} x^2|\tilde{u}(x,s)|\,dx. \tag{3.46}$$

Using Schwarz's inequality on the integral on the right-hand side in (3.46) and the fact that

$$\int_{r<|x|<r+1} |\tilde{u}(x,s)|^2\,dx \le \int_{r<|x|} |\tilde{u}(x,s)|^2\,dx, \tag{3.47}$$

we have

$$\int_{-\infty}^{\infty} x^2|\tilde{u}(x,s)|\,dx \le \sum_{r=0}^{\infty} (r+1)^2\|I_r\tilde{u}(x,s)\|_{L^2(\mathbb{R},\,dx)} \tag{3.48}$$

$$\le \frac{1}{|s|^{3/4}}\sum_{r=0}^{\infty}(r+1)^2 g(r) \tag{3.49}$$

$$\to 0 \quad \text{as } s\to\infty,\ |\arg s| \le \pi-\varepsilon. \tag{3.50}$$

Moreover, since

$$\int_{-\infty}^{\infty} x^2\tilde{u}(x,s)\,dx = \sum_{r=0}^{\infty}\int_{r<|x|<r+1} x^2\tilde{u}(x,s)\,dx \tag{3.51}$$

is, in view of the previous corollary, a sum of analytic functions on $\mathbb{C} - (-\infty, 0]$, and since

$$\left|\int_{r<|x|<r+1} x^2\tilde{u}(x,s)\,dx\right| \le \frac{(r+1)^2 g(r)}{|s|^{3/4}}, \tag{3.52}$$

it follows from (3.45) that the left-hand side of (3.51) is analytic off the negative real axis. Hence it defines $\tilde{E}(s)$ there. Then (3.40) follows from (3.50).

Proof of Lemma 3.1 (for $\pi/3 < \varepsilon < \pi/2$). Clearly it suffices to establish the lemma for $L^2(\mathbb{R}, e^{2V}dx)$, where we have the self-adjointness of L^*. We expand $(I-(s-\lambda)R_\lambda)^{-1}$ in (3.43) in a Neumann series,

$$R_s = \left(I + (s-\lambda)R_\lambda + (s-\lambda)^2R_\lambda^2 + \cdots\right)R_\lambda \tag{3.53}$$

$$= \sum_{j=0}^{\infty} A^jR_\lambda \tag{3.54}$$

$$= \sum_{j=0}^{n-1} A^jR_\lambda + (I-A)^{-1}A^nR_\lambda, \tag{3.55}$$

where $A = (s-\lambda)R_\lambda$. Note that A is a convolution operator, i.e., with $\psi(x) = -(s-\lambda)\tilde{u}(x,\lambda)$,

$$(A\phi)(x) = \psi * \phi = \int_{-\infty}^{\infty}\psi(x-y,s)\phi(y)\,dy, \tag{3.56}$$

and that $\|A\| < 1$. The key observation is the following. For $\phi \in L^2(\mathbb{R}, e^{2V}\,dx)$,

$$\|I_{r_1+\sigma_1}A\phi\|_{L^2(\mathbb{R}, e^{2V}dx)} \le \|I_{r_1}\phi\|_{L^2(\mathbb{R}, e^{2V}dx)} + e^{2(V_{\max}-V_{\min})}\|I_{\sigma_1}\psi\|_{L^1(\mathbb{R}, dx)}\|\phi\|_{L^2(\mathbb{R}, e^{2V}dx)}. \tag{3.57}$$

This follows easily from the fact that

$$I_{r_1+\sigma_1}A\phi = I_{r_1+\sigma_1}AI_{r_1}\phi + I_{r_1+\sigma_1}(I_{\sigma_1}\psi) * ((1-I_{r_1})\phi). \tag{3.58}$$

By (3.42), there exists a $\gamma > 0$ and an $r_0 > 0$ such that for $|s| > \varepsilon$ and $r \ge r_0$,

$$e^{2(V_{\max}-V_{\min})}\|I_r\psi\|_{L^1(\mathbb{R}, dx)} \le e^{-\gamma r}, \tag{3.59}$$

$$\|I_r\tilde{u}(\cdot, \lambda)\|_{L^2(\mathbb{R}, e^{2V}dx)} \le e^{-\gamma r}, \tag{3.60}$$

$$\|A\| \le e^{-\gamma}. \tag{3.61}$$

Suppose $r > r_0^2$. Then using (3.55) with $n = [\sqrt{r}]$, by induction on (3.57) with $\sigma_1 = \sqrt{r}$ starting on $r_1 = \sqrt{r}$ and $\phi = \tilde{u}(\cdot, \lambda)$, we obtain

$$\|I_{(j+1)\sqrt{r}}A^j\tilde{u}\|_{L^2(\mathbb{R}, e^{2V}dx)} \le (j+1)e^{-\gamma\sqrt{r}}\|\tilde{u}\|_{L^2(\mathbb{R}, e^{2V}dx)}. \tag{3.62}$$

Hence for $r > r_0^2$ and $|s| > \varepsilon$,

$$\left\|\sum_{j=0}^{n-1} A^j\tilde{u}(\cdot, \lambda)\right\|_{L^2(\mathbb{R}, e^{2V}dx)} \le re^{-\gamma\sqrt{r}}\|\tilde{u}\|_{L^2(\mathbb{R}, e^{2V}dx)}, \tag{3.63}$$

and for the remainder term,

$$\left\|(I-A)^{-1}A^n\tilde{u}\right\|_{L^2(\mathbb{R}, e^{2V}dx)} \le \frac{1}{1-\|A\|}e^{-\gamma[\sqrt{r}]}\|\tilde{u}\|_{L^2(\mathbb{R}, e^{2V}dx)}, \tag{3.64}$$

so that using (3.55) and Corollary 3.5,

$$\|I_r\tilde{u}(\cdot, s)\|_{L^2(\mathbb{R}, e^{2V}dx)} \le \frac{Cre^{-\gamma\sqrt{r}}}{|s|^{3/4}}, \tag{3.65}$$

where C is a constant independent of s and r. The lemma follows easily from (3.65) □

For $d = 2$ and 3, where we no longer have the explicit formula (3.42) the key equations (3.59) and (3.60) nonetheless follow easily from Corollary 3.1 by writing

$$\int_0^\infty e^{-at}u(\mathbf{x}, t)\,dt = \int_0^{r/\sqrt{a}} e^{-at}u(\mathbf{x}, t)\,dt + \int_{r/\sqrt{a}}^\infty e^{-at}u(\mathbf{x}, t)\,dt. \tag{3.66}$$

We remark that the property of the MSD described in Theorem 3.2 is not generic for functions of t that are asymptotically linear for t large and small. In fact, the addition of an arbitrarily small term which decays, for example, like $1/t^2$ for large t, destroys the property.

4. Convergence to Brownian motion. As mentioned in the Introduction, for stationary random ergodic $V(\mathbf{x}, \omega)$, the diffusion process $\mathbf{X}_t$ obeying (2.1) behaves asymptotically like Brownian motion $W_t(\mathsf{D})$ with positive definite effective diffusion tensor $\mathsf{D} \equiv \mathsf{D}(V)$, which does not depend on the particular sample of the random potential, i.e., D is independent of ω [as an $L^2(\Omega, P)$ function]. More precisely, with $\mathbf{X}_t^\varepsilon = \varepsilon \mathbf{X}_{t/\varepsilon^2}$, $(\mathbf{X}_t^\varepsilon)_{t\geq 0}$ converges weakly "in ρ-measure" as $\varepsilon \to 0$ to $\mathbf{W}_t(\mathsf{D})$, $(\mathbf{X}_t^\varepsilon)_{t\geq 0} \Rightarrow \mathbf{W}_t(\mathsf{D})$; that is, for any $T > 0$ and any bounded continuous function F on $C[0, T]$, the space of $\mathbb{R}^d$-valued continuous functions on $[0, T]$,

$$E\big(F((\mathbf{X}_t^\varepsilon)_{0\leq t\leq T})\big) \underset{\varepsilon\to 0}{\longrightarrow} E\big(F((\mathbf{W}_t(\mathsf{D}))_{0\leq t\leq T})\big) \tag{4.1}$$

in ρ-measure.

Now consider the diffusion process $\mathbf{Z}_t$ in $\mathbb{R}^d$ governed by the stochastic differential equation

$$d\mathbf{Z}_t = -\nabla\big(V(\mathbf{Z}_t, \omega) + B(\mathbf{Z}_t)\big)\, dt + d\mathbf{W}_t, \tag{4.2}$$

where V is stationary random ergodic and B is a "bump," i.e., a thrice continuously differentiable function of compact support. The forward and backward generators of (4.2) are

$$L_B = \tfrac{1}{2}\Delta - \nabla(V + B)\cdot\nabla, \tag{4.3}$$

$$L_B^* = \tfrac{1}{2}\Delta + \nabla\cdot\nabla(V + B). \tag{4.4}$$

We prove that the perturbed process $\mathbf{Z}_t$ has the same asymptotic behavior as the unperturbed process $\mathbf{X}_t$, which is stated as

THEOREM 4.1. *For $\mathbf{Z}_t$ in $\mathbb{R}^d$ obeying* (4.2),

$$\mathbf{Z}_t^\varepsilon \underset{\varepsilon\to 0}{\Longrightarrow} \mathbf{W}_t(\mathsf{D}), \tag{4.5}$$

with D as in (4.1), *where $\mathbf{Z}_t^\varepsilon = \varepsilon \mathbf{Z}_{t/\varepsilon^2}$.*

We prove Theorem 4.1 first in one dimension and then using a different method in higher dimensions.

PROOF. For $d = 1$, we prove the stronger result that we have weak convergence for almost every (in ρ) potential V. Consider the function h that is harmonic with respect to L_B, i.e.,

$$L_B h = 0. \tag{4.6}$$

For $d = 1$, (4.6) can be solved with

$$\frac{dh}{dx} = \exp\big(2(V(x) + B(x))\big). \tag{4.7}$$

We shall exploit the fact that $h(Z_t)$ is a martingale. An equation for $h(Z_t)$ can be found using Itô's formula,

$$dh(Z_t) = h'\, dZ_t + \tfrac{1}{2}h''(dZ_t)^2, \tag{4.8}$$

which becomes after standard manipulations in stochastic calculus,

$$dh(Z_t) = (L_B h)\, dt + h'\, dW_t \tag{4.9}$$

or

$$dh(Z_t) = \exp(2(V(Z_t) + B(Z_t)))\, dW_t, \tag{4.10}$$

so that with $h(Z_0) = 0$,

$$h_t = h(Z_t) = \int_0^t \exp(2(V(Z_s) + B(Z_s)))\, dW_s. \tag{4.11}$$

The idea of the proof is to write Z_t as

$$Z_t = \frac{h(Z_t)}{h(Z_t)/Z_t}, \tag{4.12}$$

which after scaling becomes

$$\varepsilon Z_{t/\varepsilon^2} = \frac{\varepsilon h(Z_{t/\varepsilon^2})}{h(Z_{t/\varepsilon^2})/Z_{t/\varepsilon^2}}. \tag{4.13}$$

Then the result will follow essentially from the observations that (i) $\varepsilon h(Z_{t/\varepsilon^2})$ converges weakly to Brownian motion as $\varepsilon \to 0$ and (ii) $h(Z_{t/\varepsilon^2})/Z_{t/\varepsilon^2}$ is arbitrarily close to a constant for large enough $|Z_{t/\varepsilon^2}|$, where the Brownian motion in (i) and the constant in (ii) are the same as for the unperturbed process X_t.

Observation (i) follows from a martingale central limit theorem [13, Theorem 5.1] and we need only show that $h(Z_t)$ satisfies the hypotheses of this theorem. In particular, we must show that the variance process A_t of $h(Z_t)$ converges in distribution (under P), for any t, upon scaling to $\sigma^2 t$, where σ^2 will be the diffusion constant of the limiting Brownian motion of $h(Z_t)$. For $h(Z_t)$ in (4.11), the variance process is

$$A_t = \int_0^t (dh_s)^2. \tag{4.14}$$

Under scaling of $h_t \to \varepsilon h_{t/\varepsilon^2}$,

$$A_t^\varepsilon = \varepsilon^2 \int_0^{t/\varepsilon^2} (dh_s)^2. \tag{4.15}$$

By identifying $(dW_s)^2$ with ds, we have from (4.10),

$$A_t^\varepsilon = \varepsilon^2 \int_0^{t/\varepsilon^2} \exp(4(V(Z_s) + B(Z_s)))\, ds. \tag{4.16}$$

Under the change of variables $\tau = t/\varepsilon^2$,

$$A_t^\varepsilon = t\frac{1}{\tau} \int_0^{\tau} \exp(4(V(Z_s) + B(Z_s)))\, ds. \tag{4.17}$$

The expression corresponding to (4.17) for the unperturbed process X_t is

$$A_t^\varepsilon = t\frac{1}{\tau} \int_0^{\tau} e^{4V(X_s)}\, ds. \tag{4.18}$$

A_t^ε can be expressed in terms of the environment process $V_t(\cdot) = V(X_t + \cdot\,)$, the potential field seen by the particle at time t, through $V(X_s) = V_s(0)$. The measure $d\mu = e^{-2V}\, d\rho / \int_\Omega e^{-2V}\, d\rho$ is stationary for the environment process. By the ergodicity under spatial translations of ρ, μ is also ergodic for the environment process. Thus by the ergodic theorem and Fubini's theorem for ρ-a.e. potential V,

$$A_t^\varepsilon \underset{\varepsilon\to 0}{\Longrightarrow} \sigma^2 t, \quad P\text{-a.e.,} \tag{4.19}$$

$$\sigma^2 = \int_\Omega e^{2V}\,d\rho \Big/ \int_\Omega e^{-2V}\,d\rho. \tag{4.20}$$

By the a priori estimates on the transition density of Z_t discussed in Section 3, most paths (in P measure) spend only a total time of order $\sqrt{\tau}$ in the support of B. Then it is not hard to see that we still obtain (4.19) and (4.20) when the perturbed process Z_t is put into (4.18), and it is easy to see that we still have (4.19) and (4.20) for (4.17).

To see observation (ii), write

$$\frac{h(Z_{t/\varepsilon^2})}{Z_{t/\varepsilon^2}} = \frac{1}{Z_{t/\varepsilon^2}} \int_0^{Z_{t/\varepsilon^2}} \exp(2(V(x) + B(x)))\,dx. \tag{4.21}$$

Then (ii) follows immediately since

$$\begin{aligned} \lim_{|z|\to\infty} \frac{h(z)}{z} &= \lim_{|z|\to\infty} \frac{1}{z}\int_0^z \exp(2(V(x) + B(x)))\,dx \\ &= \lim_{|z|\to\infty} \frac{1}{z}\int_0^z e^{2V(x)}\,dx = \int_\Omega e^{2V}\,d\rho, \end{aligned} \tag{4.22}$$

by the ergodicity of ρ under translations. Writing

$$\int_\Omega e^{2V}\,d\rho = \alpha, \tag{4.23}$$

convergence of Z_t^ε to Brownian motion now follows from

$$\left[Z_t^\varepsilon - \frac{\varepsilon}{\alpha} h(Z_{t/\varepsilon^2})\right] \underset{\varepsilon\to 0}{\Longrightarrow} 0. \tag{4.24}$$

(Here, $\Rightarrow$ means convergence in distribution for ρ-a.e. V.) Statement (4.24) follows by writing

$$\left|\varepsilon Z_{t/\varepsilon^2} - \frac{\varepsilon}{\alpha} h(Z_{t/\varepsilon^2})\right| = \left|\left(\frac{Z_{t/\varepsilon^2}}{h(Z_{t/\varepsilon^2})} - \frac{1}{\alpha}\right)\varepsilon h(Z_{t/\varepsilon^2})\right|, \tag{4.25}$$

and noting that $\varepsilon h(Z_{t/\varepsilon^2}) \Rightarrow W_t(\sigma^2)$, with σ^2 in (4.20), and that (4.22) holds, since $z/h(z)$ is bounded and if z is not large, $\varepsilon h(z)$ is small.

Note the diffusion constant of the limiting Brownian motion can be computed via (4.20) and (4.23), which yield

$$D = \frac{\int_\Omega e^{2V}\,d\rho}{\int_\Omega e^{-2V}\,d\rho\left(\int_\Omega e^{2V}\,d\rho\right)^2} \tag{4.26}$$

or

$$D = \frac{1}{\int_\Omega e^{-2V}\,d\rho \int_\Omega e^{2V}\,d\rho}. \tag{4.27}$$

We also remark that our proof of (4.5) has been a repeat of one proof of (4.1), with the additional burden of showing that the bump has no net effect on the result.

Now consider $d \geq 2$. Without loss of generality we assume that the support of B is contained in the unit ball $S(1)$ of $\mathbb{R}^d$. Under the scaling $\mathbf{Z}_t \to \mathbf{Z}_t^\varepsilon$, the perturbation region lies inside $S(\varepsilon) = \varepsilon S(1)$. Fix $\tau > 0$ and $T > 0$, with $\tau \ll T$. The idea of the proof is to show that the initial piece of the path $\mathbf{Z}_t^\varepsilon$ up to time τ is insignificant (as $\tau \to 0$), and that after time τ the particle "sees" $S(\varepsilon)$ only with a probability that vanishes as $\varepsilon \to 0$. Then the invariance principle for $\mathbf{X}_t^\varepsilon$ will yield the result. More precisely, we show the following three facts:

(i) $\lim_{\tau \to 0} \sup_{0 \leq t \leq \tau} |\mathbf{Z}_t^\varepsilon| = 0$ in P-probability, uniformly in ε and ω.
(ii) $P[\mathbf{Z}_t^\varepsilon$ hits $S(\varepsilon)$ in $[\tau, T]] \to 0$ (in ρ-measure) as $\varepsilon \to 0$.
(iii) $\hat{\mathbf{Z}}_t^\varepsilon \equiv \mathbf{Z}_{t+\tau}^\varepsilon - \mathbf{Z}_\tau^\varepsilon \Rightarrow \mathbf{W}_t(\mathsf{D})$ (in ρ-measure) as $\varepsilon \to 0$.

To show (i), we first observe that as a consequence of the Nash estimates (in particular, Corollary 3.1),

$$\sup_{\varepsilon, \omega, t} P\left[\frac{|\mathbf{Z}_t^\varepsilon|}{\sqrt{t}} > \lambda\right] = \xi(\lambda) \xrightarrow[\lambda \to \infty]{} 0, \tag{4.28}$$

since these estimates are invariant under the scaling $\mathbf{x} \to \varepsilon \mathbf{x}$ and $t \to \varepsilon^2 t$, and are uniform in $\omega \in \Omega$. Then we may write

$$\begin{aligned} P\left[\sup_{0 \leq t \leq \tau} |\mathbf{Z}_t^\varepsilon| > \eta\right] &= P\left[\left\{\sup_{0 \leq t \leq \tau} |\mathbf{Z}_t^\varepsilon| > \eta\right\} \cap \left\{|\mathbf{Z}_\tau^\varepsilon| > \eta/2\right\}\right] \\ &\quad + P\left[\left\{\sup_{0 \leq t \leq \tau} |\mathbf{Z}_t^\varepsilon| > \eta\right\} \cap \left\{|\mathbf{Z}_\tau^\varepsilon| \leq \eta/2\right\}\right] \end{aligned} \tag{4.29}$$

$$\leq \xi\left(\eta/2\sqrt{\tau}\right) + \xi\left(\eta/2\sqrt{\tau}\right). \tag{4.30}$$

The estimate in (4.30) for the second term in (4.29) follows by noting that for the corresponding event to occur, $|\mathbf{Z}_t^\varepsilon|$ must reach η, for the first time at, say t^*, and then in time $\tau - t^*$, must jump by at least $\eta/2$. The probability for this to happen is estimated using the same version of the strong Markov property used to express the distribution of the sup up to time t for Brownian motion in terms of the distribution at time t. Since the right-hand side of (4.30) vanishes as $\tau \to 0$, (i) is proven.

To prove (ii), we let $A_{\tau,T}^X(\varepsilon)$ be the event that $\mathbf{X}_t^\varepsilon$ hits $S(\varepsilon)$ in $[\tau, T]$ and $A_{\tau,T}^Z(\varepsilon)$ be the event that $\mathbf{Z}_t^\varepsilon$ hits $S(\varepsilon)$ in $[\tau, T]$. We will first prove that $P[A_{\tau,T}^X(\varepsilon)] \to_{\varepsilon \to 0} 0$ in ρ-measure and then estimate $P[A_{\tau,T}^Z(\varepsilon)]$ in terms of $P[A_{\tau,T}^X(\varepsilon)]$ to obtain (ii). Now, let $0 < \varepsilon < \varepsilon' < \varepsilon''$ and let $I(\mathbf{x})$ be 1 for $|\mathbf{x}| \leq \varepsilon'$, 0 for $|\mathbf{x}| > \varepsilon''$, and continuously interpolate between 1 and 0 as $|\mathbf{x}|$ goes from ε' to ε''. We use I to define the expression

$$P\left[A_{\tau,T}^X(\varepsilon', \varepsilon'')\right] \equiv E\left[I\left(\inf_{t \in [\tau, T]} |\mathbf{X}_t^\varepsilon|\right)\right]. \tag{4.31}$$

Then

$$P\left[A_{\tau,T}^X(\varepsilon)\right] \leq P\left[A_{\tau,T}^X(\varepsilon', \varepsilon'')\right]. \tag{4.32}$$

Let $A_{\tau,T}^W(\varepsilon)$ and $P[A_{\tau,T}^W(\varepsilon', \varepsilon'')]$ be defined for $\mathbf{W}_t$ similarly to $A_{\tau,T}^X(\varepsilon)$ and

$P[A^X_{\tau,T}(\varepsilon', \varepsilon'')]$. We have via (4.32) and the fact that $\mathbf{X}^\varepsilon_t \Rightarrow_{\varepsilon\to 0} \mathbf{W}_t$,

$$\begin{aligned} \limsup_{\varepsilon\to 0} P\big[A^X_{\tau,T}(\varepsilon)\big] &\le \lim_{\varepsilon\to 0} P\big[A^X_{\tau,T}(\varepsilon', \varepsilon'')\big] \\ &= P\big[A^W_{\tau,T}(\varepsilon', \varepsilon'')\big] \\ &\le P\big[A^W_{\tau,T}(\varepsilon'')\big], \end{aligned} \tag{4.33}$$

where the limits above are in ρ-measure. But,

$$\lim_{\varepsilon''\to 0} P\big[A^W_{\tau,T}(\varepsilon'')\big] = P[W_t \text{ hits } 0 \text{ in } [\tau, T]] = 0, \tag{4.34}$$

which proves that $P[A^X_{\tau,T}(\varepsilon)] \to 0$. We remark that because $P[W_t$ hits 0 in $[\tau, T]] \neq 0$ in $d = 1$, the present proof does not work there.

To control $P[A^Z_{\tau,T}(\varepsilon)]$, we first write

$$P\big[A^X_{\tau,T}(\varepsilon)\big] = \int_{\mathbb{R}^d} \nu^{X,\varepsilon}_\tau(d\mathbf{x})\phi_\varepsilon(\mathbf{x}), \tag{4.35}$$

$$P\big[A^Z_{\tau,T}(\varepsilon)\big] = \int_{\mathbb{R}^d} \nu^{Z,\varepsilon}_\tau(d\mathbf{x})\phi_\varepsilon(\mathbf{x}), \tag{4.36}$$

where

$$\phi_\varepsilon(\mathbf{x}) = P\big[X^\varepsilon_t \text{ hits } S(\varepsilon) \text{ in } [0, T-\tau] | \mathbf{X}^\varepsilon_0 = \mathbf{x}\big], \tag{4.37}$$

and $\nu^{X,\varepsilon}_\tau(d\mathbf{x}) = u_\varepsilon(\mathbf{x}, \tau)\, d\mathbf{x}$ with u_ε the density for $\mathbf{X}^\varepsilon_\tau$, with a similar definition for $\mathbf{Z}^\varepsilon_\tau$. To obtain (ii), we bound $\nu^{Z,\varepsilon}_\tau(d\mathbf{x})$ above by a multiple of $\nu^{X,\varepsilon}_\tau(d\mathbf{x})$ as follows. The Nash estimates (Corollary 3.1) for u_ε involve a constant K_1 which depends on the max and min of V (but not on ε). Similar bounds hold for the density $v_\varepsilon(\mathbf{x}, \tau)$ for $\mathbf{Z}^\varepsilon_\tau$, with a constant K_2 that depends only on the max and min of $V + B$. For any compact $K \subset \mathbb{R}^d$, there exists a $C(K) > 0$ such that

$$\frac{K_2}{\tau^{d/2}} \exp\left(\frac{|\mathbf{x}|^2}{-\tau K_2}\right) \le \frac{C(K)}{K_1 \tau^{d/2}} \exp\left(-K_1 \frac{|\mathbf{x}|^2}{\tau}\right), \qquad \mathbf{x} \in K. \tag{4.38}$$

Consequently, for any compact $K \subset \mathbb{R}^d$,

$$\nu^{Z,\varepsilon}_\tau(d\mathbf{x}) \le C(K)\nu^{X,\varepsilon}_\tau(d\mathbf{x}), \qquad \mathbf{x} \in K, \tag{4.39}$$

and

$$\delta(k) \equiv \sup_{\varepsilon,\omega} \int_{\mathbb{R}^d - K} \nu^{Z,\varepsilon}_\tau(d\mathbf{x}) \to 0 \quad \text{as } K \to \mathbb{R}^d. \tag{4.40}$$

Then we have

$$P\big[A^Z_{\tau,T}(\varepsilon)\big] \le C(K) P\big[A^X_{\tau,T}(\varepsilon)\big] + \delta(K). \tag{4.41}$$

Taking $\varepsilon \to 0$ and then $K \to \mathbb{R}^d$ proves (ii).

Finally, we shall prove (iii), namely, $\hat{\mathbf{Z}}^\varepsilon_t \equiv \mathbf{Z}^\varepsilon_{t+\tau} - \mathbf{Z}^\varepsilon_\tau \Rightarrow_{\varepsilon\to 0} \mathbf{W}_t$ (in ρ-measure). Consider (iii′), $\hat{\mathbf{X}}^\varepsilon_t \equiv \mathbf{X}^\varepsilon_{t+\tau} - \mathbf{X}^\varepsilon_\tau \Rightarrow_{\varepsilon\to 0} \mathbf{W}_t$. The crucial difference between (iii) and (iii′) is that the distribution of $\mathbf{Z}^\varepsilon_\tau$ is generally different from that of $\mathbf{X}^\varepsilon_\tau$, so that $\hat{\mathbf{Z}}^\varepsilon_t$ and $\hat{\mathbf{X}}^\varepsilon_t$ start in different random environments, with distributions $\hat{\nu}^{Z,\varepsilon}_{\tau,\omega}$ and $\hat{\nu}^{X,\varepsilon}_{\tau,\omega}$ on Ω, respectively. These distributions on Ω are the images of $\nu^{Z,\varepsilon}_\tau$ and $\nu^{X,\varepsilon}_\tau$ under the map $G_\omega: \mathbb{R}^d \to \Omega$ defined by $G_\omega(\mathbf{x}) = \tau_{-\mathbf{x}}\omega$, where $\tau_\mathbf{x}$ is the translation group on Ω. In order to conclude (iii), we must show that after conditioning on the environment seen by $\mathbf{X}_t^\varepsilon$ at time τ, (iii′) still holds for $\hat{\nu}_{\tau,\omega}^{X,\varepsilon}$-a.e. environment. Then we must dominate $\hat{\nu}_{\tau,\omega}^{Z,\varepsilon}$ by $\hat{\nu}_{\tau,\omega}^{X,\varepsilon}$ in an appropriate way. But from (4.39) we have what is needed for the domination, namely,

$$\hat{\nu}_{\tau,\omega}^{Z,\varepsilon} \leq C(K)\hat{\nu}_{\tau,\omega}^{X,\varepsilon} + \hat{\alpha}_{\tau,\omega,K}^{Z,\varepsilon}. \tag{4.42}$$

In (4.42), $\hat{\alpha}_{\tau,\omega,K}^{Z,\varepsilon}$ is the image of $\nu_{\tau,\omega}^{Z,\varepsilon}$ restricted to K^c under G_ω so that

$$\sup_{\varepsilon,\omega} \int_\Omega \hat{\alpha}_{\tau,\omega,K}^{Z,\varepsilon}(d\hat{\omega}) = \delta(K) \to 0 \quad \text{as } K \to \mathbb{R}^d \tag{4.43}$$

[with $\delta(K)$ and $C(K)$ independent of ε and ω].

Now consider the process $\overline{\mathbf{Z}}_{t+\tau}^\varepsilon$, $t \geq 0$, with $\overline{\mathbf{Z}}_\tau^\varepsilon = \mathbf{Z}_\tau^\varepsilon$, which evolves like the $\mathbf{X}$-process, i.e., it ignores the bump. [Until $\mathbf{Z}_{t+\tau}^\varepsilon$ hits $S(\varepsilon)$, it agrees with $\mathbf{Z}_{t+\tau}^\varepsilon$.] Let $\hat{\mathbf{Z}}_t^\varepsilon = \overline{\mathbf{Z}}_{t+\tau}^\varepsilon - \overline{\mathbf{Z}}_\tau^\varepsilon$. It follows from (ii) that we are done once we show that $(\hat{\mathbf{Z}}_t^\varepsilon)_{0 \leq t < T-\tau} \Rightarrow_{\varepsilon \to 0} (\mathbf{W}_t)_{0 \leq t \leq T-\tau}$ in ρ-measure, i.e., given a bounded continuous function F on $C[0, T-\tau]$,

$$E_\omega\Big(F\Big(\big(\hat{\mathbf{Z}}_t^\varepsilon\big)_{0\leq t\leq T-\tau}\Big)\Big) \underset{\varepsilon\to 0}{\longrightarrow} E\big(F(\mathbf{W}_t)_{0\leq t<T-\tau}\big) \quad \text{in } \rho\text{-measure}, \tag{4.44}$$

where E_ω is the expectation starting in the environment ω. Now

$$E_\omega\Big(F\Big(\big(\hat{\mathbf{Z}}_t^\varepsilon\big)_{0\leq t< T-\tau}\Big)\Big) = \int_{\mathbb{R}^d} \nu_{\tau,\omega}^{Z,\varepsilon}(d\mathbf{x}) E_{\tau_{\mathbf{x}}\omega} F\big((\mathbf{X}_t^\varepsilon)_{0\leq t\leq T-\tau}\big) \tag{4.45}$$

$$= \int_\Omega \hat{\nu}_{\tau,\omega}^{Z,\varepsilon}(d\hat{\omega}) E_{\hat{\omega}} F\big((\mathbf{X}_t^\varepsilon)_{0\leq t\leq T-\tau}\big), \tag{4.46}$$

and it will suffice to show that

$$\psi_\varepsilon(\omega) = \int_\Omega \hat{\nu}_{\tau,\omega}^{Z,\varepsilon}(d\hat{\omega})\theta_\varepsilon(\hat{\omega}) \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{in } \rho\text{-measure}, \tag{4.47}$$

where

$$\theta_\varepsilon(\omega) = E_\omega\big(F\big((\mathbf{X}_t^\varepsilon)_{0\leq t<T-\tau}\big)\big) - E\big(F\big((\mathbf{W}_t)_{0\leq t<T-\tau}\big)\big). \tag{4.48}$$

By the invariance principle for $\mathbf{X}_t^\varepsilon$, we know that $\theta_\varepsilon(\omega) \to 0$ in ρ-measure. Now, the measure $d\mu = e^{-2V(0)}\, d\rho$ is stationary for the environment process $\omega_t = \tau_{-\mathbf{X}_t}\omega$. Thus $\theta_\varepsilon(\omega) \to 0$ in μ-measure and $\theta_\varepsilon(\omega) \to 0$ in $L^1(\mu)$. Stationarity of μ implies that

$$\hat{\nu}_{\tau,\mu}^{X,\varepsilon} = \int_\Omega \mu(d_\omega)\hat{\nu}_{\tau,\omega}^{X,\varepsilon} = \mu, \tag{4.49}$$

so that

$$\int_\Omega \hat{\nu}_{\tau,\mu}^{X,\varepsilon}(d\hat{\omega})|\theta_\varepsilon(\hat{\omega})| \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{4.50}$$

It follows from (4.42) that

$$\hat{\nu}_{\tau,\mu}^{Z,\varepsilon} \leq C(K)\hat{\nu}_{\tau,\mu}^{X,\varepsilon} + \hat{\alpha}_{\tau,\mu,K}^{Z,\varepsilon}, \tag{4.51}$$

with $C(K)$ as before and $\|\hat{\alpha}_{\tau,\mu,K}^{Z,\varepsilon}\| \leq \delta(K)$, where $\|\cdot\|$ is the variation norm.

Thus as $\varepsilon \to 0$

$$\int_{\omega} \hat{\nu}_{\tau,\mu}^{Z,\varepsilon}(d\hat{\omega})|\theta_\varepsilon(\hat{\omega})| \to 0 \tag{4.52}$$

or

$$\int_{\Omega} \mu(d\omega) \int_{\Omega} \hat{\nu}_{\tau,\omega}^{Z,\varepsilon}(d\hat{\omega})|\theta_\varepsilon(\hat{\omega})| \to 0. \tag{4.53}$$

But the second integral in (4.53) dominates $|\psi_\varepsilon(\omega)|$, so that $\psi_\varepsilon(\omega) \to 0$ in $L^1(\mu)$, hence $\psi_\varepsilon(\omega) \to 0$ in μ-measure, hence $\psi_\varepsilon(\omega) \to 0$ in ρ-measure. The proof is completed in view of (i) and (iii) by taking $\tau \to 0$. □

Since the Nash estimates easily provide the requisite uniform integrability, we have as an immediate consequence of Theorem 4.1 the following

COROLLARY 4.1. *For* $\mathbf{Z}_t$ *in* $\mathbb{R}^d$ *obeying* (4.2),

$$\frac{E\left[(Z_t^i - Z_0^i)(Z_t^j - Z_0^j)\right]}{t} \underset{t\to\infty}{\longrightarrow} D_{ij}(V), \qquad i, j = 1, \dots, d, \tag{4.54}$$

where D_{ij} *are components of* $\mathsf{D}(V)$ *in* (4.1) *and the convergence is in* ρ*-measure.*

REMARK 4.1. In three and higher dimensions, where the process is presumably nonrecurrent so that there is a "last time" that the process visits the perturbation, one might think that it is possible to give a much simpler proof of Theorem 4.1. However, our proof already proceeds by establishing an effective nonrecurrence (ii), so that actual nonrecurrence would not simplify matters significantly.

REMARK 4.2. In one dimension it is crucial that the perturbation be local in V, not just in ∇V, for a local perturbation of ∇V can yield a nonlocal perturbation of V which would in general destroy asymptotic Brownian behavior.

REMARK 4.3. In two and higher dimensions, presumably the theorem can be extended to include local perturbations of ∇V, i.e., perturbations of the drift which need not be of gradient form. All that would be required for our proof to apply in this case are Nash-type estimates for the perturbed process.

REMARK 4.4. The arguments and techniques used in this paper apply as well to diffusion processes with generator $L = \nabla \cdot \mathsf{a}\nabla$ as well as to the case of $L = b\nabla \cdot a\nabla$, with $a, b > 0$.

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