Section 15.7

7. $\nabla f(x, y) = \left\langle -4e^{3y}\sin 2x, 6e^{3y}\cos 2x \right\rangle$;

$\nabla f\left(\frac{\pi}{3}, 0\right) = \left\langle -2\sqrt{3}, -3 \right\rangle$

Tangent plane: $z+1 = -2\sqrt{3}\left(x - \frac{\pi}{3}\right) - 3(y-0)$,

or $2\sqrt{3}x + 3y + z = \dfrac{\left(2\sqrt{3}\pi - 3\right)}{3}$.

15.6 –

30. a. $D_{\mathbf{u}} f = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle \cdot \left\langle f_x \cdot f_y \right\rangle = -6$, so

$3f_x - 4f_y = -30$.

$D_{\mathbf{v}} f = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle \cdot \left\langle f_x, f_y \right\rangle = 17$, so

$4f_x + 3f_y = 85$.

The simultaneous solution is

$f_x = 10,\ f_y = 15$, so $\nabla f = \langle 10, 15 \rangle$.

b. Without loss of generality, let $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$. If θ and ϕ are the angles between u and ∇f, and between v and ∇f, then:

1. $\theta + \phi = \frac{\pi}{2}$ (if ∇f is in the 1st quadrant).
2. $\theta = \frac{\pi}{2} + \phi$ (if ∇f is in the 2nd quadrant).
3. $\phi + \theta = \frac{3\pi}{2}$ (if ∇f is in the 3rd quadrant).
4. $\phi = \frac{\pi}{2} + \theta$ (if ∇f is in the 4th quadrant).

15.7 – 13. Let

$F(x, y, z) = x^2 - 2xy - y^2 - 8x + 4y - z = 0$;

$\nabla F(x, y, z) = \langle 2x - 2y - 8, -2x - 2y + 4, -1 \rangle$

Tangent plane is horizontal if $\nabla F = \langle 0, 0, k \rangle$ for any $k \neq 0$.

$2x - 2y - 8 = 0$ and $-2x - 2y + 4 = 0$ if $x = 3$ and $y = -1$. Then $z = -14$. There is a horizontal tangent plane at $(3, -1, -14)$.

15.7 – 16

16. $(1, 1, 1)$ satisfies each equation, so the surfaces intersect at $(1, 1, 1)$. For

$z = f(x, y) = x^2 y : \nabla f(x, y) = \langle 2xy, x^2 \rangle$;

$\nabla f(1, 1) = \langle 2, 1 \rangle$, so $\langle 2, 1, -1 \rangle$ is normal at $(1, 1, 1)$.

For $F(x, y, z) = x^2 - 4y + 3 = 0$;

$\nabla f(x, y, z) = \langle 2, -4, 0 \rangle$;

$\nabla f(1, 1, 1) = \langle 2, -4, 0 \rangle$ so $\langle 2, -4, 0 \rangle$ is normal at $(1, 1, 1)$.

$\langle 2, 1, -1 \rangle \cdot \langle 2, -4, 0 \rangle = 0$, so the normals, hence tangent planes, and hence the surfaces, are perpendicular at $(1, 1, 1)$.

15.8 – 11

11. We do not need to use calculus for this one. $3x$ is minimum at 0 and $4y$ is minimum at -1. $(0, -1)$ is in S, so $3x + 4y$ is minimum at $(0, -1)$; the minimum value is -4. Similarly, $3x$ and $4y$ are each maximum at 1. $(1, 1)$ is in S, so $3x + 4y$ is maximum at $(1, 1)$; the maximum value is 7. (Use calculus techniques and compare.)

15.8 — **24.** The lines are skew since there are no values of s and t that simultaneously satisfy $t - 1 = 3s$, $2t = s + 2$, and $t + 3 = 2s - 1$. Minimize f, the square of the distance between points on the two lines.

$f(s, t) = (3s - t + 1)^2 + (s + 2 - 2t)^2 + (2s - 1 - t - 3)^2$

Let

$\nabla f(s, t) = \langle 2(3s - t + 1)(3) + 2(s - 2t + 2)(1) + 2(2s - t - 4)(2), 2(3s - t + 1)(-1) + 2(s - 2t + 2)(-2) + 2(2s - t - 4)(-1)\rangle$

$= \langle 28s - 14t - 6, -14s + 12t - 28\rangle = \langle 0, 0\rangle$.

Solve $28s - 14t - 6 = 0$, $-14s + 12t - 2 = 0$, obtaining $s = \frac{5}{7}, t = 1$.

$D = f_{ss}f_{tt} - f_{st}^2 = (28)(12) - (-14)^2 > 0$; $f_{ss} = 28 > 0$. (local minimum)

The nature of the problem indicates the global minimum occurs here.

$$f\left(\frac{5}{7}, 1\right) = \left(\frac{15}{7}\right)^2 + \left(\frac{5}{7}\right)^2 + \left(-\frac{25}{7}\right)^2 = \frac{875}{49}$$

Conclusion: The minimum distance between the lines is $\frac{\sqrt{875}}{7} \approx 4.2258$. (For another way of doing this problem see Problem 21, Section 14.4.)

15.9 — **2.** $\langle y, x\rangle = \lambda\langle 8x, 18y\rangle$

$y = 8\lambda x$, $x = 18\lambda y$, $4x^2 + 9y^2 = 36$

Critical points are $\left(\frac{3}{\sqrt{2}}, \pm\frac{2}{\sqrt{2}}\right), \left(-\frac{3}{\sqrt{2}}, \pm\frac{2}{\sqrt{2}}\right)$.

Maximum value of 3 occurs at $\left(\pm\frac{3}{\sqrt{2}}, \pm\frac{2}{\sqrt{2}}\right)$.

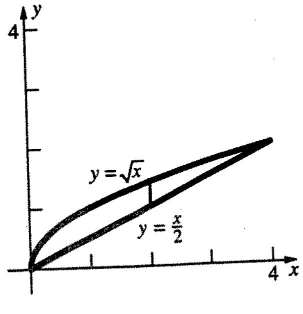
16.2 —

32. $$\int_0^1 \int_0^{\sqrt{3}} 8x(x^2 + y^2 + 1)^{-2}\,dx\,dy = \int_0^1 [-4(x^2 + y^2 + 1)^{-1}]_{x=0}^{\sqrt{3}}\,dy = 4\int_0^1 \left[\frac{-1}{4 + y^2} + \frac{1}{1 + y^2}\right] dy$$

$$= 4\left[-\frac{1}{2}\arctan\left(\frac{y}{2}\right) + \arctan(y)\right]_0^1 = 4\left[\left(-\frac{1}{2}\arctan\left(\frac{1}{2}\right) + \frac{\pi}{4}\right) - 0\right] = \pi - 2\arctan\left(\frac{1}{2}\right) \approx 2.2143$$

16.2 —

32. $\int_0^4 \int_{x/2}^{\sqrt{x}} f(x, y)\,dy\,dx$

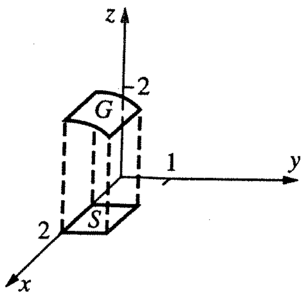


16.4 —

13. $\int_0^{\pi/4} \int_0^2 (4 + r^2)^{-1} r\,dr\,d\theta = \left(\frac{\pi}{8}\right)\ln 2 \approx 0.2722$

16.6 —

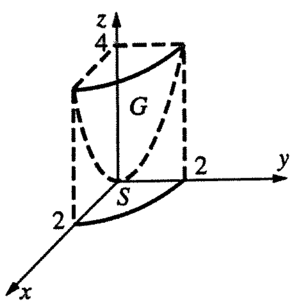
3.



$z = f(x, y) = (4 - y^2)^{1/2}$; $f_x(x, y) = 0$;
$f_y(x, y) = -y(4 - y^2)^{-1/2}$

$$A(G) = \int_0^1 \int_1^2 \sqrt{y^2(4-y^2)^{-1}+1}\, dx\, dy$$
$$= \int_0^1 \int_1^2 \frac{2}{\sqrt{4-y^2}}\, dx\, dy$$
$$= \int_0^1 \frac{2}{\sqrt{4-y^2}}\, dy = \left[2\sin^{-1}\left(\frac{y}{2}\right)\right]_0^1 = 2\left(\frac{\pi}{6}\right) - 2(0)$$
$$= \frac{\pi}{3} \approx 1.0472$$

16.6 — **6.**



Let $z = f(x, y) = x^2 + y^2$; $f_x(x, y) = 2x$;
$f_y(x, y) = 2y$.

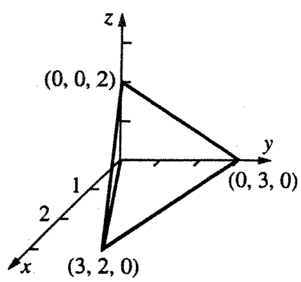
$$A(G) = 4\int_0^2 \int_0^{\sqrt{4-y^2}} \sqrt{4x^2+4y^2+1}\, dy\, dx$$
$$= 4\int_0^{\pi/2} \int_0^2 (4r^2+1)^{1/2} r\, dr\, d\theta$$
$$= 4\int_0^{\pi/2} \left[\frac{(4r^2+1)^{3/2}}{12}\right]_0^2 dr = \frac{(17^{3/2}-1)}{3}\frac{\pi}{2}$$
$$\approx 36.1769$$

16.6 **15.** $\bar{x} = \bar{y} = 0$ (by symmetry)

Let $h = \frac{h_1 + h_2}{2}$. Planes $z = h_1$ and $z = h$ cut out the same surface area as planes $z = h$ and $z = h_2$. Therefore, $\bar{z} = h$, the arithmetic average of h_1 and h_2.

16.7 – **5.** $\int_0^2 \int_1^z x^2\, dx\, dz = \int_0^2 \left(\frac{1}{3}\right)(z^3 - 1)\, dz = \frac{2}{3}$

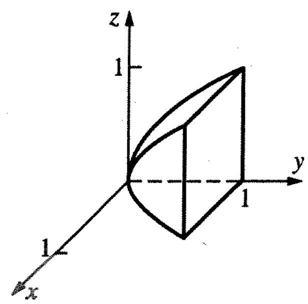
16.7 – **15.**



Using the cross product of vectors along edges, it is easy to show that $\langle 2,6,9 \rangle$ is normal to the upward face. Then obtain that its equation is
$2x + 6y + 9z = 18$.

$$\int_0^3 \int_{2x/3}^{(9-x)/3} \int_0^{(18-2x-6y)/9} f(x, y, z)\, dz\, dy\, dx$$

16.7 – **21.** $V = 4\int_0^1 \int_0^{\sqrt{y}} \int_0^{\sqrt{y}} 1\, dz\, dx\, dy = 4\int_0^1 \int_0^{\sqrt{y}} \sqrt{y}\, dx\, dy$
$= 4\int_0^1 \sqrt{y}\sqrt{y}\, dy = [2y^2]_0^1 = 2$



16.8 —

5. Let $\delta(x, y, z) = 1$.
(See write-up of Problem 24, Section 16.7.)

$$m = \int_0^{2\pi} \int_0^2 \int_{r^2}^{12-2r^2} r\, dz\, dr\, d\theta = 24\pi$$

$$M_{xy} = \int_0^{2\pi} \int_0^2 \int_{r^2}^{12-2r^2} zr\, dz\, dr\, d\theta = 128\pi$$

$\bar{z} = \frac{16}{3}$
$\bar{x} = \bar{y} = 0$ (by symmetry)