Name. _____

Calculus III 2210-4
Final Exam Study Guide
Exam Date: Monday 12 December 2005

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Vector calculus)** Complete two of the following.

   (a) Let $\mathbf{r}(t) = \begin{pmatrix} t \\ 2t \end{pmatrix}$. Define $\mathbf{u}(t) = (\mathbf{r}(t) \cdot \mathbf{r}'(t))\mathbf{r}(3t + 1)$. Find $\mathbf{u}'(0)$.

   (b) Let $\mathbf{r}(t) = \begin{pmatrix} \sin t \\ t \end{pmatrix}$. Find $\int_0^\pi \mathbf{r}(t)dt$.

   (c) True or false: $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$. Justify.

2. **(Vector algebra)** Complete two of the following.

   (a) Report three different vectors orthogonal to $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

   (b) Find all vectors $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} - \mathbf{j}$.

   (c) State the five dot product properties.

3. **(Differential geometry)** Complete two of the following.

   (a) State a formula for the curvature of a space curve $\mathbf{r}(t)$. Use it to justify why $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 0$ characterizes zero curvature.

   (b) Let $\mathbf{r}(t) = t\mathbf{i} + 2\mathbf{j} + (t + 1)\mathbf{k}$. Find the principal normal $\mathbf{N}$ at $t = 0$.

   (c) Let $\mathbf{r}(t) = t\mathbf{i} + (t + 1)\mathbf{k}$. Find the tangential component of acceleration a_T at $t = 0$.

4. **(Graphing)**
   Name and sketch the graph of $x^2 + y^2 - z^2 = 1$.

5. **(Planes)** Complete one of the following.

   (a) Find the equation of the plane parallel to $x + 2y + 9z = 2$ passing through the point $(2, -2, 4)$.

   (b) Find the acute angle between the planes $2x - 4y + z = 7$ and $3x + 2y - 5z = 9$.

6. **(Vector geometry in $\mathcal{R}^3$)** Complete two of the following.

   (a) Find the volume of a parallelepiped with edges $3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $-\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$. See 14.3-18.

   (b) Prove $|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2$. See 14.3-25.

   (c) Find the parametric equation of the line through $(1, 1, 0)$ normal to the plane $2x + 4y + z = 5$.

   (d) Find the equation of the plane through the three points $(1, 1, 0)$, $(1, 2, 0)$, $(0, 1, 1)$.

7. **(Coordinate systems)** Complete two of the following.

   (a) Change $x^2 - y^2 = 25$ to cylindrical coordinates.

   (b) Change $x^2 + y^2 + 2z^2 = 25$ to spherical coordinates.

   (c) Find the center and radius of the sphere with equation $x^2 + y^2 + z^2 - 6x + 2y - 8z = 0$.

8. **(Level sets)** Complete two of the following.
   (a) Sketch the level curves of $z = x^2/y$ for $z = -1, 1, 2$.
   (b) Describe geometrically the level surface $16x^2 + 16y^2 - 9z^2 = k$ for all real values of k.
   (c) Find a normal to the level curve $x^2/y = 4$ at each point (x, y).

9. **(Derivatives)** Complete two of the following.
   (a) Find f_x, f_{xy} for $f = (x^3 + y^2)^5$.
   (b) Let $f(x, y) = xy^2/(x + 2 + y^4)$. Is f continuous at $(0, 0)$? Discuss continuity of the partial derivative f_{xy}.
   (c) Define f_x using Newton quotients and limits.
   (d) Find the gradient of $f(x, y, z) = (x + 2y + 3z)^2 e^{3x+4y+5z}$ at $x = y = z = 0$.

10. **(Chain rule)** Complete two of the following.
    (a) Suppose $x \sin y + y \cos x = 10$. Find a formula for dy/dx.
    (b) Let $w = e^{x^2+y^2}$, $x = t^3$, $y = t^2$, $z = t$. Find dw/dt.
    (c) Let $w = u/v$, $u = x^2 - 3y$, $v = xyz$. Find the partials of w in variables x, y, z.

11. **(ch15)** Complete two of the following.
    (a) Find the directional derivative of $f(x, y, z) = x^3 y - y^2 z^2$ at $(-2, 1, 3)$ in the direction of $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. See 15.5-7.
    (b) Let $w = f(x, y, z)$ and $x = r - s$, $y = s - t$, $z = t - r$. Derive from the chain rule that $\partial_r w + \partial_s w + \partial_t w = 0$. See 15.6-30.
    (c) Find all points on the surface $z = x^2 - 2xy - y^2 - 8x + 4y$ where the tangent plane is parallel to the xy-plane. See 15.7-13.
    (d) Find a point (x_0, y_0, z_0) where surfaces $z = x^2 y$ and $y = \frac{1}{4}x^2 + \frac{3}{4}$ intersect and such that the tangent planes at (x_0, y_0, z_0) are perpendicular. See 15.7-16.

12. **(Maxima and Minima)** Complete two of the following.
    (a) State the critical point theorem and the second partials test. See 15.8.
    (b) Find the global maximum of $f(x, y) = 3x + 4y$ on the rectangle $0 \le x \le 1$, $|y| \le 1$. See 15.8-11.
    (c) Find the minimum distance between the two lines with vector equations $\mathbf{r}_1(t) = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r}_2(t) = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. See 15.8-24.
    (d) Find the maximum of $f(x, y) = xy$ subject to $4x^2 + 9y^2 = 36$. See 15.9-2 or convert it to a one-variable max problem by solving for y in terms of x.

13. **(Double Integrals)** Complete two of the following.
    (a) Define the double integral of $f(x, y)$ over a rectangle Q. See 16.1.
    (b) Let $f(x, y) = 1$ on R_1, $f(x, y) = 3$ on R_2, $f(x, y) = 5$ on R_3 and let R be the union of the three rectangles R_1, R_2, R_3. Suppose each rectangle has area 4. Find $\int \int_R f(x, y) dA$.
    (c) Let R be defined by $0 \le x \le \pi/2$, $0 \le y \le \pi/2$. Evaluate $\int \int_R \sin(x + y) dA$.
    (d) Evaluate $\int_0^{\sqrt{3}} \int_0^1 \frac{8x}{(x^2 + y + 2 + 1)^2} dy dx$. See 16.2-32.
    (e) Find the volume of the solid in the first octant bounded by the surface $9z = 36 - 9x^2 - 4y^2$

and the coordinate planes. See 16.3-29.

(f) Let R be the region in quadrant one bounded by $x^2 + y^2 = 4$ and the lines $y = 0$ and $y = x$. Evaluate $\int \int_R f(x, y) dA$ using polar coordinates, given $f(x, y) = 1/(4 + x^2 + y^2)$. See 16.4-13.

14. **(Surface Area)** Complete two of the following.

(a) Find the area of the part of the surface $z = \sqrt{4 - y^2}$ directly above the xy-plane square $1 \le x \le 2$, $0 \le y \le 1$. See 16.6-3.

(b) Find the area of the part of the paraboloid $z = x^2 + y^2$ that is cut off by the plane $z = 4$. See 16.6-6 and example 2 page 712.

(c) Find the center of mass of the homogeneous sphere $x^2 + y^2 + z^2 = a^2$ between the planes $z = a/2$ and $z = a/4$ ($a > 0$ assumed). See 16.6-15.

15. **(Triple Integrals)** Complete two of the following.

(a) Evaluate $\int_0^2 \int_1^z \int_0^{\sqrt{x/z}} 2xyz \, dy \, dx \, dz$. See 16.7-5.

(b) Evaluate $\int \int \int_S dx \, dy \, dz$ where S is the tetrahedron with vertices $(0, 0, 0)$, $(3, 2, 0)$, $(0, 3, 0)$, $(0, 0, 2)$. See 16.7-15.

(c) Find the volume of the solid in the first octant bounded by the 3D surfaces $x^2 = y$ and $z^2 = y$ and $y = 1$. See 16.7-21.

(d) Find the center of mass of the homogeneous solid bounded above by $z = 12 - 2x^2 - 2y^2$ and below by $z = x^2 + y^2$. See 16.8-5.

16. **(Line Integrals)** Complete one of the following.

(a) Evaluate $\oint_C (2x + 9z) ds$ for the curve C given by $x = t$, $y = t^2$, $z = t^3$ on $0 \le t \le 1$. See 17.2-5.

(b) Find the work done by vector force $\mathbf{F} = (x + y)\mathbf{i} + (x - y)\mathbf{j}$ where path C is the quarter ellipse $x = a \cos t$, $y = b \sin t$, $0 \le t \le \pi/2$. See 17.2-21.

17. **(Divergence and Curl)** Complete one of the following.

(a) Compute the divergence and curl for vector function $\mathbf{F} = x^2\mathbf{i} - 2xy\mathbf{j} + yz^2\mathbf{k}$. See 17.1-13.

(b) Define divergence and curl. See 17.1.

(c) Expand the two sides of the identity $\mathbf{div}(f\mathbf{F}) = (f)(\mathbf{div}\,\mathbf{F}) + (\mathbf{grad}\,f) \cdot \mathbf{F}$ to present a proof of the identity. See 17.1-20(c).

18. **(Green's Theorem)** Complete one of the following.

(a) State Green's theorem in the plane. See 17.4, Theorem A.

(b) Evaluate the line integral using Green's theorem: $\oint_C (x^2 + 4xy) dx + (2x^2 + 3y) dy$ where C is the ellipse $9x^2 + 16y^2 = 144$. See 17.4-3.

19. **(Gauss' Divergence Theorem)** Complete one of the following.

(a) State Gauss' divergence theorem in the plane. See 17.4, page 751. Give the flux interpretation.

(b) State Gauss' divergence theorem in space. See Theorem A, 17.6, page 759. Give the flux interpretation.