Birational geometry of algebraic varieties

Christopher Hacon

May-June, 2018

Outline of the talk

Introduction

- The goal of the MMP is to classify complex projective algebraic varieties up to birational isomorphism.
- $X \subset \mathbb{P}^N_{\mathbb{C}}$ is defined by homogeneous polynomials $P_1, \ldots, P_r \in \mathbb{C}[x_0, \ldots, x_N]$.
- Two varieties X, X' are **birational** if they have isomorphic fields of rational (meromorphic) functions $\mathbb{C}(X) \cong \mathbb{C}(X')$ or equivalently if they have isomorphic open subsets $U \cong U'$.
- By Hironaka's Theorem any projective variety is birational to a smooth one. Thus, at least in the beginning we will assume that X is smooth of dimension $\dim X = d$.
- Let $\omega_X = \wedge^d T_X^\vee$ be the **canonical line bundle**.
- Note that complex projective manifolds have no non-constant global holomorphic functions, so it is natural to consider global sections of line bundles.

The canonical ring

- There is only one natural choice: the canonical line bundle!
- For any $m > 0$ consider the **pluricanonical forms** $H^0(\omega_X^{\otimes m})$.
- Locally $s = f \cdot (dz_1 \wedge \ldots \wedge dz_d)^{\otimes m}$.
- If $s_0, \ldots, s_N$ is a basis of $H^0(\omega_X^{\otimes m})$, then $x \to [s_0(x) : \ldots : s_N(x)]$ defines a rational map $\phi_m : X \dashrightarrow \mathbb{P}^N$.
- $R(\omega_X) = \oplus_{m \geq 0} H^0(\omega_X^{\otimes m})$ is the **canonical ring**.
- $\kappa(X) = tr.deg._{\mathbb{C}} R(\omega_X) - 1$ is the **Kodaira dimension**.
- We have $\kappa(X) = \max\{\dim \phi_m(X)\} \in \{-1, 0, 1, \ldots, \dim X\}$.

Birational invariance of the canonical ring

- Note that $R(\omega_X)$ is a birational invariant. In fact by work of Wlodarczyk (1999) and Abramovich-Karu-Matsuki-Wlodarczyk (2002), the birational equivalence relation is generated by blow ups along smooth centers.
- If $\nu : X' \to X$ is a birational morphism given by the blow up of a smooth subvariety V, then $E := \nu^{-1}(V) \cong \mathbb{P}(N_X V)$ and $K_{X'} = \nu^* K_X + cE$ where $c = \dim X - \dim V - 1$.
- For example, if $\dim X = 2$, $V = P$ is a point, then $K_{X'} = \nu^* K_X + E$ where $E^2 = K_X \cdot E = -1$ and $E \cong \mathbb{P}^1$.
- It follows that the fibers F of $E \to V$ are isomorphic to $\mathbb{P}^c$ and $\mathcal{O}_{X'}(E)|_F \cong \mathcal{O}_{\mathbb{P}^c}(-1)$.
- Since E is exceptional, $\nu_* \mathcal{O}_{X'}(cE) \cong \mathcal{O}_X$ and by the projection formula $\nu_* \mathcal{O}_{X'}(mK_{X'}) \cong \mathcal{O}_X(mK_X)$ for any $m > 0$.

MMP

- The goal of the MMP is to find a well understood sequence of birational maps (flips and divisorial contractions) $X = X_0 \dashrightarrow X_1 \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_M$, such that either
- X_M is a **minimal model** and hence ω_{X_M} is nef (i.e. $K_{X_M} \cdot C \geq 0$ for any curve $C \subset X_M$), or
- X_M is a **Mori fiber space** so that there is a morphism $X_M \to Z$ with $\dim Z < \dim X_M$ and $-K_{X_M}$ is ample over Z.
- In this last case, the fibers are Fano varieties so that $-K_F$ is ample.
- The geometry of Fano varieties is well understood they are simply connected, and covered by rational curves.
- This implies (by the easy addition formula) that $H^0(mK_X) = 0$ for all $m > 0$ and so $\kappa(X) < 0$.

Dimension 1

- When $d = 1$, the birational equivalence relation is trivial (for smooth curves) so that X is birational to X' iff it is isomorphic.
- X is topologically a Riemann surface of genus g and there are 3 main cases:
- $\kappa(X) = -1$: Then $X \cong \mathbb{P}^1_{\mathbb{C}}$ is a **rational curve**. Note that $\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ and so $R(\omega_X) \cong \mathbb{C}$.
- $\kappa(X) = 0$: Then $\omega_X \cong \mathcal{O}_X$ and X is an elliptic curve. There is a one parameter family of these $x^2 = y(y-1)(y-s)$. In this case $R(\omega_X) \cong \mathbb{C}[t]$.

Curves of general type

- If $\kappa(X) = 1$, then we say that X is a curve of **general type**. They are Riemann surfaces of genus $g \geq 2$. For any $g \geq 2$ there is a $3g - 3$ irreducible algebraic family of these. We have $\deg(\omega_X) = 2g - 2 > 0$.
- By Riemann Roch, it is easy to see that $\omega_X^{\otimes m}$ is **very ample** for $m \geq 3$. This means that if $s_0, \ldots, s_N$ are a basis of $H^0(\omega_X^{\otimes m})$, then

$$\phi_m : X \to \mathbb{P}^N, \qquad x \to [s_0(x) : s_1(x) : \ldots : s_N(x)]$$

  is an embedding.
- Thus $\omega_X^{\otimes m} \cong \phi_m^* \mathcal{O}_{\mathbb{P}^N}(1)$ and in particular $R(\omega_X)$ is finitely generated.

Curves

- **Ex.** Show that not all elliptic curves are isomorphic.
- **Ex.** Show that any elliptic curve is a plane cubic.
- **Ex.** Show that if $g \geq 2$ (resp. ≥ 3), then $3K_X$ (resp. $2K_X$) is very ample.
- **Ex.** Show that any curve can be embedded in $\mathbb{P}^3$ but some curves can not be embedded in $\mathbb{P}^2$.
- **Ex.** Show that if $X \subset \mathbb{P}^2$ is a smooth curve of genus g and degree d, then $g = (d-1)(d-2)/2$. What is the formula for singular curves?

MMP for surfaces

- If $\dim X = 2$, the birational equivalence relation is non-trivial.
- In dimension 2, any two smooth birational surfaces become isomorphic after finitely many blow ups of smooth points (Zariski, 1931).
- By **Castelnuovo's Criterion**, if X contains a -1 curve ($E \cong \mathbb{P}^1$ and $K_X \cdot E = -1$), then there exists a morphism of smooth surfaces $X \to X_1$ contracting E to a point $P \in X_1$ and inducing an isomorphism over $X_1 \setminus P$.
- In over words, X is the blow up of X_1 at P.
- We repeat this procedure $X \to X_1 \to X_2 \to \ldots \to X_M$ until we obtain a surface X_M containing no -1 curves.
- If K_{X_M} is nef, we declare this to be a **minimal model**.
- Otherwise, either $X \cong \mathbb{P}^2$ or there is a morphism $X \to C$ where $\dim C = 1$ and the fibers are rational curves.

MMP for surfaces

- **Ex.** Show that if there is a morphism $X \to C$ where $\dim C = 1$ and the fibers are rational curves then $\kappa(X) < 0$.
- **Ex.** Show that if there is a rational curve passing through any point of an open subset of a complex projective variety X, then $\kappa(X) < 0$.
- **Ex.** Show that if there is a morphism $X \to C$ where $\dim C = 1$ and the fibers are elliptic curves then $\kappa(X) < 0$.
- **Ex.** Show that $\mathbb{F}_n$ is minimal iff $n \neq 1$. Deduce that minimal models are not unique for surfaces with $\kappa(X) < 0$.
- **Ex.** Show that if a surface X has a minimal model with K_X nef, then this minimal model is unique.

Singular minimal models

- If $\dim X \geq 3$, it is easy to see that there are smooth projective manifolds that do not admit a smooth minimal model or Mori fiber space.
- The easiest example is $V = A/ < -1_A >$ where A is an abelian threefold and -1_A is the involution induced by the inverse $-1_A(a) = -a$.
- -1_A has 2^6 fixed points.
- Locally around any of these fixed points, this map is given by $(x, y, z) \to (-x, -y, -z)$.
- **Ex.** Show that the singularities are locally isomorphic to cones over a Veronese surface and can be resolved by one blow up at the vertex.

Singular minimal models

- if $\nu : V' \to V$ is the resolution given by blowing up these singular points and $E \cong \mathbb{P}^2$ is the exceptional divisor, then $\mathcal{O}_{V'}(E)|_E \cong \mathcal{O}_{\mathbb{P}^2}(-2)$.
- By adjunction $K_E = (K_{V'} + E)|_E = (\nu^* K_V + (a+1)E)|_E$ so that $a = 1/2$ and $K_{V'} = \nu^* K_V + \frac{1}{2}E$.
- We will see later that if V' has a minimal model $\mu : V' \dashrightarrow V_M$, then V_M is isomorphic to V in codimension 1.
- But then $K_{V'} - \frac{1}{2}E = \nu^* K_V = \mu^* K_{V_M}$. If V_M is smooth, then K_{V_M} is Cartier. This is a contradiction. (Use the negativity lemma to show that $\nu^* K_V = \mu^* K_{V_M}$.)
- Therefore, if we wish to run the MMP in dimension ≥ 3 we must allow some mild singularities.

Singularities of the MMP

- A **pair** (X, B) is a normal variety and a $\mathbb{R}$-divisor B such that $K_X + B$ is $\mathbb{R}$-Cartier.
- Let $\nu : X' \to X$ be a **log resolution** so that X' is smooth, ν is projective, $\mathrm{Ex}(\nu)$ is a divisor and $\mathrm{Ex}(\nu) + \mathrm{Supp}(\nu_*^{-1}B)$ has simple normal crossings.
- We write $K_{X'} + B_{X'} = \nu^*(K_X + B_X)$, then $a_E(X, B) = -\mathrm{mult}_E(B_{X'})$ is the **discrepancy** of (X, B) along the prime divisor E.
- If E is a divisor on X, then $a_E(X, B) = -\mathrm{mult}_E(B)$.
- If $P \in X$ is a smooth point and $\nu : X' \to X$ is the blow up of this point with exceptional divisor E, then $a_E(X, 0) = \dim X - 1$.

Singularities of the MMP

- This can be seen via the following local computation.
- If $x_1, \ldots, x_n$ are local coordinates at $O \in X$, then the blow up $X' \to X$ is induced by the projection $X \times \mathbb{P}^{n-1}$.
- In local coordinates, X' is defined by equations $x_i X_j - x_j X_i$ for $1 \leq i, j \leq n$ where $[X_1 : \ldots : X_n]$ are homogeneous coordinates on $\mathbb{P}^{n-1}$.
- On the chart $X_1 \neq 0$ we may set $X_1 = 1$ and $X_i = x_i/x_1$, so that $x_1, X_2, \ldots, X_n$ are local coordinates on X' and $f : X' \to X$ is given by $x_i = X_i x_1$.
- But then $dx_1 \wedge \ldots \wedge dx_n = dx_1 \wedge (X_2 dx_1 + x_1 dX_2) \wedge \ldots \wedge (X_n dx_1 + x_1 dX_n) = x_1^{n-1} dx_1 \wedge dX_2 \wedge \ldots \wedge dX_n$.
- Thus $f^*\omega_X = \omega_{X'}(-(n-1)E)$ as required.

Singularities of the MMP

- We say that (X,B) is **log canonical** or LC if $B \geq 0$ if $a_E(X,B) \geq -1$ for any E over X.
- We say that (X,B) is **Kawamata log terminal** if $B \geq 0$ or KLT if $a_E(X,B) > -1$ for any E over X.
- We say that X is **canonical** (resp. **terminal**) if $B \geq 0$ if $a_E(X,0) \geq 0$ (resp. $a_E(X,0) > 0$) for any exceptional divisor E over X.
- Note that the LC and KLT conditions can be checked one or any log resolution, whilst the canonical/terminal conditions must be checked for all eceptional divisors.
- If we do not assume that $B \geq 0$, then we say that (X,B) is sub-LC, sub-KLT etc.

Singularities of the MMP

- If $\dim X = 2$, then X is terminal iff it is smooth and X is canonical iff it has du Val (or RDP) singularities. Note that if X is a surface with canonical singularities, then K_X is Cartier.
- It is easy to see that if X and X' are birational and have canonical singularities, then $R(K_X) \cong R(K_{X'})$.
- Pick a common resolution $p : Y \to X$, $q : Y \to X'$, it suffices to show that $H^0(mK_X) \cong H^0(mK_Y)$ but this follows easily since by definition $K_Y - p^*K_X$ is effective and p-exceptional.

Singularities of the MMP

- It is easy to see that KLT pairs are LC and if X is terminal, then it is canonical and that if $B=0$ canonical implies KLT.
- The idea is that bigger discrepancies correspond to milder singularities (terminal singularities are the mildest, and smooth varieties are always terminal).
- It is known that KLT singularities are rational so that if $\nu : X' \to X$ is a resolution, then $R\nu_*\mathcal{O}'_X = \mathcal{O}_X$.
- We let $a(X,B) = \inf\{a_E(X,B)\}$ where E is a divisor over X, be the **minimal discrepancy**.
- Similarly if $P \in X$ is a Grothendieck point, we let $a_P(X,B) = \inf\{a_E(X,B)\}$ where E is a divisor with center $P \in X$, be the **minimal discrepancy over** P.
- It is easy to see that (X,B) is LC (resp. KLT) iff $a(X,B) \geq -1$ (resp. $a(X,B) > -1$).

Singularities of the MMP

- Give examples of singularities that are LC but not KLT; KLT but not canonival; canonical but not terminal.
- Suppose that X is d dimensional with a singularity given by $x_0^n + \ldots + x_d^n = 0$. For which values of n is it terminal, canonical, LT, LC. Compute the blow up of the singular point and the discrepancy along the corresponding exceptional divisor.
- Let X be a smooth variety. Show that if $B \geq 0$ is a $\mathbb{Q}$-divisor with $\mathrm{mult}_x(B) \geq \dim X$, then (X, B) is not klt at $x \in X$.
- Give an example of a rational LC singularity with boundary $B = 0$ which is not klt.

Singularities of the MMP

Claim

If there is a divisor E over X with $a_E(X,B) < -1$, then $a(X,B) = -\infty$.

- Assume in fact that there is a divisor with $a_E(X,B) < -1$.
- Let $f : X' \to X$ be a resolution such that E is a divisor on X', then $K_{X'} + (1+b)E + G = f^*(K_X + B)$ where G and E have no common components and $b > 0$.
- If we blow up a general codimension 1 point on E, then we obtain a new divisor E_1 with coefficient b.
- If we blow up the intersection of E_1 with the strict transform of E, we obtain a divisor E_2 with coefficient $2b = -a_{E_2}(X,B)$.
- Proceeding this way, we get divisor E_i with coefficient $ib = -a_{E_i}(X,B)$. Thus $a(X,B) = \inf a_{E_i}(X,B) = -\infty$.

Singularities of the MMP

- It is instructive to work out in detail some easy two dimensional examples.
- Let X be the cone over a rational curve C of degree n, then blowing up the vertex $P \in X$ we obtain a resolution $f : X' \to X$ with exceptional divisor $E \cong f^{-1}(P) \cong \mathbb{P}^1$ such that $E^2 = -n$.
- By adjunction we have $K_E \cong (K_{X'} + E)|_E = (f^*(K_X) + aE + E)|_E = (1 + a)E|_E$ and so $-2 = n(-a - 1)$ i.e. $a = -1 + 2/n$.
- Note that then if $L \subset X$ is a line through P then $(K_X + L)|_L \cong (K_{X'} + f_*^{-1}L + (1 - \frac{1}{n})E)|_E = K_L + (1 - \frac{1}{n})P$.
- Here we have used that $f^*L = f_*^{-1}L + \frac{1}{n}E$ (this can be checked by intersecting with E).

Singularities of the MMP

- The above example illustrates two facts.
- First, the discrepancies can be rational numbers with arbitrarily large denominators, however we expect minimal discrepancies to only have accumulation points from above. This is **Shokurov's ACC for MLD's conjecture**.
- Shokurov also conjectures several other interesting properties. According to his conjecture on the **semicontinuity of MLD's**, log discrepancies over closed points are supposed to be lower semicontinuous (they jump down at special points which are hence considered more singular).
- Finally it is expected that $x \in X$ is a smooth closed point not contained in the support of $B \geq 0$ iff $\min\{a_E(X, B)|\nu(E) = x\} = \dim X - 1$.

Singularities of the MMP

- Notice that by the semicontinuity of MLD's, since a smooth point not contained in the support of $B \geq 0$ satisfies $a_x(X,B) = \min\{a_E(X,B) | \nu(E) = x\} = \dim X - 1$, it conjecturally follows that $\min\{a_E(X,B) | \nu(E) = x'\} \leq \dim X - 1$ for all $x' \in X$.
- However, it is even not known if there is a number M such that for any d-dimensional projective pair (X,B), we have $a_x(X,B) \leq M$ for all $x \in X$.

Singularities of the MMP

- Secondly, adjunction on singular varieties is more complicated.
- If $(X, S + B)$ is log canonical, then we consider a log resolution $f : X' \to X$.
- Let $S' = f^{-1}_* S$, write $K_{X'} + S' + B' = f^*(K_X + S + B)$ and $K_{S'} + B'_{S'} = (K_{X'} + S' + B')|_{S'}$ so that $B_{S'} = B'|_{S'}$.
- Let $\nu : S^\nu \to S$ be the normalization and $g : S' \to S^\nu$ the induced morphism.
- We then define $K_{S^\nu} + B_{S^\nu} := g_*(K_{S'} + B_{S'}) = (K_X + S + B)|_S$.

Singularities of the MMP

- The $\mathbb{Q}$-divisor $B_{S^\nu} = \mathrm{Diff}_{S^\nu}(B)$ is the **different** of $(X, S+B)$ along S^ν.
- It is an effective $\mathbb{Q}$-divisor whose coefficients lie in the set $\{1 + \frac{b-1}{n} | n \in \mathbb{N} \cup +\infty,\ b = \sum b_i\}$ where $B = \sum_i b_i B_i$ (this can be checked from the classification of 2-dimensional log canonical singularities).
- Note that if the coefficients of B are in a DCC set, then so are those of $\mathrm{Diff}_{S^\nu}(B)$.

Singularities of the MMP

- There are two other kinds of singularities that we wish to introduce at this point.
- (X, B) is DLT if $B = \sum b_i B_i$ with $0 \leq b_i \leq 1$ and there is an open subset $U \subset X$ such that $(U, B|_U)$ is log smooth (ie. has SNC) and $a_E(X, B) > -1$ for any E with center contained in $Z = X \setminus U$.
- By a result of Szabo, this is equivalent to the existence of a log resolution that extracts only divisors with $a_E(X, B) > -1$.
- (X, B) is PLT if it is DLT and $\lfloor B \rfloor = S$ is irreducible (so that $a_E(X, B) > -1$ for any exceptional divisor over X).
- Not that as S is a minimal non-klt center, it is normal by Kawamata's result.

Singularities of the MMP

- An important and interesting result is that $(X, S+B)$ is PLT iff (S, B_S) is KLT (near S).
- By definition if $(X, S+B)$ is PLT, then $\lfloor B' \rfloor = 0$ and so $\lfloor B'_{S'} \rfloor = 0$ i.e. (S, B_S) is KLT.
- Conversely if (S, B_S) is KLT, then we know that $\lfloor B'_{S'} \rfloor = 0$ so $\lfloor B' \rfloor \cap S' = 0$. To see that $(X, S+B)$ is PLT we must in fact check that $\lfloor B' \rfloor = 0$ (over S).
- This follows immediately from the Connectedness Lemma of Kollár and Shokurov.

Theorem

Let (X, B) be a pair and $f : X' \to X$ a proper birational morphism, then $f^{-1}(x) \cap (\cup_{E \subset X' | a_E(X,B) \leq -1} E)$ is connected.

Proof of the Connectedness Lemma

- Let $f : X' \to X$ be a log resolution, $K_{X'} + B' = f^*(K_X + B)$, $S := \lfloor B' \rfloor - \lfloor (B')^{<1} \rfloor$.
- Consider the short exact sequence

$$0 \to \mathcal{O}_{X'}(-\lfloor B' \rfloor) \to \mathcal{O}_{X'}(-\lfloor (B')^{<1} \rfloor) \to \mathcal{O}_S(-\lfloor (B')^{<1} \rfloor) \to 0.$$

- Since

$$-\lfloor B' \rfloor = K_{X'} - \lfloor K_{X'} + B' \rfloor = K_{X'} - (K_{X'} + B') + \{K_{X'} + B'\}$$

where $(X', \{B'\})$ is klt and $-(K_{X'} + B') \equiv 0$ is $f-$nef and f-big, it follows from K-V vanishing that $R^1 f_* \mathcal{O}_{X'}(-\lfloor B' \rfloor) = 0$.
- Thus $f_* \mathcal{O}_{X'}(-\lfloor (B')^{<1} \rfloor) \to f_* \mathcal{O}_S(-\lfloor (B')^{<1} \rfloor)$ is surjective.

Proof of the Connectedness Lemma

- Since $-\lfloor (B')^{<1} \rfloor$ is effective and f-exceptional, we have $f_*\mathcal{O}_{X'}(-\lfloor (B')^{<1} \rfloor) \cong \mathcal{O}_X$.
- The induced surjection $\mathcal{O}_X \to f_*\mathcal{O}_S(-\lfloor (B')^{<1} \rfloor)$ factors through $\mathcal{O}_{f(S)} \to f_*\mathcal{O}_S$ and so $f_*\mathcal{O}_S(-\lfloor (B')^{<1} \rfloor) = f_*\mathcal{O}_S$. It follows that $\mathcal{O}_{f(S)} \to f_*\mathcal{O}_S$ is surjective and hence $S \to f(S)$ has connected fibers. $\square$

Non-KLT loci

- If (X, B) is a pair and E is a divisor over X such that $a_E(X, B) \leq -1$ (resp. < -1), then we say that E is a **non-KLT place** (resp. a **non-LC place**).
- The image $V = f(E)$ is a **non-KLT center** (resp. **non-LC center**).
- The **non-KLT locus** $\mathrm{NKLT}(X, B)$ is the union of the non-KLT centers.
- If (X, B) is log canonical then we may refer to the non-KLT locus as the **log canonical locus**.
- Next we recall several important results about the structure of non-KLT centers.

Non-KLT loci

Theorem (Kawamata)

If (X, B_0) is KLT and (X, B) is LC and W_1, W_2 are non-KLT centers, then every irreducible component of $W_1 \cap W_2$ is a non-KLT center. In particular there are minimal log canonical centers W at any point $x \in X$ (so that if $x \in W'$ is any LC center, then $W \subset W'$).

Theorem (Kawamata)

If (X, B_0) is KLT and (X, B) is LC and W is a minimal log canonical center, then W is normal and for any ample $\mathbb{Q}$-divisor H on X there is a KLT pair (W, B_W) such that

$$(K_X + B + H)|_W = K_W + B_W.$$

Log canonical thresholds

- If (X, B) is a LC pair and $D \geq 0$ a non-zero $\mathbb{R}$-Cartier divisor. We let
$$\mathrm{lct}((X, B); D) := \sup\{t \geq 0 | (X, B + tD) \text{ is LC}\}.$$
- For example if $X = \mathbb{C}^2$, $B = 0$ and D is the cusp $x^2 - y^3 = 0$, then $\mathrm{lct}((X, B); D) = 5/6$.
- We can check this on a log resolution of $(X, B + D)$ given by blowing up the origin, then the intersection of the exceptional divisor and the strict transform of the cusp and finally the intersection of the 2 exceptional divisors.
- Then we have $K_{X'/X} = E_1 + 2E_2 + 4E_3$ and $f^*D = f_*^{-1}D + 2E_1 + 3E_2 + 6E_3$ so that
$$K_{X'} + \frac{5}{6} f_*^{-1} D + \frac{2}{3} E_1 + \frac{1}{2} E_2 + E_3 = f^*(K_X + \frac{5}{6} D).$$

Log canonical thresholds

- To check this via a local computation, let $x = yt$ then $x^2 - y^3 = y^2(t^2 - y)$ where $\{y = 0\} = E_1$ and and $t^2 - y = 0$ is the strict transform of D.
- The substitution $y = ts$ gives $y^2(t^2 - y) = t^3 s^2(t - s)$ where $t = 0$ is the second exceptional divisor E_2, $\{y = 0\}$ is the strict transform of E_1 and $t = s$ is the strict transform of D.

The Cone Theorem

- In order to discuss the MMP we will need to first recall the Cone Theorem.
- If X is a $\mathbb{Q}$-factorial projective variety (so that every Weil divisor D has an integral multiple mD which is Cartier), then there is a natural equivalence relation on $\mathrm{Div}_{\mathbb{R}}(X)$ the space of $\mathbb{R}$-divisors on X.
- If $D, D' \in \mathrm{Div}_{\mathbb{R}}(X)$, then we say that $D \equiv D'$ iff $D \cdot C = D' \cdot C$ for any curve $C \subset X$.
- We let $N^1(X) = \mathrm{Div}_{\mathbb{R}}(X)/\equiv$.

The Cone Theorem

- Similarly, there is an equivalence relation on the space of curves $A_1(X)$ and we let $N_1(X) = A_1(X)/\equiv$ so that N_1 and N^1 are dual $\mathbb{R}$-vector spaces of dimension $\rho = \rho(X)$.
- We let $NE(X) \subset N_1(X)$ be the cone generated by linear combinations of curves $C = \sum c_i C_i$ where $c_i \in \mathbb{R}_{\geq 0}$.

The Cone Theorem

Theorem

Let (X, B) be a KLT pair. Then

1. *There is a countable collection of rational curves C_i such that* $\overline{NE}(X) = \overline{NE}(X)_{(K_X+B)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$.
2. *If $\epsilon > 0$ and H is ample, then* $\overline{NE}(X) = \overline{NE}(X)_{(K_X+B+\epsilon H)\geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i]$.
3. *If $F \subset \overline{NE}(X)$ is a $K_X + B$ negative extremal face, then there is a contraction morphism $c_F : X \to W$ which contracts a curve C iff $[C] \in F$.*
4. *If L is Cartier such that $L \cdot C = 0$ for all curves C such that $[C] \in F$, then $L = c_F^* L_W$ for some Cartier divisor L_W on W.*

The Cone Theorem Examples

- $X = \mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$.
- $\overline{NE}(X) =< F, \Sigma >$ where F is a fiber and Σ is a section with $\Sigma^2 = -n$.
- We have $F \cong \mathbb{P}^1$ and $K_X \cdot F = -2$ and the contraction of F is the projection $\mathbb{F}_n \to \mathbb{P}^1$.
- $\Sigma \cong \mathbb{P}^1$, $(K_X + (1-\epsilon)\Sigma) \cdot \Sigma = -2 - \epsilon\Sigma^2 = -2 + \epsilon n < 0$ for $0 < \epsilon \ll 1$ and the contraction of Σ is a morphism to a cone over a rational curve of degree n.
- **Ex.** If $Z \subset X$ is a smooth subvariety of a smooth variety, then show that the blow up $\mathrm{B}_Z X \to X$ is a contraction of an extremal ray.
- **Ex.** If $E \subset X$ is a -1 curve on a surface, then show that it generates a negative extremal ray.

The Cone Theorem Examples

- More generally if $X \subset \mathbb{P}^r$ is a smooth variety and Y is the cone over X and Y' is the blow up of this vertex, then the strict transform of any line through the vertex is a $K_{Y'}$-negative extremal ray whose contraction correspond to the projection $Y' \to X$.
- A general intersection of ample divisors gives a curve C on the exceptional divisor of $Y' \to Y$ which generates the $K_{Y'} + (1-\epsilon)E$ negative extremal ray corresponding to this morphism.

The Cone Theorem Examples

- Assume that $K_X \equiv \lambda H$ where H is the hyperplane class, then K_Y is $\mathbb{Q}$-Cartier.
- We have $K_{Y'}|_E \cong (K_{Y'} + E - E)|_E \cong K_X + H = (1+\lambda)H$.
- If $1+\lambda < 0$, then Y is klt, $K_{Y'} \cdot C < 0$ and C is numerically equivalent to a multiple of a rational curve.
- This fails if $1+\lambda > 0$.

The Cone Theorem Examples

- Let $X \subset \mathbb{P}^n$ be a smooth projective variety and $C \subset \mathbb{A}^{n+1}$ the cone over X with vertex $v \in C$.
- **Ex.** Show that X is normal iff $H^0(\mathcal{O}_{\mathbb{P}}(m)) \to H^0(\mathcal{O}_X(m))$ is surjective for every $m \geq 0$.
- **Ex.** If D is a divisor on X and C_D the corresponding cone divisor on C, then show that (C, C_B) is a pair (i.e. $K_C + C_B$ is $\mathbb{R}$-Cartier) iff $K_X + B \sim_{\mathbb{R}} rH$ where $r \in \mathbb{R}$ and H is the hyperplane class.
- **Ex.** For what values of r is (C, C_B) LC, KLT, canonical, terminal?
- **Ex.** Show that if X is Fano, then $\mathrm{NE}(X)$ is a generated by finitely many rays.

The Cone Theorem Examples

- Let X be the blow up of $\mathbb{P}^2$ along 9 points given by the intersection of two general cubics.
- This way we obtain an elliptic fibration $X \to \mathbb{P}^1$ where each exceptional divisor E_i corresponds to a section.
- Translating by elements of the generic fiber (which is an elliptic curve) gives infinitely many sections that are always -1 curves.
- Similar examples come from K3 surfaces with $\rho \geq 3$ and no -2 curves or abelian surfaces with $\rho \geq 3$.

Running the MMP

- **Input**: a pair (X, B) with $\mathbb{Q}$-factorial KLT (or canonical or terminal) singularities.
- If $K_X + B$ is nef, then X is a minimal model and we stop.
- Otherwise, by the Cone Theorem, there is a $K_X + B$ negative extremal ray F so that $F = \mathbb{R}_{\geq 0}[C]$, $(K_X + B) \cdot C < 0$ and a nef divisor H such that $F = \overline{NE}(X) \cap H^{\perp}$.
- Consider $c_F : X \to W$ the contraction defined by the Cone Theorem.
- If $\dim W < \dim X$, then $-(K_X + B)$ is ample over W and we have a Mori fiber space (note that moreover $\rho(X/W) = 1$).
- The fibers are KLT Fano's and hence $K_X + B$ is not pseudo-effective and X is uniruled.

Running the MMP

- If $\dim W = \dim X$ and $X \to W$ contracts a divisor, then W is $\mathbb{Q}$-factorial with klt singularities.
- To see this, let E be an exceptional divisor.
- **Ex.** Show that E is irreducible.
- By the negativity lemma E is not numerically trivial over W and so E generates $\mathrm{Pic}(X/W)$. For any divisor D_W on W with strict transform $D_X = (c_F^{-1})_* D_W$ we can pick $e \in \mathbb{Q}$ such that $D_X + eE \equiv_W 0$.
- But then $D_X + eE \sim_{\mathbb{Q},W} c_F^* D_W$ and D_W is $\mathbb{Q}$-Cartier by the Cone Theorem.
- Thus W is $\mathbb{Q}$-factorial.

Running the MMP

- We have $B_W = c_{F*}(B)$ and $K_X + B = c_F^*(K_W + B_W) + aE$.
- Thus $-aE$ is ample over W and so $a > 0$.
- It then follows easily that for any divisor F over W we have $-1 < a_F(X, B) \leq a_F(W, B_W)$ with strict inequality if the center of F is contained in E. Thus (W, B_W) is KLT.
- Thus we may replace (X, B) by (W, B_W) and repeat the process (this is the analog of contracting -1 curves on surfaces).
- If $\dim W = \dim X$ and $X \to W$ contracts no divisor, then $K_W + B_W$ is not $\mathbb{R}$-Cartier and so W is not $\mathbb{Q}$-factorial with KLT singularities.

Running the MMP

- Assume to the contrary that $K_W + B_W$ is $\mathbb{R}$-Cartier, then $c_F^*(K_W + B_W) = K_X + B$ so that $0 = C \cdot c_F^*(K_W + B_W) = C \cdot (K_X + B)$ a contradiction.
- Thus we can not replace (X, B) by (W, B_W) and so we look for an alternative approach.
- Suppose that $R(K_X + B/W) := \oplus_{m \geq 0} c_{F,*}\mathcal{O}_X(m(K_X + B))$ is a finitely generated $\mathcal{O}_W$-algebra, then we define $X^+ = \mathrm{Proj} R(K_X + B/W)$.
- The rational map $\phi : X \dashrightarrow X^+$ is a small birational morphism (an isomorphism over $X \setminus \mathrm{Ex}(c_F)$) such that $X^+ \to W$ contracts no divisors), $K_{X^+} + B^+$ is ample over Z and $a_E(X, B) \leq a_E(X^+, B^+)$ for all E over X.

Running the MMP

- Let $p : X' \to X$ be a log resolution of $R(K_X + B/Z)$, then $p^*(K_X + B) = M + F$ where $kM = q^*\mathcal{O}_{X^+}(1)$ is free for some $k > 0$ and $kF = \mathrm{Fix}|k(M + F)| \geq 0$ and $q : X' \to X^+$.
- We may assume that in fact $krF = \mathrm{Fix}|kr(M + F)|$ for $r \in \mathbb{N}$.
- It is easy to see that since $|tkM| + tk(F + E) = |tkM + tk(F + E)|$ for all $t > 0$, and any p-exceptional divisor E, then $q : X' \to X^+$ contracts E and so $X^+ \to Z$ is also small.
- **Ex.** Verify the above assertion.
- But then $q_*M \equiv K_{X^+} + B^+$ and $a_E(X, B) \leq a_E(X^+, B^+)$ for every divisor E over X.
- The proof that X^+ is $\mathbb{Q}$-factorial with $\rho(X) = \rho(X^+)$ is similar to the divisorial contraction case

Running the MMP

- We may thus replace (X, B) by (X^+, B^+).
- It is not clear that this procedure terminates, however one sees that there are at most finitely many divisorial contractions.
- In fact a divisorial contraction decreases $\rho(X)$ by 1, but $\rho(X)$ is not changed by flips.
- The existence of flips was proven in [BCHM2010] and so one of the main open problems in the MMP is to show that there are no infinite sequences of flips.

Finite generation

Theorem (Birkar, Cascini, Hacon, M^{c}Kernan, Siu 2010)

Let $f: X \to Z$ be a projective morphism and (X, B) a KLT pair, then the canonical ring $R(K_X + B/Z)$ is finitely generated (as an $\mathcal{O}_Z$ algebra).

Corollary (Birkar, Cascini, Hacon, M^{c}Kernan)

If (X, B) a KLT pair and $f: X \to Z$ a flipping contraction, then the flip $\phi: X \dashrightarrow X^+$ exists.

Finite generation

Corollary (Birkar, Cascini, Hacon, M^{c}Kernan)

Let $f : X \to Z$ be a a projective morphism and (X, B) a KLT pair, such that $K_X + B$ is big (resp. $K_X + B$ is not PSEF) then there is a canonical model $X_{\rm can}$ and a minimal model $X_{\rm min}$ for (X, B) (resp. there is a Mori fiber space $f : X_N \to Z$).

Canonical models

- If $K_X + B$ is big, then the **canonical model** $X_{\mathrm{can}} := \mathrm{Proj}(R(K_X + B))$ is a distinguished "canonical" (unique) representative of the birational equivalence class of X which is defined by the generators and relations in the finitely generated ring $R(K_X + B)$.
- X_{can} may be singular, but its singularities are mild (klt or even canonical if $B = 0$). In particular they are cohomologically insignificant so that e.g. $H^0(\mathcal{O}_X) \cong H^0(\mathcal{O}_{X_{\mathrm{can}}})$.
- Even if X is smooth and $B = 0$, the "canonical line bundle" is now a $\mathbb{Q}$-line bundle which means that $n(K_{X_{\mathrm{can}}} + B_{\mathrm{can}})$ is Cartier for some $n > 0$.
- $K_X + B$ is ample so that $m(K_X + B) = \phi_m^* \mathcal{O}_{\mathbb{P}^N}(1)$ for some $m > 0$.

Goals

- The primary gol of this lecture series is to prove that the existence of minimal models in dimension $n-1$ implies the existence of flips in dimension n.
- To do this we will 1) prove several delicate extension theorems, 2) We will use Shokurov's existence of Pl-flips (not proven in these notes but see [Corti], 3) we will show that the existence of pl-flips is implied by the finite generation of the restricted algebra, and 4) we will show that the existence of minimal models in dimension $n-1$ implies the finiteness of such minimal models.
- A secondary goal is to introduce the MMP with scaling and explain why every mmp with scaling has to terminate (under mild hypothesis). Note that to do this we will assume the existence of minimal models in dimension n, however for the

Outline of the talk

Kawamata Viehweg vanishing

- We will not prove the Cone Theorem here (see Kawamata-Matsuda-Matsuki and Kollár-Mori (1998)) however let us prove the Non-Vanishing Theorem and give an idea of the proof of the Base Point Free Theorem which are key steps in proving the Cone Theorem.
- Before doing this, we recall the Kawamata-Viehweg vanishing theorem.

Theorem (Kawamata-Viehweg Vanishing)

Let $f: X \to Y$ be a proper morphism, (X, B) a klt pair and N a $\mathbb{Q}$-Cartier integral Weil divisor such that $N - (K_X + B)$ is f-nef and f-big, then $R^i f_* N = 0$ for all $i > 0$.

Kawamata Viehweg vanishing

- Recall that an $\mathbb{R}$-Cartier divisor N is nef if $N \cdot C \geq 0$ for any curve C.
- In general N is big if $\limsup(h^0(mD)/m^d) > 0$ where $d = \dim X$. It is well known that if $\limsup(h^0(mD)/m^d) > 0$ then $\limsup(h^0(mD)/m^d) = \lim(h^0(mD)/m^d)$.
- It is also known that N is big iff $N \equiv A + E$ where A is ample and $E \geq 0$.
- If N is nef then it is nef and big iff $N^{\dim X} > 0$.
- N is pseudo-effective iff $N + A$ is big for any ample Q-divisor A.
- The relative versions are defined similarly. N is f-nef if $N \cdot C \geq 0$ for all f-vertical curves.

Kawamata Viehweg vanishing

- Note that if N is f-ample, then $N + f^*H$ is ample for any sufficiently ample divisor H on Y.
- However it is not the case that if N is f-nef, then $N + f^*H$ is ample for any sufficiently ample divisor H.
- Consider in fact $X = E \times E$ the product of a general elliptic curve with itself and $f : X \to E$ the first projection.
- Let F be the class of a fiber which is an extremal ray of $N_E(X)$ cut out by f^*H where H is ample on E.
- Since $N_E(X)$ is a circular cone, we may pick N such that $N \cdot F = 0$ and $N^{\perp}$ intersects the interior of $N_E(X)$.
- It is easy to see that $(N + kf^*H)^{\perp}$ is not nef for any $k > 0$.

Kawamata Viehweg vanishing

- Note that one can deduce this result easily from the traditional statement where X is a smooth variety B is SNC and $Y = \mathrm{Spec}(\mathbb{C})$.
- We assume for simplicity that Y is projective.
- **Step 1:** We may assume (X, B) is log smooth and N is f-ample.
- Since $N - (K_X + B)$ is f-big, we may assume that $N - (K_X + B) \equiv_Y A + E$ where A is ample and $E \geq 0$.
- Thus $N - (K_X + B + \epsilon E) \equiv_Y (1 - \epsilon)(N - (K_X + B)) + \epsilon A$ is ample.
- Replacing B by $B + \epsilon E$ for some $0 < \epsilon \ll 1$, we may assume that $N - (K_X + B)$ is f-ample.

Kawamata Viehweg vanishing

- If $\nu : X' \to X$ is a log resolution and $K_{X'} + B_{X'} = \nu^*(K_X + B)$, then we let $N' = \lceil \nu^* N - B_{X'} - \epsilon F \rceil$ and $B' = \{B_{X'} - \nu^* N + \epsilon F\}$ for $0 < \epsilon \ll 1$.
- Thus (X', B') is klt and

$$N' - (K_{X'} + B') = \nu^* N - B_{X'} - \epsilon F - K_{X'} = \nu^*(N - K_X - B) - \epsilon F$$

is $f \circ \nu$-ample.

Kawamata Viehweg vanishing

- Since (X, B) is klt, $\lfloor B_{X'} \rfloor \leq 0$ and ν-exceptional so $N' - \lfloor \nu^* N \rfloor = \lceil \nu^* N - B_{X'} \rceil - \lfloor \nu^* N \rfloor \geq 0$ is ν-exceptional, it follows that $\nu_* \mathcal{O}_{X'}(N') = \mathcal{O}_X(N)$.
- The inclusion $\subset$ is clear. The converse follows since if $g \in \mathbb{C}(X)$ satisfies $(g) + N \geq 0$, then $\nu^*(g) + \lfloor \nu^* N \rfloor \geq 0$ (as $\nu^*(g)$ is Cartier) and so $\nu^*(g) + N' \geq 0$.
- By the log smooth case $R^i \nu_* \mathcal{O}_{X'}(N') = 0$ for $i > 0$.
- But then $R^j f_* \mathcal{O}_X(N) = R^j (f \circ \nu)_* \mathcal{O}_{X'}(N') = 0$ for $j > 0$.

Kawamata Viehweg vanishing

- **Step 2:** We may assume (X, B) is log smooth $Y = \mathrm{Spec}(\mathbb{C})$.
- Let H be ample on Y, then $N + kf^*H - (K_X + B)$ is ample for any $k \gg 0$ and so by assumption $H^i(\mathcal{O}_X(N + kf^*H)) = 0$ for any $i > 0$.
- We may assume that $H^i(R^j f_* \mathcal{O}_X(N) \otimes \mathcal{O}_Y(kH)) = 0$ for $i > 0$ and $R^j f_* \mathcal{O}_X(N) \otimes \mathcal{O}_Y(kH)$ is generated by global sections.
- But then, for $i > 0$, $0 = H^i(\mathcal{O}_X(N + kf^*H)) = H^0(R^i f_* \mathcal{O}_X(N) \otimes \mathcal{O}_Y(kH))$.
- As $R^i f_* \mathcal{O}_X(N) \otimes \mathcal{O}_Y(kH)$ is globally generated, it must also vanish.
- Since $\mathcal{O}_Y(kH)$ is invertible, $R^i f_* \mathcal{O}_X(N) = 0$. $\square$

Non vanishing

Theorem (Non-vanishing theorem)

Let (X, B) be a projective sub-KLT pair and D a nef Cartier divisor such that $aD - (K_X + B)$ is nef and big for some $a > 0$, then $H^0(\mathcal{O}_X(mD - \lfloor B \rfloor)) \neq 0$ for all $m \gg 0$.

Note that then $a'D - (K_X + B)$ is nef and big for any $a' \geq a$ (as the sum of a nef and big divisor and a big divisor is nef and big).

Non vanishing

- **Step 0.** We may assume that X is smooth and $aD-(K_X+B)$ is ample.
- To this end, let $f:X'\to X$ be a log resolution, $D'=f^*D$ and $K_{X'}+B'=f^*(K_X+B)$.
- Then $aD'-(K_{X'}+B')=f^*(aD-(K_X+B))$ is nef and big so that $aD'-(K_{X'}+B')\sim_{\mathbb{Q}} A+F$ where A is ample and $F\geq 0$.
- For any $0<\epsilon\ll 1$, $(X',B'+\epsilon F)$ is sub-KLT and

$$aD'-(K_{X'}+B'+\epsilon F)\sim_{\mathbb{Q}}(1-\epsilon)(D'-(K_{X'}+B'))+\epsilon A$$

  is ample.
- As $f_*(B'+\epsilon F)\geq B$, we have

$$h^0(\mathcal{O}_{X'}(mD'-\lfloor B'+\epsilon F\rfloor))\leq h^0(\mathcal{O}_X(mD-\lfloor B\rfloor)).$$

Non vanishing

- **Step 1.** We may assume that $D \not\equiv 0$.
- Suppose $D \equiv 0$, then for $k, t \in \mathbb{Z}$, we have

$$kD - \lfloor B \rfloor \equiv K_X + \{B\} + tD - (K_X + B).$$

- Since $tD - (K_X + B)$ is ample, by K-V vanishing

$$h^0(mD - \lfloor B \rfloor) = \chi(mD - \lfloor B \rfloor) = \chi(-\lfloor B \rfloor) = h^0(-\lfloor B \rfloor) \neq 0.$$

Non vanishing

- **Step 2.** For any $x \in (X \setminus \mathrm{Supp}(B))$ there is an integer q_0 such that if $q \in \mathbb{N}_{\geq q_0}$, then there exists a $\mathbb{Q}$-divisor

$$M(q) \equiv qD - (K_X + B), \qquad \text{with} \qquad \mathrm{mult}_x M(q) > 2 \dim X.$$

- If $d = \dim X$ and A is ample, then $D \cdot A^{d-1} \neq 0$ (as $D \not\equiv 0$).
- Since D is nef, $D^e \cdot A^{d-e} \geq 0$ and so $(qD - K_X - B)^d = ((q-a)D + aD - K_X - B)^d > d(q-a)D \cdot (aD - K_X - B)$.

Non vanishing

- Thus $\lim(RHS) = +\infty$ as $q \to +\infty$. By Riemann-Roch, we have

$$h^0(\mathcal{O}_X(e(qD - K_X - B))) > \frac{e^d}{d!}(2d)^d + O(e^{d-1}).$$

- Vanishing at x to multiplicity $> 2de$ imposes at most

$$\binom{2de+d}{d} = \frac{(2de)^d}{d!} + O(e^{d-1}) \qquad \text{conditions.}$$

Non vanishing

- Thus there is a divisor

  $$M(q,e) \in |e(qD - K_X - B)| \qquad \text{with} \qquad \mathrm{mult}_x M(q,e) > 2de.$$

- We let $M = M(q) = M(q,e)/e$.
- **Step 3.** Let $f : X' \to X$ be a log resolution of $(X, B+M)$, $F \geq 0$ an f-exceptional such that $-F$ is $f-$ *ample*.
- Fix $0 < \epsilon \ll 1$ and let

  $$t := \sup\{\tau > 0 | (X', B' + \epsilon F + \tau f^* M) \text{ is LC}\}.$$

  Then $t < 1/2$.
- Since $A := qD - K_X - B$ is ample, we may assume that $f^*A - \epsilon F$ is ample and $(X', B' + \frac{\epsilon}{2} F + tf^*M)$ has a unique non-klt center $E = (B' + \frac{\epsilon}{2} F + tf^*M)^{=1}$.

Non vanishing

- Let $D' = f^*D$ and $K_{X'} + E + \Delta = f^*(K_X + B + tM) + \frac{\epsilon}{2}F$ so that $f_*\lfloor \Delta \rfloor \geq \lfloor B \rfloor$ and $\lfloor \Delta \rfloor \leq 0$.
- $mD' - E - \lfloor \Delta \rfloor - (K_{X'} + \{\Delta\}) \equiv$ $f^*((m - tq)D - (1 + t)(K_X + B)) - \frac{\epsilon}{2}F$ is ample and so it follows that $H^1(mD' - E - \lfloor \Delta \rfloor) = 0$.
- Note that $K_E + (\Delta)|_E = K_E + (\lfloor \Delta \rfloor + \{\Delta\})|_E$ is sub klt.
- $mD'|_E - (K_E + (\Delta)|_E) \equiv$ $(f^*((m - tq)D - (1 + t)(K_X + B)) - \frac{\epsilon}{2}F)|_E$ is ample.
- By induction $H^0(\mathcal{O}_E(mD' - \lfloor \Delta \rfloor)) \neq 0$ for all $m \gg 0$.

Non vanishing

- From the short exact sequence
$0 \to mD' - E - \lfloor \Delta \rfloor \to mD' - \lfloor \Delta \rfloor \to (mD' - \lfloor \Delta \rfloor)|_E \to 0$
we see that $|mD' - \lfloor \Delta \rfloor| \neq \emptyset$ for $m \gg 0$.
- Since $f_*(-\lfloor \Delta \rfloor) \leq -\lfloor B \rfloor$, we have $|mD - \lfloor B \rfloor| \neq \emptyset$ for $m \gg 0$. $\square$

Base Point Free Theorem

The next result that plays an important role in the Cone Theorem is the Base Point Free Theorem

Theorem

Let $f : X \to Z$ be a proper morphism, (X, B) a klt pair and D an f-nef divisor such that $aD - (K_X + B)$ is f-nef and f-big for some $a > 0$, then $|bD|$ is f-free for all $b \gg 0$ (i.e. $f^*(f_*\mathcal{O}_X(bD)) \to \mathcal{O}_X(bD)$ is surjective.

- The main idea is that by the non-vanishing theorem $|bD| \neq \emptyset$.
- Let $G \in |D|_{\mathbb{Q}}$ be a very general element and assume that $(X, B + tG)$ is LC but not klt.

Base Point Free Theorem

- We may write $aD - (K_X + B) \sim_{\mathbb{Q}} A + E$ where A is ample and $E \geq 0$.
- Thus $mD - (K_X + B + \epsilon F) = (m-a)D + (1-\epsilon)(aD - (K_X + B)) + \epsilon A$ is ample for $m \geq a$.
- By a perturbation argument, we may assume that $(X, B + tG)$ has a unique log canonical place E whose center is contained in $\mathbf{B}(D)$.
- Assume for simplicity that E is a smooth divisor on X, then there is a short exact sequence

$$0 \to mD - E \to mD \to mD|_E \to 0.$$

Base Point Free Theorem

- Let $aD - K_X - B \sim_{\mathbb{Q}} A + F$ where $F \geq 0$ and A is ample.
- Since $mD - E \sim_{\mathbb{Q}} (m-t)D + K_X + \{B + tG\}$ where $(X, \{B + tG\} + \epsilon F)$ is klt for any $0 < \epsilon \ll 1$ and

$$(m-t)D - \epsilon F \sim_{\mathbb{Q}} (m-t-\epsilon)D + \epsilon A$$

  is ample for $m > t + \epsilon$, $H^1(mD - E) = 0$ and so $|mD| \to |mD|_E|$ is surjective.
- By induction on the dimension we have that $|mD|_E| \neq \emptyset$ and so E is not contained in $\mathrm{Bs}(mD)$ a contradiction.
- We will not give the proof of the Cone Theorem. We refer the reader to Kollár-Mori 1998 for the details.

Base Point Free Theorem

- However we make the following remarks:
- If F is a $K_X + B$-negative extremal face, then there is a nef divisor N such that $\overline{NE}(X) \cap N^{\perp} = F$.
- It follows that $(aN - (K_X + B)) \cdot C > \epsilon H \cdot C$ for all curves in $\overline{NE}(X)$ and so $aN - (K_X + B)$ is ample.
- By the Base Point Free theorem N is semiample and so it defines a morphism $c_F : X \to Z$ contracting curves in F.
- Again by the Base Point Free theorem, if L is Cartier such that $L \cdot C = 0$ iff $[C] \in F$, then L is nef and big over Z.
- By the relative version of the BPF theorem $mL = c_F^*(L_{Z,m})$ for all $m \gg 0$ where $L_{Z,m}$ is Cartier and so $L = c_F^*(L_{Z,m+1} \otimes L_{Z,m}^{\vee})$.

Outline of the talk

MMP with scaling

- We start with a $\mathbb{Q}$-factorial klt pair $(X, B+H)$ such that $K_X + B + H$ is nef and B is big.
- Typically we may choose H to be sufficiently ample.
- If $K_X + B$ is big and KLT but B is not big, pick a divisor $G \in |K_X + B|_{\mathbb{Q}}$. Then $K_X + B + \epsilon G \sim_{\mathbb{Q}} (1+\epsilon)(K_X + B)$.
- Then $(X, B + \epsilon G)$ is KLT for $0 < \epsilon \ll 1$ and every step of the $K_X + B + \epsilon G$ MMP is a step of the $K_X + B$ MMP. Therefore, replacing B by $B + \epsilon G$, we may assume that B is big.
- Let $t = \inf\{t' \geq 0 | K_X + B + t'H \text{ is nef}\}$.
- If $t = 0$, we are done: $K_X + B$ is nef.
- Otherwise there is an extremal ray R such that $(K_X + B + tH) \cdot R = 0$ and $H \cdot R > 0$.

MMP with scaling

- We now perform $f : X \dashrightarrow Z$ the corresponding contraction guaranteed by the Cone Theorem.
- If $\dim Z < \dim X$, then $X \to Z$ is a Mori fiber space (as $(K_X + B) \cdot R = -tH \cdot R < 0$.
- If $\dim Z = \dim X$ then we replace X by the corresponding divisorial contraction or flip $\phi : X \dashrightarrow X'$.
- Note that by the Cone Theorem $K_X + B + tH = f^*(K_Z + B_Z + tH_Z)$ and hence $K_{X'} + B' + tH' = \phi_*(K_X + B + tH)$ is also nef.
- Repeating this procedure we obtain a sequence of rational numbers $t_1 \geq t_2 \geq t_3 \geq \ldots$ and birational contractions $X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \ldots$ such that $K_{X_i} + B_i + t_i H_i$ is nef or equivalently $X \dashrightarrow X_i$ is a minimal model for $(X, B + t_i H)$.

MMP with scaling

- Suppose that we can show that there are only finitely many such minimal models for $t \in [0,1]$, then it follows that $X_i \dashrightarrow X_j$ is an isomorphism (which identifies B_i with B_j) for some $i > j$.
- This immediately leads to a contradiction because as we have seen, for any divisor E over X we have $a_E(X_i, B_i) \geq a_E(X_j, B_j)$ and the inequality is strict for any divisor with center contained in a flipping locus.
- We will now show that this is the case assuming the existence of minimal models.

Minimal Models

Definition

Let (X, B) be a klt pair and $f : X \dashrightarrow X'$ a proper birational map of $\mathbb{Q}$-factorial varieties such that ϕ^{-1} contracts no divisors. Let $p : Y \to X$ and $q : Y \to X'$ and $B' = \phi_* B$.
If $p^*(K_X + B) - q^*(K_{X'} + B') \geq 0$, then we say that ϕ is $K_X + B$ **non-positive**. If moreover the support of $p^*(K_X + B) - q^*(K_{X'} + B')$ contains all ϕ-exceptional divisors, then ϕ is $K_X + B$ **negative**.
If ϕ is $K_X + B$ negative and $K_{X'} + B'$ is nef, then we say that ϕ is a **minimal model** for (X, B). If ϕ is $K_X + B$ non-positive and $K_{X'} + B'$ is ample (resp. nef), then we say that ϕ is an **ample model** (resp. a **weak log canonical model**) for (X, B).

Minimal Models

- We have already seen that a sequence of flips yields a $K_X + B$ negative rational map.
- Conjecturally (assuming termination of flips), if $K_X + B$ is klt and pseudo-effective the MMP gives a minimal model of (X, B) say $\phi : X \dashrightarrow X'$ via a finite sequence of flips and divisorial contractions.

Theorem (Existence of Minimal Models)

If $K_X + B$ is a $\mathbb{Q}$-factorial pseudo-effective klt pair such that B is big, then there exists a minimal model $\phi : X \dashrightarrow X'$.

We will see that this result implies the termination of flips with scaling by showing a result on the finiteness of minimal models.

Minimal Models

Claim

If (X, B) is a minimal model and B is big then $R(K_X + B)$ is finitely generated.

- In fact $K_X + B$ is nef and we may write $B \sim_{\mathbb{Q}} A + E$ where A is ample and $E \geq 0$.
- But then $(X, B + \epsilon E)$ is klt for $0 < \epsilon \ll 1$ and $2(K_X + B) - (K_X + B + \epsilon E) \sim_{\mathbb{Q}} (1 - \epsilon)(K_X + B) + \epsilon A$ is ample.

Minimal Models

- By the Base Point Free Theorem $K_X + B$ is semiample. Let $f : X \to Z$ be the induced morphism with $f_*\mathcal{O}_X = \mathcal{O}_Z$, then $m(K_X + B) = f^*\mathcal{O}_Z(1)$ for some $m > 0$.
- But then $R(m(K_X + B)) \cong R(\mathcal{O}_Z(1))$ is finitely generated and hence so is $R(K_X + B)$ (by theorem of E. Noether, since $R(K_X + B)$ is integral over $R(m(K_X + B))$).

Finite generation

Claim

If (X, B) is klt then $R(K_X + B)$ is finitely generated.

- If (X, B) is klt and $R(K_X + B) \neq \mathbb{C}$, then let $f : X \dashrightarrow Z$ be the Iitaka fibration (so that $f_*\mathcal{O}_X = \mathcal{O}_Z$, $\dim Z = \kappa(K_X + B) := \mathrm{tr.deg.}_{\mathbb{C}} R(K_X + B) - 1$ and $\kappa(F) = 0$).
- We may assume that f is a morphism and by a result of Fujino-Mori $R(K_X + B) \cong R(K_Z + B_Z + M_Z)$ where (Z, B_Z) is KLT and M_Z is nef.
- Since $K_Z + B_Z + M_Z$ is big, we may write $K_Z + B_Z + M_Z \sim_{\mathbb{Q}} A + E$ where A is ample and $E \geq 0$.

Finite generation

- But then

  $$(1+\epsilon)(K_Z+B_Z+M_Z)=K_Z+B_Z+\epsilon E+M_Z+\epsilon A\sim_{\mathbb{Q}} K_Z+B'_Z$$

  where (Z,B'_Z) is klt and $K_Z+B'_Z$ is big.
- Thus $R(K_Z+B'_Z)$ is finitely generated.
- The claim now follows as for $m\in N$ sufficiently divisible, $m'=m(1+\epsilon)\in\mathbb{N}$ and $R(m'(K_X+B))\cong R(m'(K_Z+B_Z+M_Z))\cong R(m(K_Z+B'_Z))$.

Minimal Models

- Suppose that $\phi : X \dashrightarrow X'$ is a minimal model of (X, B) where B is big, then $B' = \phi_* B$ is also big and by what we have seen above $K_{X'} + B'$ is semiample.
- Let $p : W \to X$ and $q : W \to X'$ be a common resolution. Then $p^*(K_X + B) = q^*(K_{X'} + B') + E$ where E is q-exceptional.
- Since ϕ is $K_X + B$ non-positive, $p_* E \geq 0$ consists of all divisors contracted by ϕ.
- By the negativity lemma, $E \geq 0$.
- It follows that $R(K_X + B) \cong R(K_X + B')$.
- Thus $\mathbf{B}(p^*(K_X + B)) = \mathbf{B}(q^*(K_{X'} + B') + E) = E$ and $\mathbf{B}(K_X + B) = p(E)$.

Finite generation

- It follows that two different minimal models of a KLT pair (X, B) are isomorphic in codimension 1.
- In fact different minimal models are connected by flops i.e. small birational $K_X + B$-trivial maps.

Theorem (Kawamata)

Let (X, B) be a KLT pair and $\phi : X \dashrightarrow Y$, $\phi' : X \dashrightarrow Y'$ two minimal models, then there is a finite sequence of $(K_X + B)$-flops $Y = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \ldots \dashrightarrow X_N = Y'$.

We will check this later on for KLT pairs of general type. It is also known that KLT pairs of general type have finitely many minimal models (this is not true for pairs of intermediate Kodaira dimension).

Minimal Models

Theorem

Let X be a $\mathbb{Q}$-factorial projective variety, A an ample $\mathbb{Q}$-divisor, $\mathcal{C} \subset \mathrm{Div}_{\mathbb{R}}(X)$ a rational polytope such that for any $B \in \mathcal{C}$, $K_X + B$ is KLT. Then there exist finitely many proper birational maps $\phi_i : X \dashrightarrow X_i$ with $1 \le i \le k$ such that if $B \in \mathcal{C}$ and $K_X + B + A$ is PSEF, then

1. *ϕ_i is a minimal model of $(X, B + A)$ for some $1 \le i \le k$.*
2. *If $\phi : X \dashrightarrow X'$ is a minimal model of $(X, B + A)$ for some $B \in \mathcal{C}$, then $\phi_i \circ \phi^{-1} : X' \dashrightarrow X_i$ is an isomorphism for some $1 \le i \le k$.*

Proof of (1). We proceed by induction on the dimension of $\mathcal{C}$. The case when $\dim = 0$ is just the theorem on the existence of minimal models.

Minimal Models

- Note that the cone of pseudo-effective divisors is closed and $\mathcal{C} \cap \mathrm{PSEFF}(X)$ is compact thus it suffices to prove the claim locally around any point $B_0 \in \mathcal{C} \cap \mathrm{PSEFF}(X)$.
- **Step 1.** We may assume that $K_X + B_0 + A$ is nef and klt.
- Consider $\phi : X \dashrightarrow X'$ a minimal model for $(X, B_0 + A)$, then $(X', \phi_*(B + A))$ is klt and $a_E(X, B + A) < a_E(X', \phi_*(B + A))$ for any ϕ-exceptional divisor E and any $B \in \mathcal{C}$ such that $||B - B_0|| \ll 1$.
- It follows easily that if $\psi : X' \dashrightarrow Y$ is a minimal model for $K_{X'} + \phi_*(B + A)$, then $\psi \circ \phi : X \dashrightarrow Y$ is a minimal model for $K_X + B + A$.
- Thus we may replace X by X' and assume $K_X + B_0 + A$ is nef.

Minimal Models

- **Step 2.** We may assume that $K_X + B_0 + A \sim_{\mathbb{Q}} 0$.
- By the Base Point Free theorem, $K_X + B_0 + A$ is semiample. Let $\psi : X \to Z$ be the induced morphism, then $K_X + B_0 + A \sim_{\mathbb{Q}} = \psi^* H$ where H is ample on Z.
- After shrinking $\mathcal{C}$ in a neighborhood of B_0, we may assume that that any minimal model over Z for $K_X + B + A$ for $B \in \mathcal{C}$ is also a minimal model for $K_X + B + A$.
- If this is not the case then there is a curve C with $0 > (K_X + B + A) \cdot C \geq -2 \dim X$. Note that $(K_X + B_0 + t(B - B_0) + A) \cdot C = (1 - t)\psi^* H \cdot C + t(K_X + B + A) \cdot C$.

Minimal Models

- This quantity is non-negative unless $\psi_* C \neq 0$. Suppose that kH is Cartier, then $k\psi^* H \cdot C \geq 1$ and this quantity is non-negative if $t < 1/(4k \dim X)$.
- Working over Z, we may assume that $K_X + B_0 + A \sim_{\mathbb{Q},Z} 0$.

Minimal Models

- **Step 3.** Pick any $\Theta \in \mathcal{C}$ and $B \in \partial\mathcal{C}$ such that
$$\Theta - B_0 = \lambda(B - B_0) \qquad 0 \leq \lambda \leq 1.$$
- Thus $K_X + \Theta + A =$
$\lambda(K_X + B + A) + (1 - \lambda)(K_X + B_0 + A) \sim_{\mathbb{R},Z} \lambda(K_X + B + A)$
so that $K_X + \Theta + A$ is pseudo-effective over Z iff so is $K_X + B + A$.
- Similarly $\phi : X \dashrightarrow X'$ is a minimal model over Z for $K_X + \Theta + A$ iff so is $K_X + B + A$.
- Since the boundary $\partial\mathcal{C}$ has smaller dimension than $\mathcal{C}$ and we are done by induction.

Minimal Models

Theorem

Let $f : X \to Z$ be a projective morphism to a normal affine variety, $K_X + B$ a pseudo-effective klt pair such that, B big over Z. Then

1. *(X, B) has a minimal model $\phi : X \dashrightarrow Y$ over Z.*
2. *If $V \subset \mathrm{WDiv}_{\mathbb{R}}(X)$ is a finite dimensional affine subspace defined over $\mathbb{Q}$ and containing B. There exists a constant $\delta > 0$ such that if P is a prime divisor contained in $\mathbf{B}(K_X + B)$, then P is contained in $\mathbf{B}(K_X + B')$ for any $\mathbb{R}$-divisor $0 \leq qB' \in V$ with $||B - B'|| \leq \delta$.*
3. *If $W \subset \mathrm{WDiv}_{\mathbb{R}}(X)$ is the smallest affine subspace containing B defined over $\mathbb{Q}$. Then there is a real number $\eta > 0$ and an integer $r > 0$ such that if $B' \in W$, $||B - B'|| \leq \eta$ and $k > 0$ is an integer such that $k(K_X + B')/r$ is Cartier, then every component of $\mathrm{Fix}(k(K_X + B'))$ is a component of $\mathbf{B}(K_X + B)$.*

Minimal Models

- (1) We have already seen that (1) follows from the termination of the minimal model with scaling. Note that $R(K_X + B) \cong R(K_Y + \phi_* B)$. Since $K_Y + \phi_* B$ is nef and klt, it is easy to see by the Base Point Free Theorem that $K_Y + \phi_* B$ is semiample.
- Thus there is a projective morphism $\psi : Y \to W$ over Z such that $\psi_* \mathcal{O}_Y = \mathcal{O}_W$ and $m(K_Y + \phi_* B) = \psi^* H$ where H ample over Z.
- Thus $R(m(K_X + B)) \cong R(m(K_Y + \phi_* B)) \cong R(H)$ is finitely generated.
- We now turn our attention to (2). Note that if $K_X + B'$ is not pseudo-effective, then $\mathbf{B}(K_X + B') = X$. Thus we may assume that $K_X + B'$ is pseudo-effective.
- Let $\phi : X \dashrightarrow Y$ be a log terminal model of $K_X + B$.

Minimal Models

- There exists a $\delta > 0$ such that if $||B - B'|| \leq \delta$, then $K_Y + \phi_* B'$ is klt and $a_E(X, B') < a_E(Y, \phi_* B')$ for any ϕ exceptional divisor $E \subset X$.
- If $\psi : Y \to W$ is a minimal model for $(Y, \phi_* B')$, then it is a minimal model for (X, B') and so all ϕ-exceptional divisors are contained in $\mathbf{B}(K_X + B')$, and (2) follows.
- We now prove (3).
- Let $\mathcal{C} = \{B' \in W \ s.t. \ ||B - B'|| \leq \eta\}$ for some $0 < \eta \ll 1$. We claim that $\phi : X \dashrightarrow Y$ is also a minimal model for $K_X + B'$.
- Otherwise there is a $K_X + B'$ negative extremal ray R such that $(K_X + B) \cdot R = 0$. But then $B' \notin W$ a contradiction.
- Since Y is $\mathbb{Q}$-factorial, there is an integer $I > 0$ such that if G is integral on Y, then IG is Cartier.

Minimal Models

- By Kollár's effective base point free theorem, there is an integer $r > 0$ such that if $k(K_Y + \phi_* B')/r$ is Cartier, then $k(K_Y + \phi_* B')$ is generated by global sections.
- Let $p : W \to X$ and $q : W \to Y$ be a common resolution, then $p^*(K_X + B') = q^*(K_Y + \phi_* B') + E$ where $E \geq 0$ is q-exceptional and the support of $p_* E$ is the union of the ϕ-exceptional divisors.
- It follows that the support of $\mathrm{Fix}(k(K_X + B'))$ consits of divisors contained in $\mathbf{B}(K_X + B')$ i.e. of divisors in the support of $p_* E$.
- Since ϕ is a minimal model of $K_X + B$, these divisors are also contained in $\mathbf{B}(K_X + B)$.

Outline of the talk

Birational Minimal Models

- As we mentioned above, it is possible that a KLT pair has two different minimal models, however they are isomorphic in codimension one and they are related by a finite sequence of flops, that is small birational morphisms $\phi : X \dashrightarrow X^+$ which are $K+B$-trivial.
- More precisely there are projective morphisms $f : X \to Z$ and $f : X^+ \to Z$ such that $\rho(X/Z) = \rho(X^+/Z) = 1$ and $K_X + B \equiv_Z 0$ and $K_{X^+} + B^+ \equiv_Z 0$.

Theorem

Let (X, B) be a klt pair and $\phi : X \dashrightarrow Y$, $\phi' : X' \dashrightarrow Y'$ two minimal models, then there exists a finite sequence of flops $Y = Y_0 \dashrightarrow Y_1 \dashrightarrow \ldots \dashrightarrow Y_N = Y'$.

Birational Minimal Models

- We will prove this in the case that B is big (which also implies the case in which $K_X + B$ is big).
- We already know that in this case there are finitely many minimal models for (X, B).
- Let $Z = \mathrm{Proj}(R(K_X + B))$. By the base point free theorem $K_Y + B$ is semiample and so there is a morphism $\psi : Y \to Z$ such that $m(K_Y + \phi_* B) = \psi^* H_Z$ and similarly we have $\psi' : Y' \to Z$ such that $m(K_{Y'} + \phi'_* B) = \psi'^* H_Z$.
- Let $A_{Y'}$ be ample on Y' and A_Y be its strict transform on Y.
- Replacing A_Y by ϵA_Y for $0 < \epsilon \ll 1$, we may assume that $K_Y + B_Y + A_Y$ is KLT (where $B_Y = \phi_* B$).

Birational Minimal Models

- We now run the $K_Y + B_Y + A_Y$ MMP with scaling over Z.
- Since $K_Y + B_Y \equiv_Z 0$, each step is $K_Y + B_Y$ trivial.
- Since $B_Y + A_Y$ is big, this MMP ends with a minimal model $(\bar{Y}, \bar{B} + \bar{A})$ over Z.
- But then $\bar{A}$ is nef and big over Z and hence semiample over Z by the Base Point Free theore.
- So there is a small birational morphism $\nu : \bar{Y} \to Y'$ with $\bar{A} = \nu^* A_{Y'}$.
- Since Y' is $\mathbb{Q}$-factorial, ν is an isomorphism. $\square$

Sarkisov Program

- Consider now a non-pseudo-effective KLT pair (X, B) and two Mori fiber spaces $\phi : X \dashrightarrow Y$ and $\phi' : X \dashrightarrow Y'$ and morphisms $f : Y \to Z$ and $f' : Y' \to Z'$ such that $-(K_Y + B_Y)$ is f ample and $-(K_{Y'} + B_{Y'})$ is f' ample.
- We say that $f : Y \to Z$ and $f' : Y' \to Z'$ are **Sarkisov related**.
- **Link of type I.** Let X be the blow up of $\mathbb{P}^2$ at a point P, and consider $\phi : X \to \mathbb{P}^2$ and $\phi' = id_X$. We have Mori fiber spaces $f : \mathbb{P}^2 \to \mathrm{Spec}(\mathbb{C})$, and $f' : X \to \mathbb{P}^1$ the morphism induced by the natural projection.
- **Link of type II.** Let $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}} \oplus \mathcal{O}_{\mathbb{P}}(n)) \to \mathbb{P}^1$ be the Hirzebruch surface with negative section C such that $C^2 = -n$.

Sarkisov Program

- Blow up a point $P \in C$ to get $\phi : X \to \mathbb{F}_n$, then $C_X^2 = -n-1$ where $C_X = \phi_*^{-1} C$. If F is the fiber containing P, F_X is its strict transform, then $F_X^2 = -1$ and so F_X is a -1 curve which can be contracted $\phi' : X \to \mathbb{F}_{n+1}$.
- Note that if $C' = \phi'_* C$, then $(C')^2 = -(n+1)$.
- We call this birational transformation an **elementary transformation**.
- **Link of type IV.** Let $X = Y = Y' = \mathbb{P}^1 \times \mathbb{P}^1$ and f, f' be the two projections.
- It is well known that if X is rational, then its minimal models are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_n$ with $n \neq 1$. By what we have see, all of these are connected by finitely many links of type I-IV (type III is the inverse of type I).

Sarkisov Program

- In general any two Sarkisov related Mori Fano fiber spaces $f : Y \to Z$ and $f' : Y' \to Z'$ are related by one of the following 4 kinds of arbitrary elementary Sarkisov links.
- Type I: Let $W \to X$ be a divisorial contraction and $W \dashrightarrow Y'$ a sequence of flips and $Z' \to Z$ a morphism with $\rho(Z'/Z) = 1$.
- Type II: $W \to Y$ and $W' \to Y'$ divisorial contractions, $W \dashrightarrow W'$ a sequence of flips and $Z \cong Z'$.
- Tipe III: $X \dashrightarrow W$ a sequence of flips, $W \to Y'$ a divisorial contraction and $Z \to Z'$ a morphism with $\rho(Z/Z') = 1$.
- Type IV: $Y \dashrightarrow Y'$ a sequence of flips and $Z \to W$, $Z' \to W$ morphisms of relative Picard number $1 = \rho(Z/W) = \rho(Z'/W)$.

Birational Minimal Models

- We end this section with two useful constructions.
- Let (X, B) be a klt pair and $f : X' \to X$ a resolution.
- Write $K_{X'} + B' = f^*(K_X + B) + E$ where $B', E \geq 0$ and $B' \wedge E = 0$.
- Let $\mathcal{E}$ be any collection of f-exceptional divisors including the support of E and let $F = \epsilon \sum_{E_i \in \mathcal{E}} E_i$.
- We may run the $K_{X'} + B' + F$ MMP over X, say $X' \dashrightarrow \bar{X}$.
- Then $E + F$ is effective, exceptional and nef over X so that $E + F = 0$.

Birational Minimal Models

- If $\mathcal{E}$ is the set of all f-exceptional divisor, then $\bar{X} \to X$ is a small birational morphism such that $\bar{X}$ is $\mathbb{Q}$-factorial.
- We say that $\bar{X} \to X$ is a $\mathbb{Q}$-**factorialization**.
- If $\mathcal{E} = \{E_i | a_E(X, B) > 0\}$ and $\mathrm{Ex}(f)$ contains all divisors over X with $a_E(X, B) \leq 0$, then $K_{\bar{X}} + \bar{B} = f^*(K_X + B)$ s a terminal pair.
- We say that $\bar{X} \to X$ is a **terminalization**.

Outline of the talk

Flips

- Let (X, B) be a $\mathbb{Q}$-factorial KLT pair.
- A flipping contraction is a small birational projective morphism $f : X \to Z$ such that $f_*\mathcal{O}_X = \mathcal{O}_Z$, $\rho(X/Z) = 1$ and $-(K_X + B)$ is ample over Z.
- The flip of f is a small projective birational morphism $f^+ : X^+ \to Z$ such that $f_*\mathcal{O}_{X^+} = \mathcal{O}_Z$ and $K_{X_+} + B^+$ is ample over Z where $B^+ = \phi_* B^+$ and $\phi : X \dashrightarrow X^+$ is the induced birational morphism.
- By the base point free theorem, it is easy to see that X^+ is $\mathbb{Q}$-factorial and $\rho(X^+/Z) = 1$.

Flips

- Since ϕ is small, then $R(K_X + B/Z) \cong R(K_{X^+} + B^+/Z)$.
- Since $K_{X_+} + B^+$ is ample over Z, then $X^+ = \mathrm{Proj}(R(K_{X^+} + B^+/Z)) = \mathrm{Proj}(R(K_X + B/Z))$ so that the existence of a flip implies the finite generation of $R(K_X + B/Z)$.
- This also shows that the construction is local over Z and so we may assume that Z is affine and hence we can identify $R(K_X + B/Z) \cong \oplus_{m \geq 0} H^0(m(K_X + B))$.
- Shokurov shows that the existence of flips is implied by the more restrictive (and hence easier to prove) existence of **pl flips**. Here pl stands for prelimiting.

PLT pairs

- A plt pair $(X, S+B)$ is a LC pair such that $\lfloor S+B \rfloor = S$ is irreducible and S is the only non-klt place of $(X, S+B)$ i.e. if $a_E(X, S+B) \leq -1$, then $E = S$.
- Define $K_S + B|_S = (K_X + S + B)|_S$ by adjunction..
- If $f : X' \to X$ is a log resolution and $K_{X'} + S' + B' = f^*(K_X + S + B)$ where $S' = f_*^{-1}S$, then we have $K_{S'} + B'|_{S'} = (K_{X'} + S' + B')|_{S'}$.
- By definition $B_S = f_* B'|_{S'}$. It is easy to see that $B_S \geq 0$ and $K_{S'} + B'|_{S'} = f^*(K_S + B_S)$.
- If $(X, S+B)$ is PLT, then $\lfloor B' \rfloor = 0$ so that $\lfloor B'|_{S'} \rfloor = 0$ and hence (S, B_S) is KLT.
- Conversely, if (S, B_S) is KLT, then $\lfloor B'|_{S'} \rfloor = 0$ so that $\lfloor B' \rfloor|_{S'} = 0$. By the connectedness lemma $\lfloor B' \rfloor \cap f^{-1}(S) = 0$ and hence $(X, S+B)$ is PLT in a neighborhood of S.

Pl-Flips

- Let $(X, S+B)$ be a PLT pair such that $\lfloor S+B \rfloor = S$. A flipping contraction $f : X \to Z$ is a **pl flip** if $-S$ is f-ample.
- The **restricted algebra** $R_S(K_X+S+B)$ is given by

$$\oplus_{m\geq 0}\mathrm{Im}\left(H^0(m(K_X+S+B)) \to H^0(m(K_S+B_S))\right).$$

- Here we are assuming Z is affine.

Theorem (Shokurov)

The flip $f^{\cdot}X^+ \to Z$ exists if and only if the restricted algebra $R_S(K_X+S+B)$ is finitely generated.

Pl-Flips

- Clearly, if the flip exists, $R(K_X+S+B)$ is finitely generated and hence so is $R_S(K_X+S+B)$ since $R(K_X+S+B)\to R_S(K_X+S+B)$ is surjective.
- Recall the following result due to E. Noether: Let $R=\oplus_{m\geq 0}R_m$ be a graded ring and $R^{(d)}=\oplus_{m\geq 0}R_{dm}$ the truncation corresponding to an integer $d>0$. Then R is finitely generated iff so is $R^{(d)}$.
- Suppose R is finitely generated, to see that $R^{(d)}$ is finitely generated, note that it is the ring of invariants of R for the natural $\mathbb{Z}/d\mathbb{Z}$ action induced by multiplying by ζ^m in degree m where ζ is a d-th root of unity.
- Conversely if $R^{(d)}$ is finitely generated, then finite generation of R follows as R is integral over $R^{(d)}$ (for any $f\in R_m$, f satisfies the monic polynomial $x^d-f^d=0$ which lies in $R^{(d)}[x]$).

PI-flips

- We now show that finite generation of the restricted algebra gives the existence of PI-flips.
- Assume that $R_S(K_X + S + B)$ is finitely generated. Pick $S \neq S' \sim S$ and g a rational function such that $S - S' = (g)$.
- By assumption there exists integers $a, b > 0$ such that $a(K_X + S + B) - bS' \equiv_Z 0$.
- By the Base Point Free Theorem $a(K_X + S + B) - bS'$ is semiample over Z and so we may assume $a(K_X + S + B) \sim bS'$.
- By what we observed above $R_S(S')$ is finitely generated and $R(K_X + S + B)$ is finitely generated iff so is $R(S')$.

PI-flips

- We will show that the kernel of $\psi : R(S') \to R_S(S')$ is the principal ideal $(g)R(S')$.
- It then follows that if $R_S(S')$ is generated by elements $x_1, \ldots, x_k$ in degree $\leq m$ and $f_i \in R(S')$ are lifts such that $\psi(f_i) = x_i$, then it is easy to see that $R(S')$ is generated by $g^l f_i$ where $0 \leq l \leq m$ and $1 \leq i \leq k$.
- An element g_m be a rational function corresponding to a divisor in $|mS'|$ so that $(g_m) + mS' \geq 0$.
- If it is in the kernel of ψ , then $(g_m) + mS' = S + D$ where $D \geq 0$.
- Since $(g) = S - S'$, then $(g_m/g) + (m-1)S' = D \in |(m-1)S'|$.

Outline of the talk

Multiplier ideal sheaves

- Let X be a smooth variety and $B \geq 0$ a reduced divisor with simple normal crossings support and $D \geq 0$ a $\mathbb{Q}$-divisor whose support contains no strata of the support of B.
- Let $f : Y \to X$ be a log resolution of $(X, B + D)$ and write
$$K_Y + \Gamma = f^*(K_X + B) + E,$$
where $\Gamma \wedge E = 0$.
- For any $c > 0$ we define
$$\mathcal{J}_{B,c\cdot D} := f_*\mathcal{O}_Y(E - \lfloor cf^*D \rfloor).$$
- If $B = 0$, then $\mathcal{J}_{B,c\cdot D} = \mathcal{J}_{c\cdot D}$ is the usual multiplier ideal sheaf.

Multiplier ideal sheaves

- It is not hard to see that $\mathcal{J}_{B,c\cdot D}$ does not depend on the log resolution $f : Y \to X$.

Lemma

If $D \geq D'$, $B' \leq B$, and $c \geq c'$, then $\mathcal{J}_{B,c\cdot D} \subset \mathcal{J}_{B',c'\cdot D'} \subset \mathcal{O}_X$.

- This is immediate since $B' \leq B$, $E' \geq E$ and $f^*D \geq f^*D'$ so that $E - \lfloor cf^*D \rfloor \leq E' - \lfloor c'f^*D' \rfloor$.

Multiplier ideal sheaves

Lemma

If $\Sigma \geq 0$ is Cartier, $D, D' \geq 0$ are $\mathbb{Q}$-Cartier such that $D \leq \Sigma + D'$ and $\mathcal{J}_{B,D'} = \mathcal{O}_X$, then $\mathcal{I}_\Sigma = \mathcal{O}_X(-\Sigma) \subset \mathcal{J}_{B,D}$.

- Since $\mathcal{J}_{B,D'} = \mathcal{O}_X$, then $E - \lfloor f^*D' \rfloor \geq 0$ and since Σ is Cartier, $\lfloor f^*D \rfloor \leq f^*\Sigma + \lfloor f^*D' \rfloor$.
- Thus $E - \lfloor f^*D \rfloor \geq E - f^*\Sigma - \lfloor f^*D' \rfloor \geq -f^*\Sigma$.
- Finally it follows that $\mathcal{I}_\Sigma = f_*\mathcal{O}_{X'}(-f^*\Sigma) \subset f_*\mathcal{O}_{X'}(E - \lfloor f^*D \rfloor) = \mathcal{J}_{B,D}$. $\square$

Multiplier ideal sheaves

Proposition

Let X be a smooth variety, $B \geq 0$ a reduced SNC divisor and $D \geq 0$ a $\mathbb{Q}$-divisor whose support contains no strata of B. For any component S of B there is a short exact sequence

$$0 \to \mathcal{J}_{B-S,D+S} \to \mathcal{J}_{B,D} \to \mathcal{J}_{(B-S)|_S,D|_S} \to 0.$$

Proof By a result of Szabo, we may fix a log resolution $f : X' \to X$ which is an isomorphism at the generic point of each strata of B. Write $K_{X'} + B' = f^*(K_X + B) + E$ and $S' = f_*^{-1}S$. We have

$$0 \to E - \lfloor f^*D \rfloor - S' \to E - \lfloor f^*D \rfloor \to E|_{S'} - \lfloor f^*D|_{S'} \rfloor \to 0.$$

Multiplier ideal sheaves

Since $E - \lfloor f^*D \rfloor - S' \sim_{\mathbb{Q},X} K_{X'} + B' - S' + \{f^*D\}$, by Kawamata-Viehweg vanishing we have $R^1 f_* \mathcal{O}_{X'}(E - \lfloor f^*D \rfloor - S') = 0$.

Since $K_{S'} + (B' - S')|_{S'} = f^*(K_S + (B - S)|_S) + E|_{S'}$, it follows that $f_* \mathcal{O}_{S'}(E|_{S'} - \lfloor f^*D|_{S'} \rfloor) = \mathcal{J}_{(B-S)|_S, D|_S}$ and so we have the required exact sequence. $\square$

Proposition

Let (X, B) and D be as above. If $g : X \to Z$ is a proj. morphism Z is normal and affine, and N a Cartier divisor such that $N - D$ is ample, then $H^i(\mathcal{J}_{B,D}(K_X + B + N)) = 0$ for $i > 0$ and

$$H^0(\mathcal{J}_{B,D}(K_X + B + N)) \to H^0(\mathcal{J}_{(B-S)|_S, D|_S}(K_S + (B - S)|_S + N|_S))$$

is surjective.

Multiplier ideal sheaves

- Fix a log resolution $f : X' \to X$ which is an isomorphism at the generic point of each strata of B. Write $K_{X'} + B' = f^*(K_X + B) + E$ and $S' = f_*^{-1}S$
- Let $F \geq 0$ be a $\mathbb{Q}$-divisor such that $-F$ is f-ample, we may assume that $f^*(N - D) - F$ is ample and $(X', B' + \{f^*D\} + F)$ is DLT.

Multiplier ideal sheaves

- Since $E - \lfloor f^*D \rfloor + f^*(K_X + B + N) = K_{X'} + \Gamma + \{f^*D\} + F + f^*(N - D) - F$, by Kawamata-Viehweg vanishing we have $R^j f_* \mathcal{O}_{X'}(E - \lfloor f^*D \rfloor + f^*(K_X + B + N)) = 0$ for $j > 0$ and so $H^i(\mathcal{J}_{B,D}(K_X + B + N)) = H^i(E - \lfloor f^*D \rfloor + f^*(K_X + B + N)) = 0$ for $i > 0$.
- The last assertion follows similarly (by the previous Proposition). $\square$

Extending pluricanonical forms

Theorem

Let $(X, S+B+D)$ be a projective log smooth pair such that $\lfloor B \rfloor = 0$, $S \wedge B = 0$ and $(S+B) \wedge D = 0$. Suppose that $\Delta = S+B+P$ where P is nef and $K_X + \Delta \sim_{\mathbb{Q}} D$. If kP and kB are integral and H is sufficiently ample, then for any $\Sigma \in |k(K_S + (B+P)|_S)|$ and $U \in |H|_S|$ and any $l \in \mathbb{N}$, we have

$$l\Sigma + U \in |lk(K_X + \Delta) + H|_S.$$

Here $|G|_S$ denotes the image of the restriction morphism $|G| \to |G|_S|$ where G is any divisor whose support does not contain S.

Extending pluricanonical forms

Proof.

- We begin by defining a few auxiliary notions. For $m \geq 0$, let $l_m = \lfloor m/k \rfloor$ and $r_m = m - l_m k$, $P_m = kP$ if $r_m = 0$, $P_m = 0$ if $r_m \neq 0$, $B_m = \lceil mB \rceil - \lceil (m-1)B \rceil$ and

$$D_m = \sum_{i=1}^{m} (K_X + S + P_i + B_i) = m(K_X + S) + l_m kP + \lceil mB \rceil.$$

- D_m is integral and $D_m = l_m k(K_X + \Delta) + D_{r_m}$.
- Let H be sufficiently ample so that $D_j + H$ is very ample for $0 \leq j \leq k-1$, and $|D_k + H|_S = |(D_k + H)|_S|$.
- We aim to show that if $\Sigma \in |k(K_S + (B + P)|_S)|$ and $U_m \in |D_{r_m} + H|_S|$ then $l_m \Sigma + U_m \in |D_m + H|_S$.

Extending pluricanonical forms

- When $r_m = 0$, this implies the Theorem.
- We proceed by induction on m. The case $m = k$ follows from our assumptions above.
- For any $m > k$, pick $0 < \delta \ll 1$ such that $D_{r_{m-1}} + H + \delta B_m$ is ample
- For $0 < \epsilon \ll 1$, let $F = (1 - \epsilon\delta)B_m + l_{m-1}k\epsilon D$, then $(X, S + F)$ is log smooth, $\lfloor F \rfloor = 0$ and $S \not\subset \mathrm{Supp}(F)$. It follows that if $W \in |(D_{r_{m-1}} + H)|_S|$ is general and $\Phi = F|_S + (1 - \epsilon)W$, then (S, Φ) is klt.
- By induction we may assume that there is $G \in |D_{m-1} + H|$ such that $G|_S = l_{m-1}\Sigma + W$.
- If $C = (1 - \epsilon)G + F$, then $C \sim_{\mathbb{Q}} (1 - \epsilon)(D_{m-1} + H) + (1 - \epsilon\delta)B_m + l_{m-1}k\epsilon D$.

Extending pluricanonical forms

- We have $C|_S = (1-\epsilon)G|_S + F|_S \leq l_{m-1}\Sigma + \Phi \leq (l_m\Sigma + U_m) + \Phi$.
- Applying the Lemma (with $B = 0$, $\Sigma = l_m\Sigma + U_m$, $D = C|_S$ and $D' = \Phi$) one sees that $\mathcal{O}_X(-l_m\Sigma + U_m) \subset \mathcal{J}(C|_S)$.
- The divisor $A = \epsilon(D_{r_{m-1}} + H + \delta B_m) + P_m$ is ample. Thus

$$D_m + H = K_X + S + D_{m-1} + B_m + P_m + H =$$

$$K_X + S + (1-\epsilon)D_{m-1} + l_{m-1}k\epsilon(K_X + \Delta) + \epsilon D_{r_{m-1}} + B_m + P_m + H \sim_{\mathbb{Q}}$$

$$K_X + S + A + (1-\epsilon)D_{m-1} + l_{m-1}k\epsilon D + (1-\epsilon\delta)B_m + (1-\epsilon)H \sim_{\mathbb{Q}}$$

$$K_X + S + A + C.$$

- But then $l_m\Sigma + U_m \in |D_m + H|_S$ (using the Proposition with $D = C$). $\square$

Extending pluricanonical forms

Theorem

Let $(X, S+B)$ be a plt log smooth pair, A an ample $\mathbb{Q}$-divisor and $\Delta = S+A+B$. If $C \geq 0$ is a $\mathbb{Q}$-divisor such that (S, C) is canonical and $m \in \mathbb{N}$ such that mA, mB and mC are integral. Pick $q \gg 0$ such that qA is very ample $S \not\subset \mathrm{Bs}|qm(K_X + \Delta + \frac{1}{m}A)|$ and $C \leq B|_S - B|_S \wedge \frac{1}{qm}\mathrm{Fix}|qm(K_X + \Delta + \frac{1}{m}A)|_S$, then

$$|m(K_S + A|_S + C)| + m(B|_S - C) \subset |m(K_X + \Delta)|_S.$$

In particular

$$\mathrm{Fix}|m(K_S + A|_S + C)| + m(B|_S - C) \geq \mathrm{Fix}|m(K_X + \Delta)|_S.$$

Extending pluricanonical forms

- Let $f : X' \to X$ be a log resolution of $|qm(K_X + \Delta + \frac{1}{m}A)|$, $S' = f_*^{-1}S$ and
$$K_{X'} + S' + B' = f^*(K_X + S + B) + E.$$
- Then $K_{S'} + B'|_{S'} = f^*(K_S + B|_S) + E|_{S'}$ where $B'|_{S'} \wedge E|_{S'} = 0$ so that $E|_{S'}$ is $f|_{S'}$ exceptional.
- Set $\Gamma = S' + f^*A + B'$, $F_q := \frac{1}{qm}\mathrm{Fix}|qm(K_{X'} + \Gamma + \frac{1}{m}f^*A)|$, $B_q := B' - B' \wedge F_q$, $\Gamma_q = S' + B'_q + f^*A$.
- We may pick a general $D \in |K_{X'} + \Gamma_q + \frac{1}{m}f^*A|_{\mathbb{Q}}$ such that $(X', S' + B'_q + D)$ is log smooth and $D \wedge (S' + B'_q) = 0$.
- Since (S, C) is canonical, $K_{S'} + C' = f^*(K_S + C) + F$ where $C' = (f|_{S'})_*^{-1}C$.
- **Claim:** $C' \leq B'_q$.

Extending pluricanonical forms

- By what we saw above, there is a very ample divisor H on X' such that for any $\Sigma' \in |qm(K_{S'} + (B'_q + (1+\frac{1}{m})f^*A)|_{S'})|$ and $U \in |H|_{S'}|$ and $p \in \mathbb{N}$ we have
$$p\Sigma' + U \in |pqm(K_{X'} + \Gamma_q + \frac{1}{m}f^*A) + H|_{S'}.$$
- Pick $G \geq 0$ such that $\lfloor B' + \frac{1}{m}G \rfloor = 0$ and $f^*A - G$ is ample. Thus $(S', (B' + \frac{1}{m}G)|_{S'})$ is KLT.
- Let $W_1 \in |qf^*A|_{S'}|$ and $W_2 \in |H|_{S'}|$ be general sections and for $k \gg 0$, letting $W = kW_1 + W_2$ and $\Phi = B'|_{S'} + \frac{1}{m}G|_{S'} + \frac{1}{qk}W$, then (S', Φ) is klt and $A_0 = \frac{1}{mkq}(f^*A - G) - \frac{m-1}{m}H$ is ample for $k \gg 0$.
- Fix $\Sigma \in |m(K_S + A|_S + C)|$ (the section we hope to extend).

Extending pluricanonical forms

- Since $C' \leq B'_q|_{S'}$, by the claim one sees

$$qf^*\Sigma + qm(F + B'_q|_{S'} - C') + W_1 \in |qm(K_{S'} + (B'_q + (1 + \frac{1}{m})f^*A)|_{S'})|.$$

- Thus we may find $G \in |lm(K_{X'} + (B'_q + (1 + \frac{1}{m}f^*A) + H|$ such that $G \wedge T = 0$ and $G|_{S'} = kqf^*\Sigma + kqm(F + B'_q|_{S'} - C') + W$.
- Let $B_0 = \frac{m-1}{mkq}G + (m-1)(\Gamma - \Gamma_q) + B' + \frac{1}{m}G$. Then

$$m(K_{X'} + \Gamma) = K_{X'} + S' + (m-1)(K_{X'} + \Gamma + \frac{1}{m}f^*A) + \frac{1}{m}f^*A + B'$$

$$\sim_{\mathbb{Q}} K_{X'} + S' + \frac{m-1}{mkq}G + (m-1)(\Gamma - \Gamma_q) + \frac{1}{m}f^*A - \frac{m-1}{mkq}H + B'$$

$$= K_{X'} + S' + A_0 + B_0.$$

Extending pluricanonical forms

- Since $\Gamma - \Gamma_q = B' - B'_q$, then $B_0|_{S'} = \frac{m-1}{m} f^*\Sigma + (m-1)(F + B'_q|_{S'} - C' + (\Gamma - \Gamma_q)|_{S'}) + \frac{m-1}{mkq} W + B'|_{S'} + \frac{1}{m} G|_{S'}$ $\leq g^*\Sigma + m(F + B'|_{S'} - C') + \Phi$.
- Since $g^*\Sigma + m(F + B'|_{S'} - C') \in |m(K_{X'} + \Gamma)|_{S'}|$, we get $g^*\Sigma + m(F + B'|_{S'} - C') \in |m(K_{X'} + \Gamma)|_{S'}$.
- Pushing forward gives $\Sigma + m(B|_S - C) \in |m(K_X + \Delta)|_S$ as required.

Extending pluricanonical forms

- To check the claim note that since $\mathrm{Mob}(qm(K_{X'} + \Gamma + \frac{1}{m} f^* A))$ is free and $S' \wedge F_q = 0$, then

$$\frac{1}{qm}\mathrm{Fix}|qm(K_{X'} + \Gamma + \frac{1}{m} f^* A)| = F_q|_{S'}$$

and so

$$B'_q|_{S'} = B'|_{S'} - (B' \wedge F_q)|_{S'} =$$

$$B'|_{S'} - B'|_{S'} \wedge \frac{1}{qm}\mathrm{Fix}|qm(K_{X'} + \Gamma + \frac{1}{m} f^* A)|.$$

- Thus $C' \leq B'_q|_{S'}$, since $B'_q|_{S'} \geq 0$ and $C' = g_*^{-1} C$.
- Note that (S, C) is canonical and so C' is the strict transform of C.

Existence of flips

Theorem

Let X be $\mathbb{Q}$-factorial normal, $\pi : X \to U$ a projective morphism to an affine variety, A an ample $\mathbb{Q}$-divisor, $(X, S+B)$ is PLT pair such that $(S, \Omega := \mathrm{Diff}_S(B))$ is canonical and $\mathbf{B}(K_X+S+B+A)$ does not contain S. For any $m>0$ sufficiently divisible, let

$$F_m = \mathrm{Fix}(|m(K_X+S+B+A)|_S)/m \qquad \text{and} \qquad F = \lim F_{m!}.$$

Then $\Theta = \Omega - \Omega \wedge F$ is rational and if $k(K_X+S+B+A)$ and $k(K_S+\Theta+A)$ are Cartier, then

$$R_S(k(K_X+S+B+A)) \cong R(k(K_S+\Theta+A|_S)).$$

Existence of flips

- Let $V \subset \mathrm{Div}_{\mathbb{R}}(S)$ be the vector space generated by the components of Ω and $W \subset V$ be the smallest rational affine space containing Θ.
- As we have seen $\mathbf{B}(K_S + \Theta + A|_S) \subset \mathbf{B}(K_S + \Theta' + A|_S)$ if $\Theta' \in W$ and $||\Theta - \Theta'|| \ll 1$.
- Moreover if $k(K_S + \Theta' + A|_S)/r$ is Cartier, then every component of $\mathrm{Fix}(k(K_S + \Theta' + A|_S))$ is contained in $\mathbf{B}(K_S + \Theta + A|_S)$.
- By construction if $l(K_X + S + B + A)$ is Cartier and $\Theta_l = \Omega - \Omega \wedge F_l$, then $l(K_S + \Theta_l + A|_S)$ is Cartier and $|l(K_X + S + B + A)|_S \subset |l(K_S + \Theta_l + A|_S)| + l(\Omega \wedge F_l)$.
- Thus no component of Θ_l is contained in $\mathrm{Fix}(l(K_S + \Theta_l + A|_S))$ hence in $\mathbf{B}(K_S + \Theta_l + A|_S)$ and $B(l(K_S + \Theta_l + A|_S))$.

Existence of flips

- But then no component of Θ is contained in $B(I(K_S + \Theta_I + A|_S))$.
- If $\mathrm{mult}_P(\Theta) \notin \mathbb{Q}$ then $P \in \mathrm{Supp}(\Theta), \mathrm{Supp}(F)$.
- If $\mathrm{mult}_P(\Theta) \in \mathbb{Q}$, then $\mathrm{mult}_P(\Theta') = \mathrm{mult}_P(\Theta)$ for $\Theta' \in W$.
- By Diophantine approximation, there is an integer $k > 0$, an effective divisor $\Phi \in W$ such that $k\Phi/r$ and $k(K_X + S + \Delta + A)/r$ are Cartier, $\mathrm{mult}_P(\Phi) < \mathrm{mult}_P(\Omega \wedge F)$ and $||\Phi - (\Omega \wedge F)|| \ll 1$.
- Then $\Omega \wedge (1 - \epsilon/k)F \leq \Phi \leq \Omega$ for $0 < \epsilon \ll 1$.
- Thus by (β), $|k(K_S + \Omega - \Phi + A)| + k\Phi \subset |k(K_X + S + B + A)|_S$.

Existence of flips

- Since $\mathrm{mult}_P(\mathrm{Fix}(|k(K_X+S+B+A)|_S|)) \geq \mathrm{mult}_P(kF) = \mathrm{mult}_P(k(\Omega \wedge F)) > \mathrm{mult}_P(k\Phi)$, it follows from (β) that $P \in \mathrm{Supp}(\mathrm{Fix}(k(K_X+S+B+A)))$.
- Since $||(\Omega - F) - \Theta|| \ll 1$ and $k(K_S + \Omega - \Phi + A)/r$ is Cartier, P is contained in $\mathbf{B}(K_S + \Theta)$ which is impossible and so Θ is rational.
- By the extension theorem, $R_S(k(K_X+S+B+A)) \cong R(k(K_S+\Theta+A))$ now follows.

Existence of flips

- Finally we deduce the existence of pl-flips.
- Let $f : X \to Z$ be a pl flipping contraction for a plt pair $(X, S + B)$.
- We may assume that Z is affine and hence that S is mobile and B is big.
- Let $S' \sim S$ such that $S' \wedge S = 0$ and write $B \sim_{\mathbb{Q}} A + C$ where A is ample and $C \geq 0$.
- For $0 < \epsilon \ll 1$ let $A' = \epsilon A$ and $B' = (1 - \epsilon)B + \epsilon C$.
- Let $g : Y \to X$ be a log resolution and write $K_Y + \Gamma = g^*(K_X + S + B' + A') + E$ where $E, \Gamma \geq 0$ and $E \wedge \Gamma = 0$.

Existence of flips

- Let $T = g_*^{-1}S$. we may assume that (Y, Γ) is plt and $(T, (\Gamma - T)|_T)$ is terminal.
- Let $F \geq 0$ be effective, g-exceptional and $-F$ be g-ample.
- Let $B_Y = \Gamma - T + F$ and $A_Y = g^*A - F$.
- We may assume that $A_Y = g^*A - F$ is ample and $(Y, T + B_Y)$ is PLT and $(T, (B_Y)|_T)$ is terminal.
- Since $S \not\subset \mathbf{B}(K_X + S + B + A)$, it follows that $T \not\subset \mathbf{B}(K_Y + T + B_Y + A_Y)$.
- By what we have seen above there is a $\mathbb{Q}$-divisor $0 < \Theta \leq (B_Y)|_T$ on T such that $R_T(k(K_Y + B_Y + A_Y)) \cong R(k(K_T + \Theta + A|_T))$ for all $k > 0$ sufficiently divisible.
- Since $R_T(k(K_Y + B_Y + A_Y)) \cong R_S(k(K_X + B + A))$ the restricted algebra is finitely generated.$\square$

Useful references

[BCHM] C. Birkar, P. Cascini, C. Hacon, J. McKernan, *Existence of minimal models for varieties of log general type.* J. Amer. Math. Soc. 23 (2010), no. 2, 405–468.

[Corti] A. Corti, *Flips for 3-folds and 4-folds,* 76110, Oxford Lecture Ser. Math. Appl., 35, Oxford Univ. Press, 2007.

[HM] C. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type II,* J. Amer. Math. Soc.

[KMM] Y. Kawamata, K. Matsuda, and K. Matsuki, *Introduction to the minimal model program,* Algebraic Geometry, Sendai, 1987, Adv. Stud. Pure Math., vol. 10, pp. 283–360.

[KM] J. Kollár and S. Mori, *Birational geometry of algebraic varieties,* Cambridge tracts in mathematics, vol. 134, Cambridge University Press, 1998.

Thanks

- Thanks to Kenji Matsuki for many useful questions and corrections.
- Thanks to The University of Kyoto for the hospitality.
- Thanks to the NSF and the Simons foundation for financial support.