Math 6320, Assignment 2 **Due: Weekend of February 9**

1. (a) If a polynomial $f(x) := a_0 + a_1x + \cdots a_nx^n$ has a rational root p/q with $(p,q) = 1$, then p divides a_0 and q divides a_n. Prove this assertion.
   (b) Prove that a cubic polynomial in $\mathbb{Z}[x]$ with no rational roots is irreducible in $\mathbb{Q}[x]$.
   (c) Is the polynomial $x^3 - 4x^2 - (9/2)x - (5/2)$ irreducible in $\mathbb{Q}[x]$?
2. Study the irreducibility of $x^4 + 1$ in $\mathbb{Q}[x]$ and factorize it as a product of irreducible factors in $\mathbb{Q}(\zeta)[x]$, where ζ is one of its roots.
3. For each prime p and integer n, verify that the polynomial $x^n + px + p^2$ is irreducible in $\mathbb{Q}[x]$.
4. Give an example of an irreducible quadratic polynomial $f(x)$ in $\mathbb{Q}[x]$ such that $f(x^2)$ is reducible.
5. Let F be a field and $f(x), g(x)$ polynomials in $F[x]$. Let α and β be roots of $f(x)$ and $g(x)$, respectively, in some extension of F.
   (a) Verify that the polynomial $R(z) := \mathrm{Res}_x(f(x), g(z-x))$ is a polynomial in $F[z]$ having $a+b$ as a root.
   (b) More generally, for any polynomial $p(x,y)$ in $F[x,y]$, the element $p(\alpha,\beta)$ is a root of the polynomial
   $$\mathrm{Res}_x(f(x), \mathrm{Res}_y(z - p(x,y), g(y))) \quad \text{in } F[z].$$
   (c) Verify that $\alpha\beta$ is a root of the polynomial $\mathrm{Res}_x(f(x), x^n g(z/x))$, where $n = \deg(g(x))$.
   (d) For a polynomial $p(y)$ in $F[y]$, describe a polynomial over F that vanishes at $p(\alpha)$.
6. Let α be a root of the polynomial $f(x) = x^3 + 2x + 2$ in $\mathbb{Q}[x]$. Since $f(x)$ is irreducible, by the Eisentein criterion, the $\mathbb{Q}$-vector space $\mathbb{Q}(\alpha)$ has a basis $1, \alpha, \alpha^2$. Express the following elements in $\mathbb{Q}(\alpha)$ in terms of this basis:
   $$a^{-1}, \qquad 1/(a^2 + a + 1), \qquad \text{and} \qquad a^6 + 3a^4 + 2a^3 + 6a$$
   Determine the minimal polynomials, over $\mathbb{Q}$, of these elements.
7. Let R be a ring and $f(x) := a_0 + a_1x + \cdots a_nx^n$ a polynomial over R. Prove the following assertions.
   (a) $f(x)$ is a unit in $R[x]$ if and only if a_0 is a unit and each a_i is nilpotent.
   (b) $f(x)$ is nilpotent if and only if each a_i is nilpotent.
   (c) $f(x)$ is a zerodivisor if and only if there exists a nonzero element $r \in R$ such that $r \cdot f(x) = 0$.