1.3.5; Suppose A is a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 > \lambda_3 \geq \ldots$. Show that

$$\max_{\langle u,v\rangle=0} \langle Au, u\rangle + \langle Av, v\rangle = \lambda_1 + \lambda_2$$

where $||u|| = ||v|| = 1$.

Use Lagrange multipliers and look for extremals of

$$P(u, v) = \langle Au, u\rangle + \langle Av, v\rangle + \mu_1(\langle u, u\rangle - 1) + \mu_2(\langle v, v\rangle - 1) + \mu_3|\langle u, v\rangle|^2$$

Differentiate $P(u, v)$ with respect to parameters μ_1, μ_2, and μ_3 and set the derivatives to zero to learn (as expected) that $||u||^2 = ||v||^2 = 1$, and $\langle u, v\rangle = 0$. Now set $u = u_0 + h, v = v_0 + g$, expand $P(u, v)$ and require that the linear terms vanish. That is, require

$$\text{Re } (\langle Au + \mu_1 u, h\rangle + \langle Av + \mu_2 v, g\rangle) = 0$$

for all g, h. It follows that we must have

$$Au + \mu_1 u = 0, \qquad Av + \mu_2 v = 0.$$

In other words, u and v must be eigenvectors for A.

Now suppose that the normalized eigenvectors of A are $\{\phi_i\}$ with corresponding eigenvalues λ_i. Since we know that u and v must be eigenvectors, we have that

$$\begin{aligned}\max_{\langle u,v\rangle=0} \langle Au, u\rangle + \langle Av, v\rangle &= \max_{i\neq j} \langle A\phi_i, \phi_i\rangle + \langle A\phi_j, \phi_j\rangle \\ &= \max_{i\neq j} \lambda_i + \lambda_j \\ &= \lambda_1 + \lambda_2\end{aligned}$$

∎

2.2.21; Verify that the wavelet generated by the sinc function is

$$W(x) = \text{sinc}(x - \frac{1}{2}) - 2\ \text{sinc}(2x - 1).$$

Recall that

$$\begin{aligned}\text{sinc}(x) &= \sum_k \text{sinc}(\frac{k}{2})\ \text{sinc}(2x - k) \\ &= \text{sinc}(2x) + \sum_{k \text{odd}} \text{sinc}(\frac{k}{2})\ \text{sinc}(2x - k)\end{aligned}$$

so that

$$\sum_{k \text{odd}} \text{sinc}(\frac{k}{2}) \text{ sinc}(2x-k) = \text{ sinc}(x) - \text{ sinc}(2x)$$

It follows that

$$\begin{aligned} W(x) &= \sum_k (-1)^k \text{ sinc}(\frac{k-1}{2}) \text{ sinc}(2x-k) \\ &= \sum_k (-1)^{k+1} \text{ sinc}(\frac{k}{2}) \text{ sinc}(2x-1-k) \\ &= -\text{ sinc}(2x-1) + \sum_{k \text{odd}} \text{ sinc}(\frac{k}{2}) \text{ sinc}(2x-1-k) \\ &= -2 \text{ sinc}(2x-1) + \text{ sinc}(x - \frac{1}{2}). \end{aligned}$$

∎

8.3.15 (A new problem) Under what conditions is the interchange $\frac{d}{dt}\int_a^b u(x,t)dx = \int_a^b \frac{\partial u}{\partial t}dx$ valid?

Answer: Use Fubini's theorem to show that the interchange is valid if $\frac{\partial u}{\partial t}$ is absolutely integrable.