The Dynamics of Excitability

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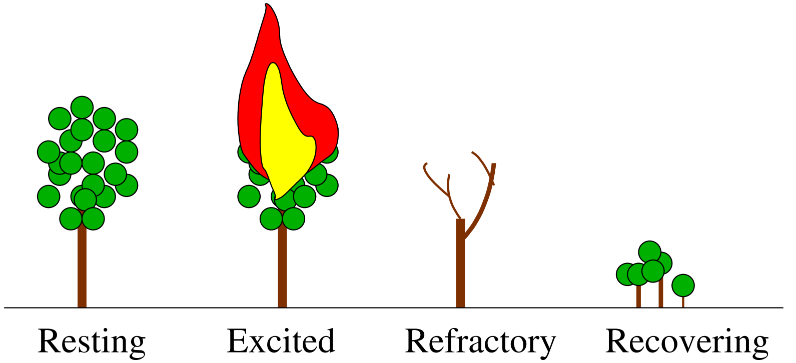
Examples of Excitable Media

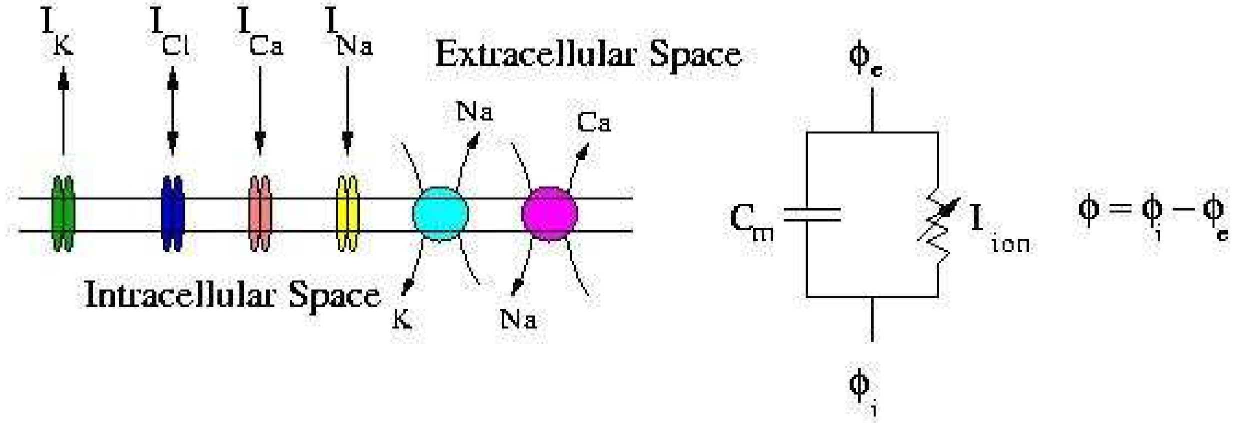
- B-Z reagent
- Nerve cells
- cardiac cells, muscle cells
- Slime mold (*dictystelium discoideum*)
- CICR (Calcium Induced Calcium Release)
- Forest Fires

Features of Excitability

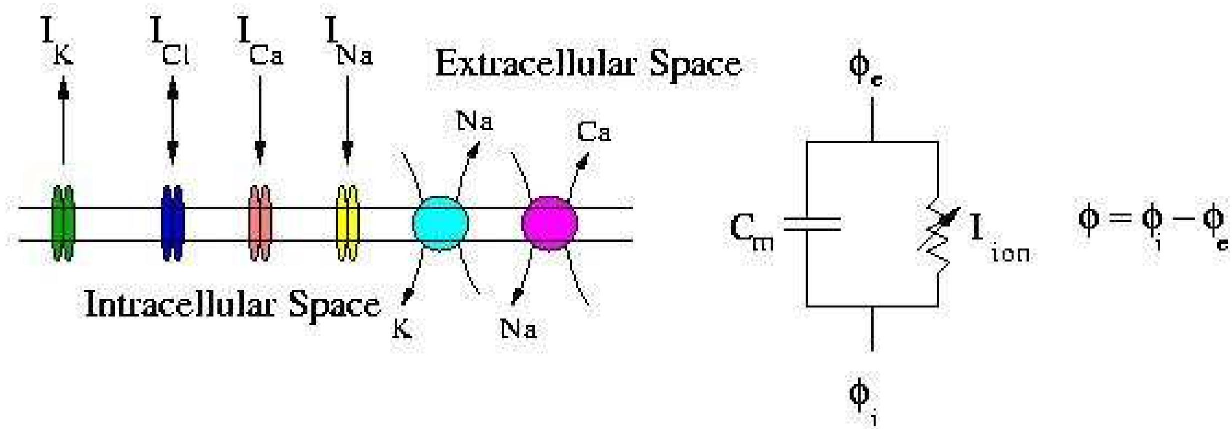
- Threshold Behavior
- Refractoriness
- Recovery



Modeling Membrane Electrical Activity



Modeling Membrane Electrical Activity



Transmembrane potential ϕ is regulated by transmembrane ionic currents and capacitive currents:

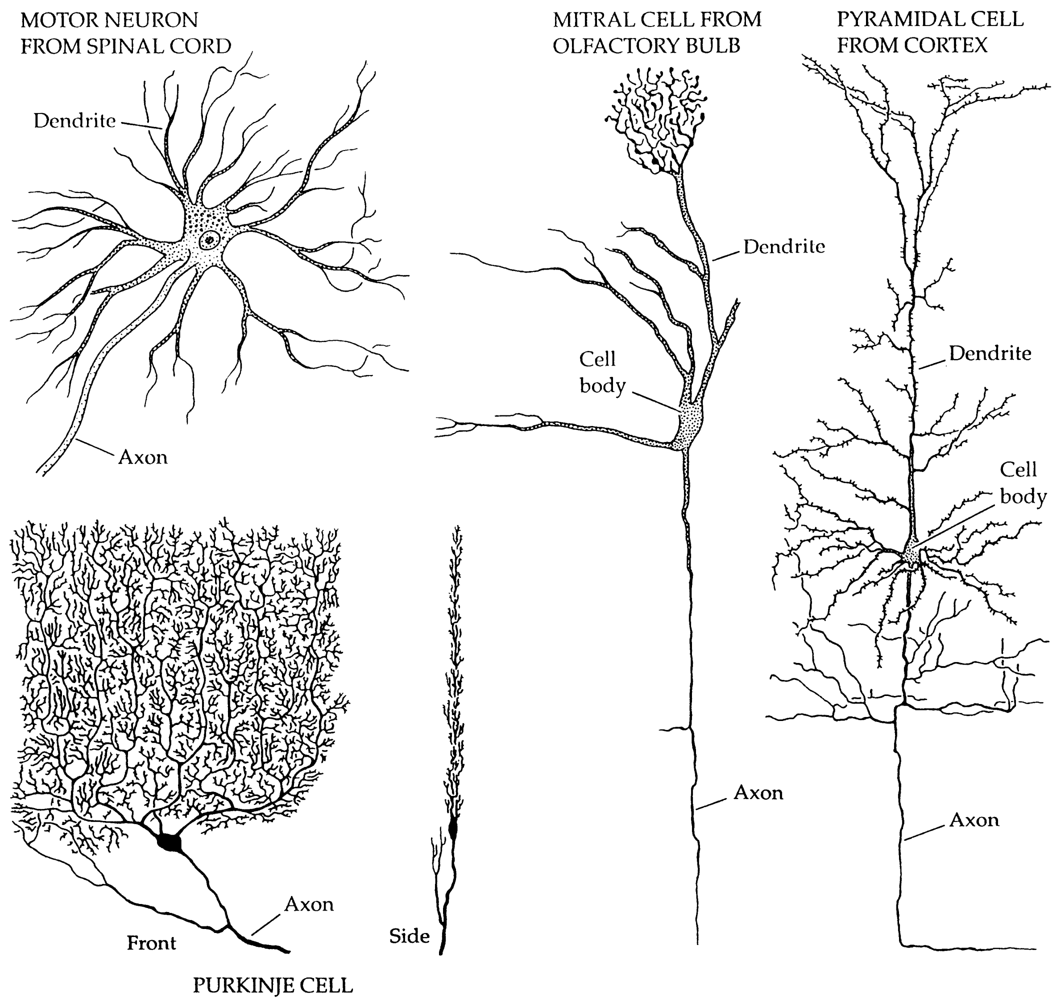
$$C_m \frac{d\phi}{dt} + I_{ion}(\phi, w) = I_{in} \quad \text{where } \frac{dw}{dt} = g(\phi, w), \qquad w \in R^n$$

Examples

Examples include:

- Neuron - Hodgkin-Huxley model
- Purkinje fiber - Noble
- Cardiac cells - Beeler-Reuter, Luo-Rudy, Winslow-Jafri, Bers
- Two Variable Models - reduced HH, FitzHugh-Nagumo, Mitchell-Schaeffer, Morris-Lecar, McKean, Puschino, etc.)

The Squid Giant Axon...



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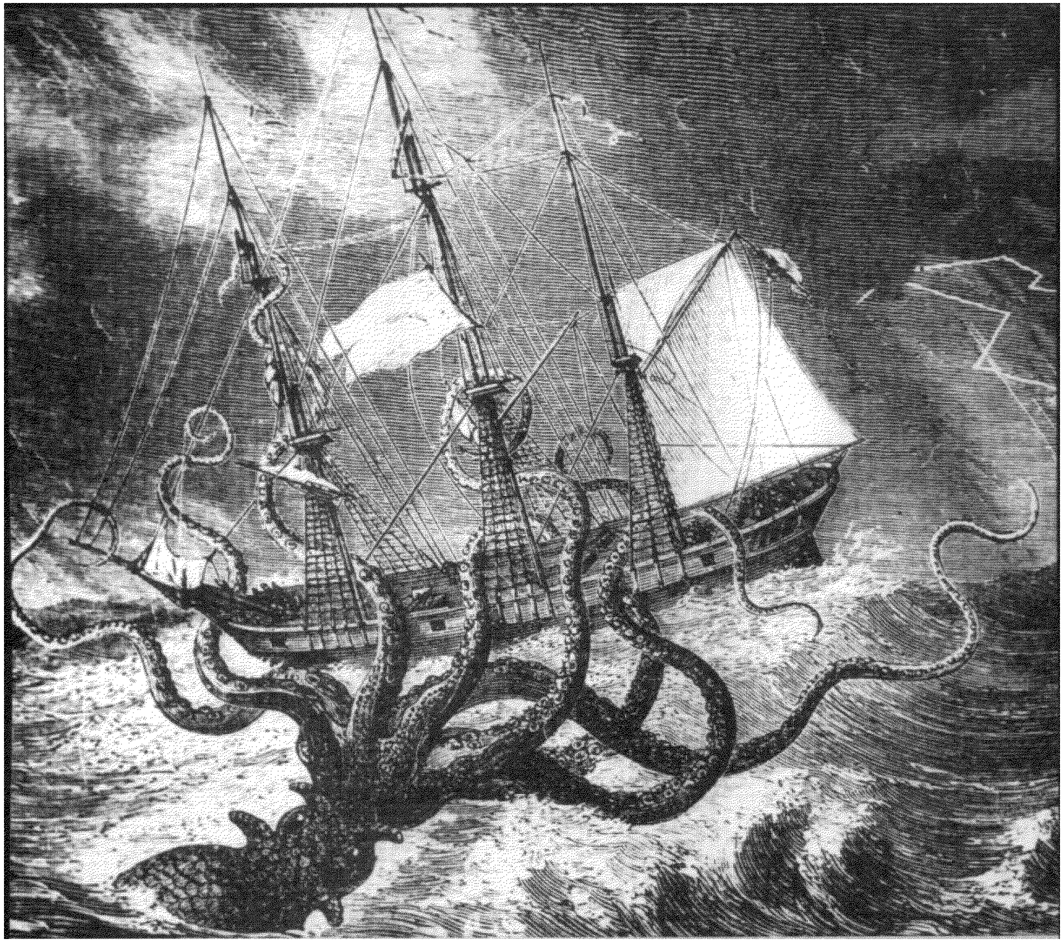
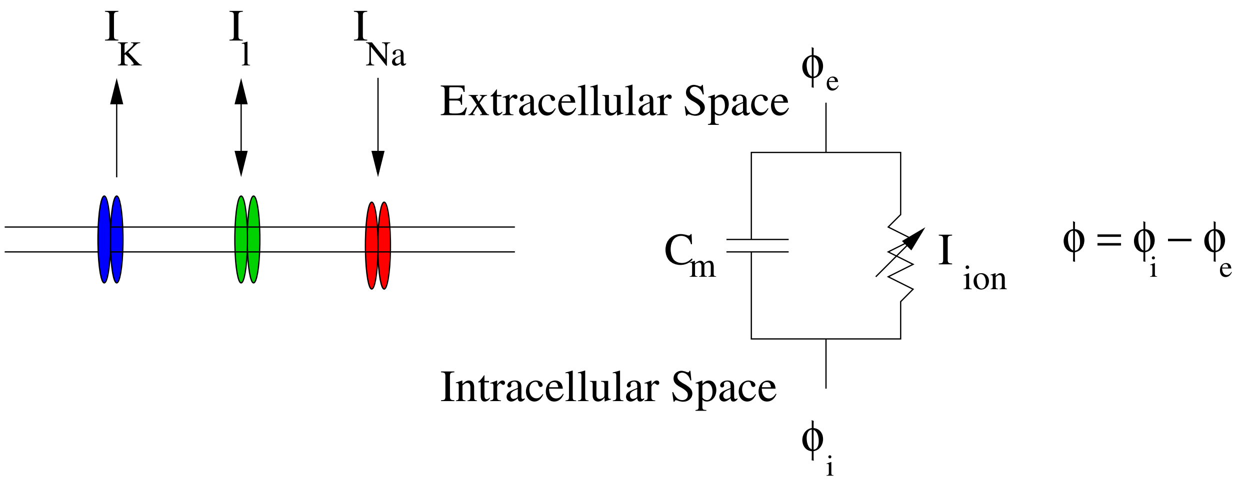
is not the Giant Squid Axon

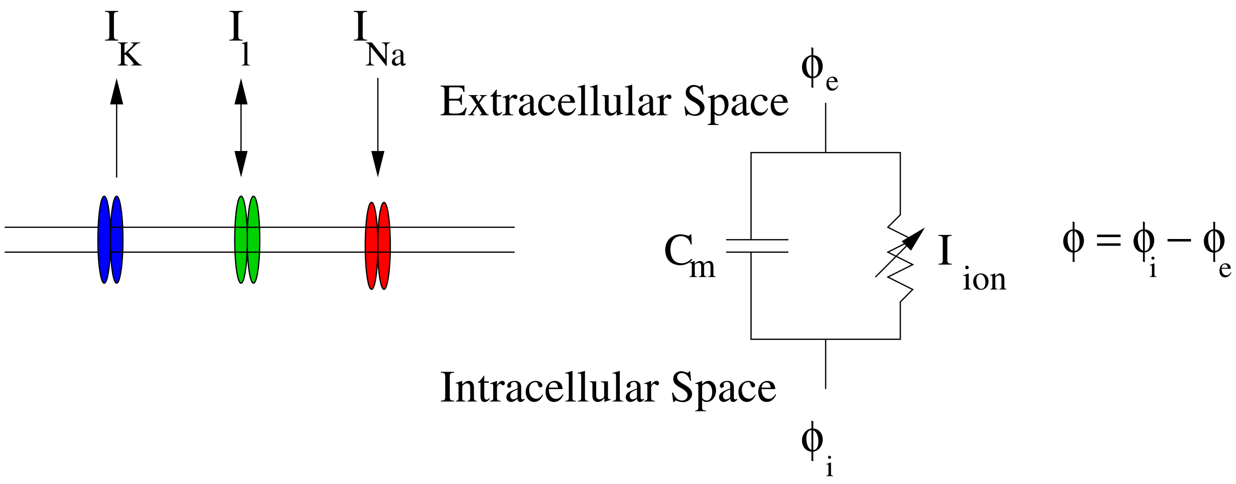


The Hodgkin-Huxley Equations



$$C_m \frac{dV}{dt} + I_{\mathrm{Na}} + I_{\mathrm{K}} + I_{\mathrm{l}} = 0,$$

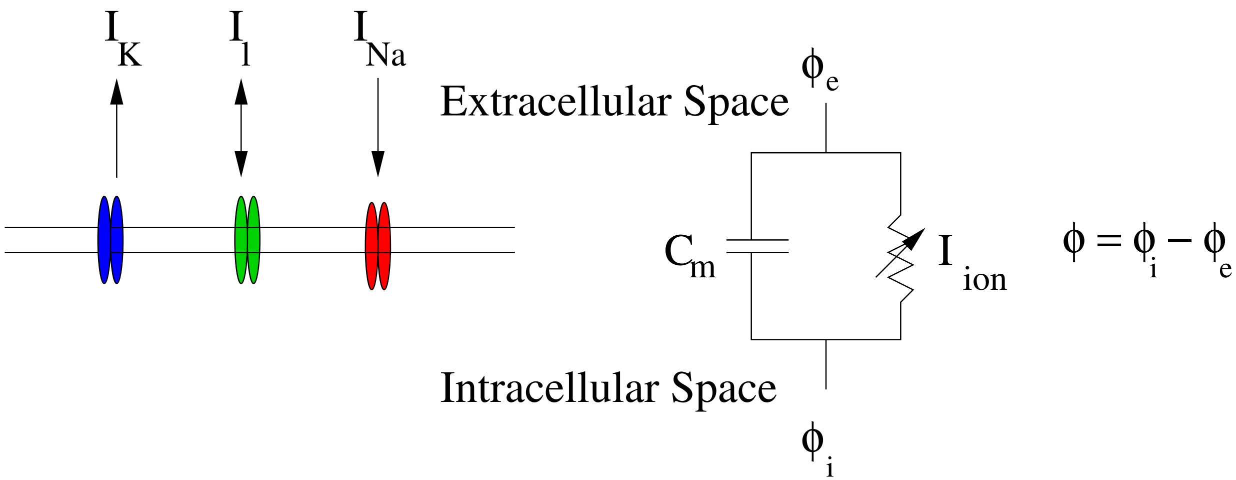
The Hodgkin-Huxley Equations



$$C_m \frac{dV}{dt} + \boxed{I_{\rm Na}} + I_{\rm K} + I_{\rm l} = 0,$$

with sodium current $I_{\rm Na}$,

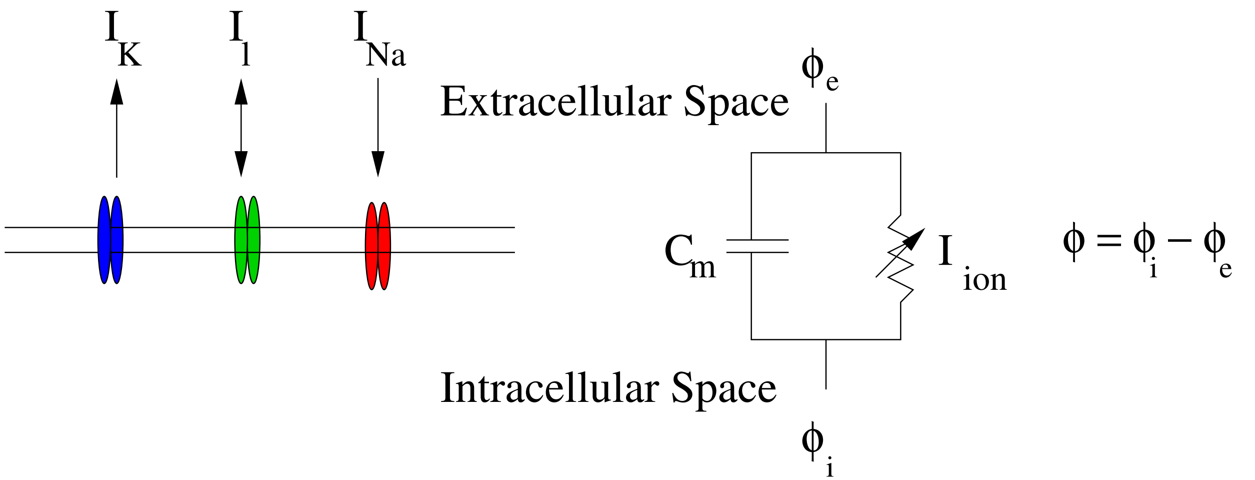
The Hodgkin-Huxley Equations



$$C_m \frac{dV}{dt} + \boxed{I_{\rm Na}} + \boxed{I_{\rm K}} + I_{\rm l} = 0,$$

with sodium current $I_{\rm Na}$, potassium current $I_{\rm K}$,

The Hodgkin-Huxley Equations



$$C_m \frac{dV}{dt} + \boxed{I_{\mathrm{Na}}} + \boxed{I_{\mathrm{K}}} + \boxed{I_{\mathrm{l}}} = 0,$$

with sodium current I_{Na}, potassium current I_{K}, and leak current I_{l}.

Ionic Currents

Ionic currents are typically of the form

$$I = \ g(\phi, t) \ \ \Phi(\phi)$$

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$$I = \boxed{g(\phi, t)}\,\boxed{\Phi(\phi)}$$

where $g(\phi, t)$ is the total number of open channels, and $\Phi(\phi)$ is the I-ϕ relationship for a single channel.

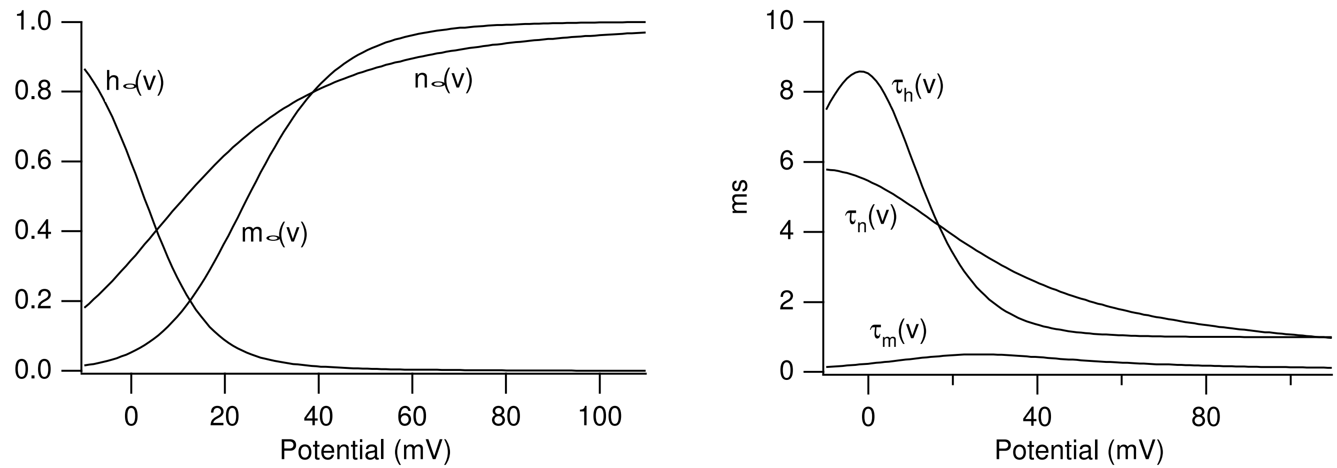
Currents

Hodgkin and Huxley found that

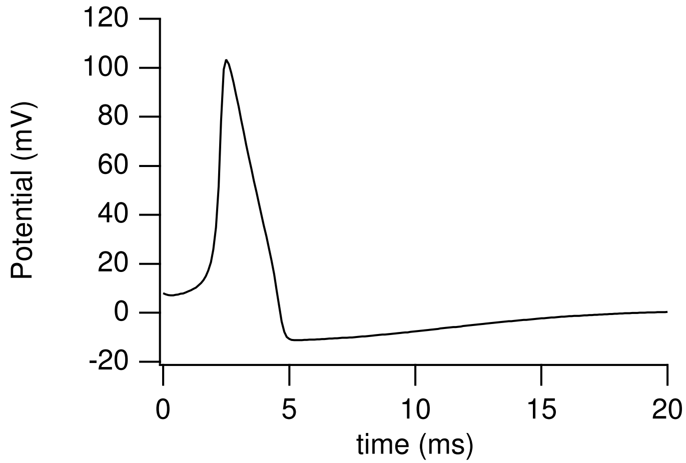
$$I_k = g_k n^4(\phi - \phi_K), \qquad I_{Na} = g_{Na} m^3 h(\phi - \phi_{Na}),$$

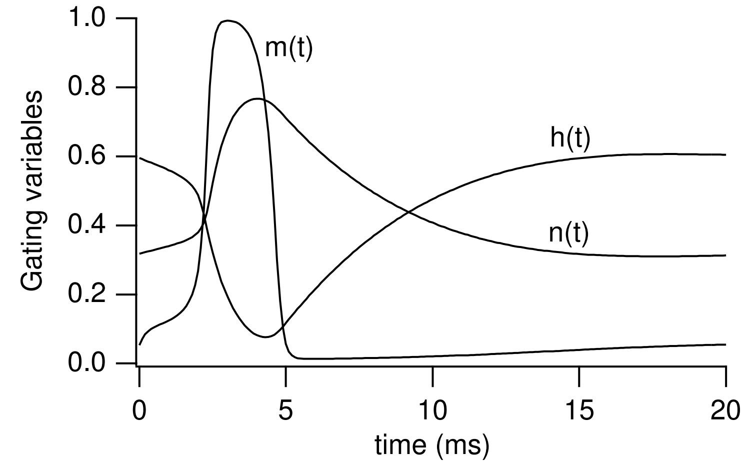
where

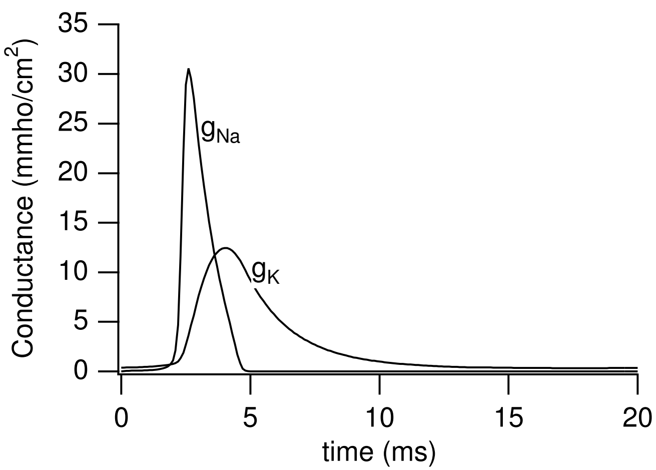
$$\tau_u(\phi)\frac{du}{dt} = u_\infty(\phi) - u, \qquad u = m, n, h$$

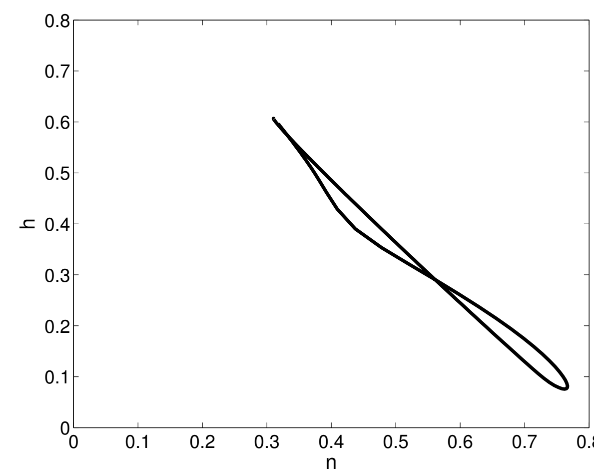


Action Potential Dynamics



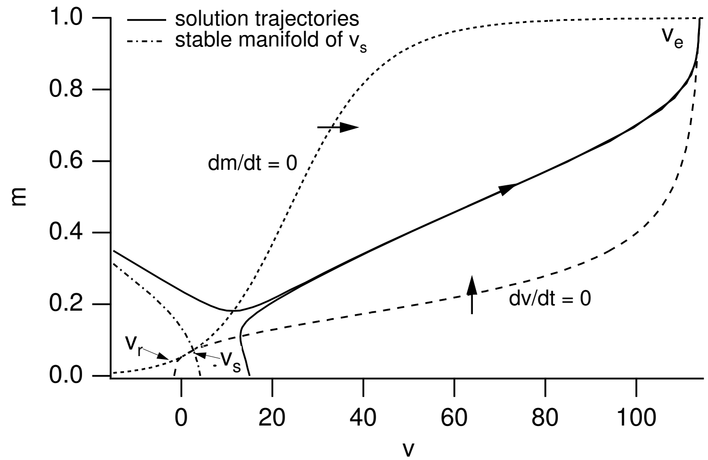


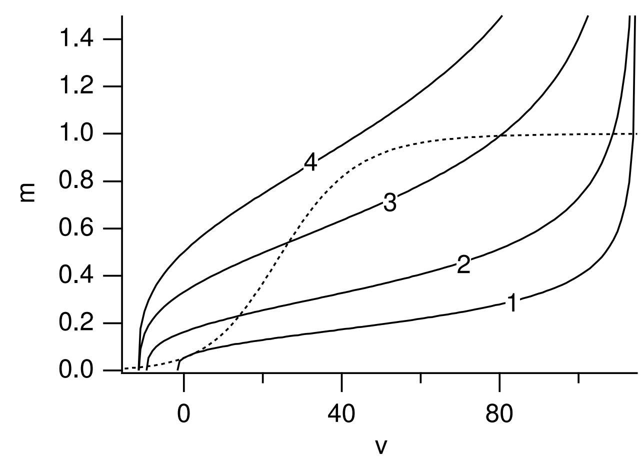




Fast-Slow Subsystem Dynamics

Observe that $\tau_m << \tau_n, \tau_h$





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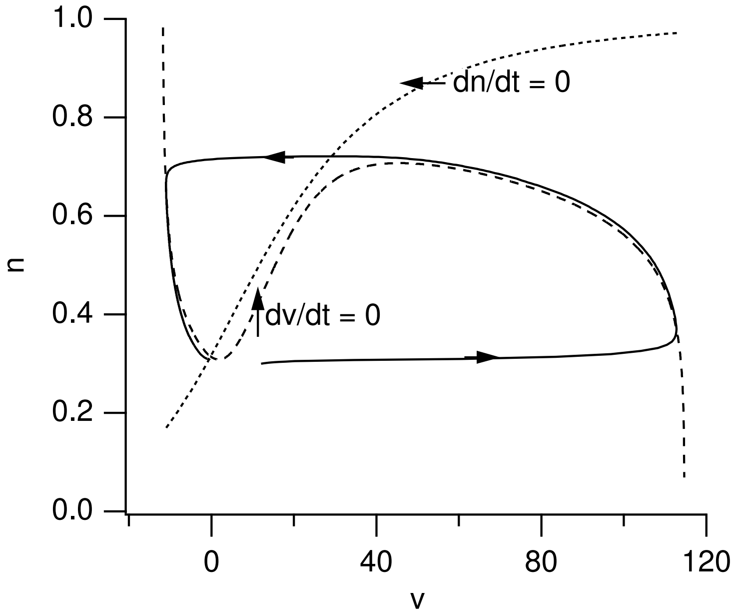
Two Variable Reduction of HH Eqns

Set $m = m_\infty(\phi)$, and set $h + n \approx N = 0.85$.
This reduces to a two variable system

$$C\frac{d\phi}{dt} = \bar{g}_K n^4(\phi - \phi_K) + \bar{g}_{Na} m_\infty^3(\phi)(N - n)(\phi - \phi_{Na}) + \bar{g}_l(\phi - \phi_L),$$

$$\tau_n(\phi)\frac{dn}{dt} = n_\infty(\phi) - n.$$



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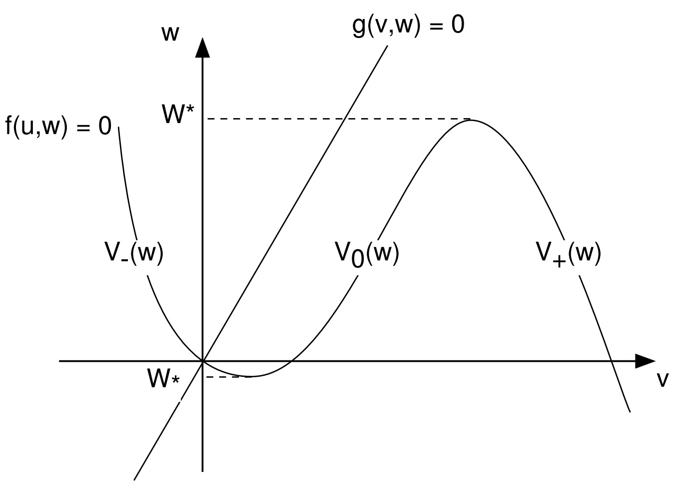
Two Variable Models

Following is a summary of two variable models of excitable media. The models described here are all of the form

$$\frac{dv}{dt} = f(v,w) + I$$
$$\frac{dw}{dt} = g(v,w)$$

Typically, v is a "fast" variable, while w is a "slow" variable.



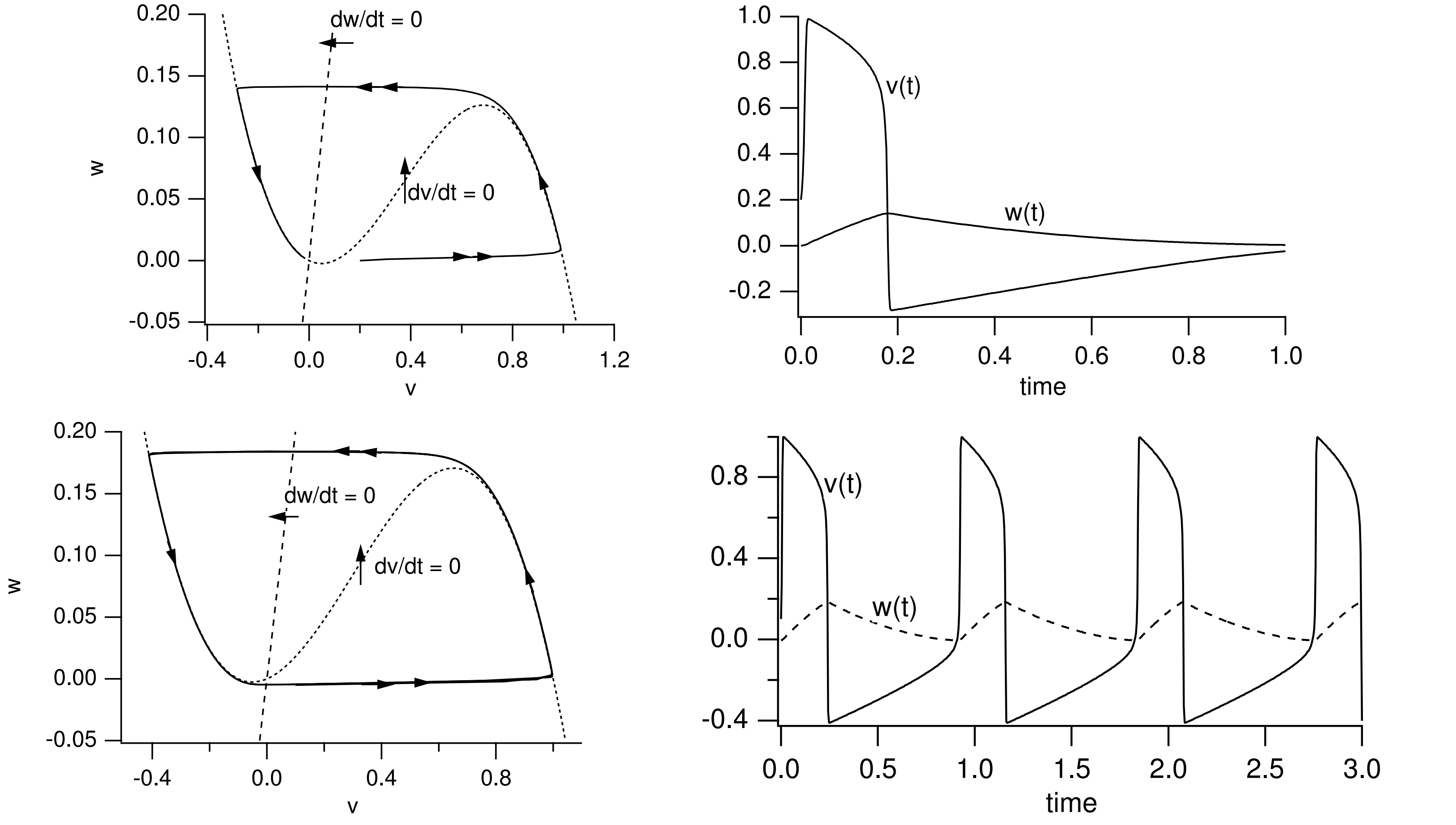
Cubic FitzHugh-Nagumo

The model that started the whole business uses a cubic polynomial (a variant of the van der Pol equation).

$$\begin{aligned} F(v,w) &= Av(v-\alpha)(1-v)-w, \\ G(v,w) &= \epsilon(v-\gamma w). \end{aligned}$$

with $0 < \alpha < \frac{1}{2}$, and ϵ “small”.

FitzHugh-Nagumo Equations



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Mitchell-Schaeffer

Mitchell-Schaeffer two-variable model (also in a slightly different but equivalent form by Karma)

$$
\begin{aligned}
F(v,w) &= \frac{1}{\tau_{in}} w v^2 (1-v) - \frac{v}{\tau_{out}}, \\
G(v,w) &= \begin{cases} \frac{1}{\tau_{open}}(1-w) & v < v_{gate} \\ -\frac{w}{\tau_{close}} & v > v_{gate} \end{cases}
\end{aligned}
$$

Notice that $F(v,w)$ is cubic in v, and w is an inactivation variable (like h in HH).

Mitchell-Schaeffer Revised

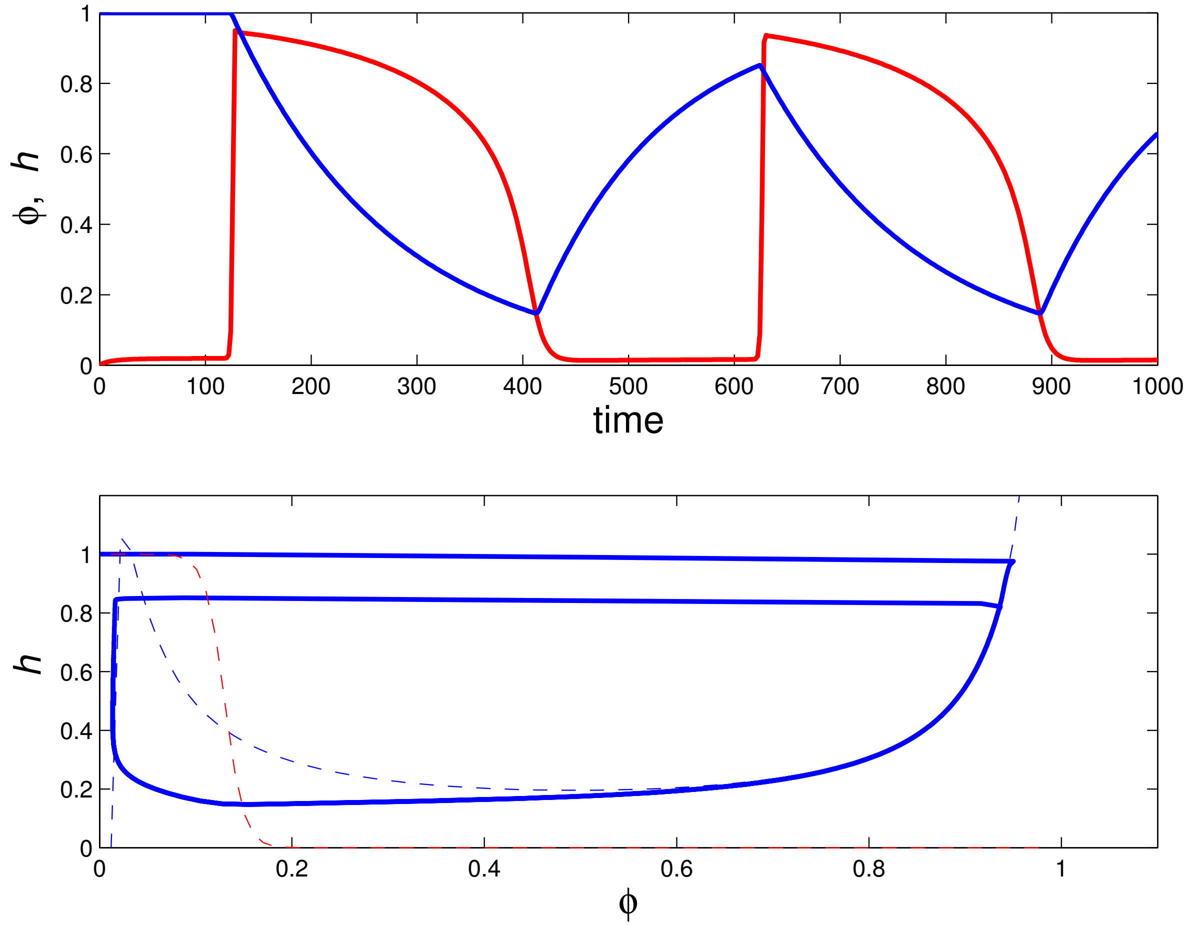
To make the Mitchell-Schaeffer look like an ionic model, take

$$\begin{aligned} C_m \frac{dv}{dt} &= g_{Na} h m^2 (V_{Na} - v) + g_K (V_K - v), \\ \tau_h \frac{dh}{dt} &= h_\infty(v) - h \end{aligned}$$

where

$$m(v) = \begin{cases} 0, & v < 0 \\ v, & 0 < v < 1 \\ 1, & v > 1 \end{cases}, \quad \begin{aligned} h_\infty &= 1 - f(v), \\ \tau_h &= \tau_{open} + (\tau_{close} - \tau_{open}) f(v) \\ f(v) &= \frac{1}{2}(1 + \tanh(\kappa(v - v_{gate}))), \end{aligned}$$

Mitchell-Schaeffer Revised-II



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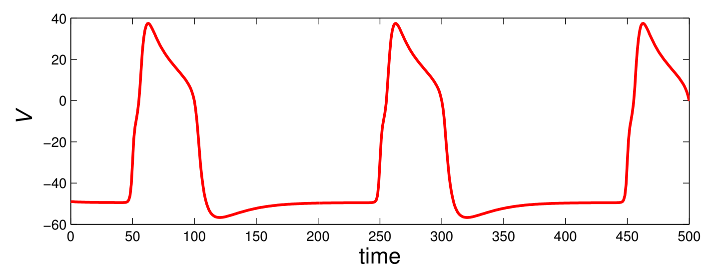
Morris-Lecar

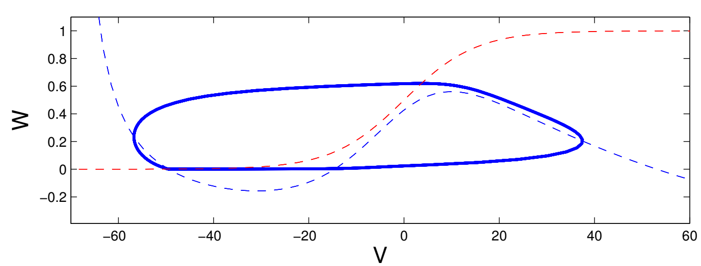
This model was devised for barnacle muscle fiber.

$$F(v,w) = -g_{ca}m_\infty(v)(v-v_{ca}) - g_k w(v-v_k) - g_l(v-v_l) + I_{app}$$

$$G(v,w) = \phi\cosh\left(\frac{1}{2}\frac{v-v_3}{v_4}\right)(w_\infty(v)-w),$$

$$m_\infty(v) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{v-v_1}{v_2}\right), \qquad w_\infty(v) = \left(1+\tanh\left(\frac{v-v_3}{2v_4}\right)\right).$$





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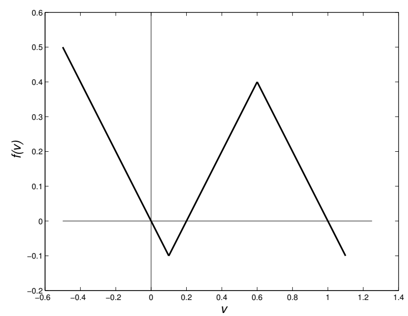
McKean

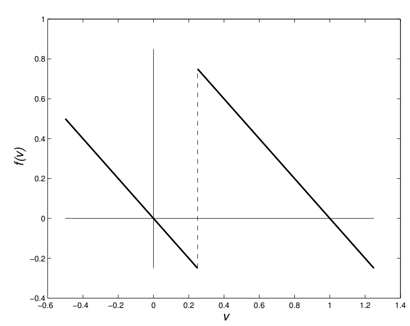
McKean suggested two piecewise linear models with $F(v,w) = f(v) - w$ and $G(v,w) = \epsilon(v - \gamma w)$. For the first,

$$f(v) = \begin{cases} -v & v < \frac{\alpha}{2} \\ v - \alpha & \frac{\alpha}{2} < v < \frac{1+\alpha}{2} \\ 1 - v & v > \frac{1+\alpha}{2} \end{cases}$$



where $0 < \alpha < \frac{1}{2}$.
The second model suggested by McKean had

$$f(v) = \begin{cases} -v & v < \alpha \\ 1 - v & v > \alpha \end{cases}$$



and $\gamma = 0$.

Barkley

A model devised to give very fast 2D computations (the code is known as EZspiral)

$$F(v,w) = v(1-v)(v - \frac{w+b}{a}),$$

$$G(v,w) = \epsilon(v-w).$$

Puschino

A piecewise linear model devised to match cardiac restitution properties

$$F(v,w) = f(v) - w$$

$$G(v,w)\frac{1}{\tau(v)}(v-w)$$

where

$$f(v) = \begin{cases} -30v, & v < v_1 \\ \gamma v - 0.12, & v_1 < v < v_2, \\ -30(v-1), & v > v_2 \end{cases}, \qquad \tau(v) = \begin{cases} 2 & v < \\ 16.6 & v > \end{cases}$$

with $v_1 = \frac{0.12}{30+\gamma}$, $v_2 = \frac{30.12}{30+\gamma}$. (Go Back)

Aliev

For the Aliev model,

$$F(v,w) = g_a(v-\beta)(v-\alpha)(1-v) - vw$$
$$G(v,w) = -\epsilon(v,w)(w + g_s(v-\beta)(v-\alpha-1)$$

where $\epsilon(v,w) = \epsilon_1 + \mu_1 \frac{w}{v+\mu_2}$.

Reasonable parameter values are $\beta = 0.0001$, $\alpha = 0.05$, $g_a = 8.0$, $g_s = 8.0$, $\mu_1 = 0.05$, $\mu_2 = 0.3$, $\epsilon_1 = 0.03$, $\epsilon_2 = 0.0001$.

Tyson-Fife

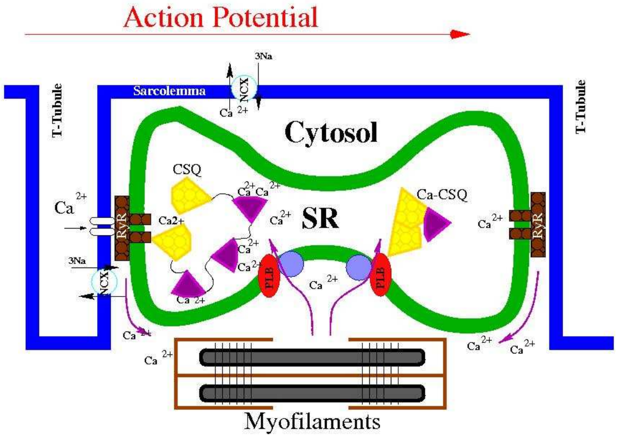
These dynamics describe the oxidation-reduction of malonic acid. For this system,

$$F(v,w) = v - v^2 - (fw + \phi_0)\frac{v-q}{v+q} \qquad \text{(-10)}$$

$$G(v,w) = \epsilon(v - w) \qquad \text{(-10)}$$

with typical parameter values $\epsilon = 0.05$, $q = 0.002$, $f = 3.5$, $\phi_0 = 0.01$.

Intracellular Calcium



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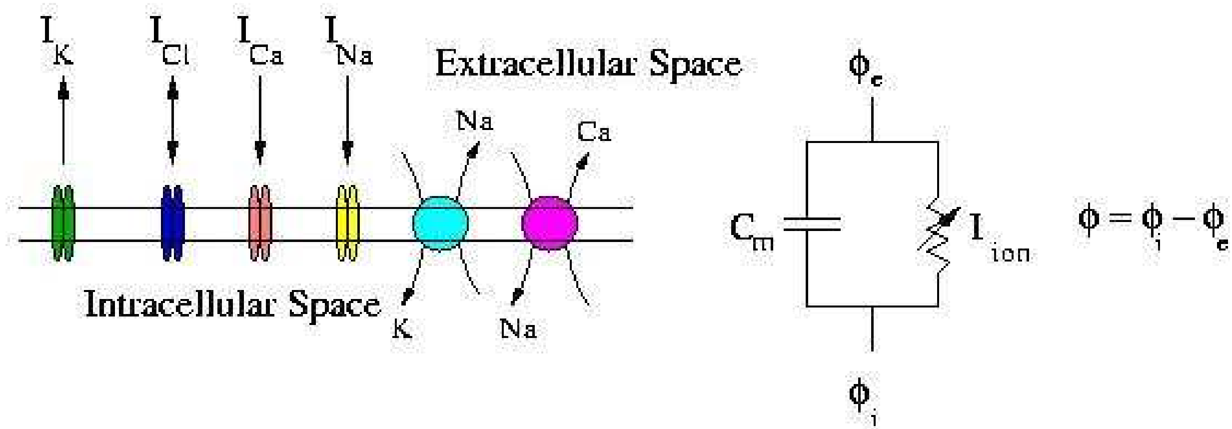
Features of Excitable Systems

Threshold Behavior, Refractoriness

Alternans

Wenckebach Patterns

Cardiac Models

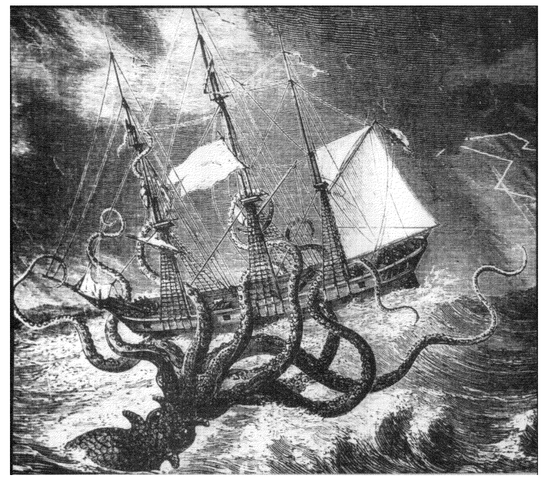


All cardiac models are of the form

$$C_m \frac{d\phi}{dt} + I_{ion}(\phi, w, [Ion]) = I_{in}$$

with currents, gating and concentrations for sodium, potassium, calcium, and chloride ions.

Hodgkin-Huxley Equations



$$C_m \frac{dV}{dt} = -g_{Na}(V - V_{Na}) - g_K(V - V_K) - g_L(V - V_L) + I_{\text{app}}, \quad \text{(-11)}$$

where

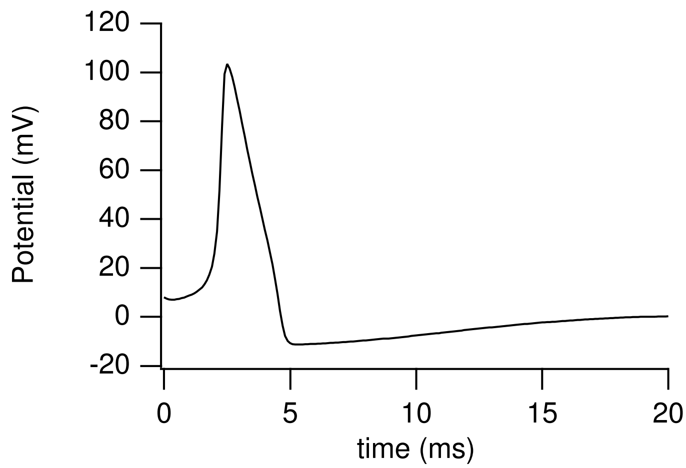
$$g_{Na}(v) = \bar{g}_{Na} m^3 h, g_K = \bar{g}_K n^4 \quad \text{(-11)}$$

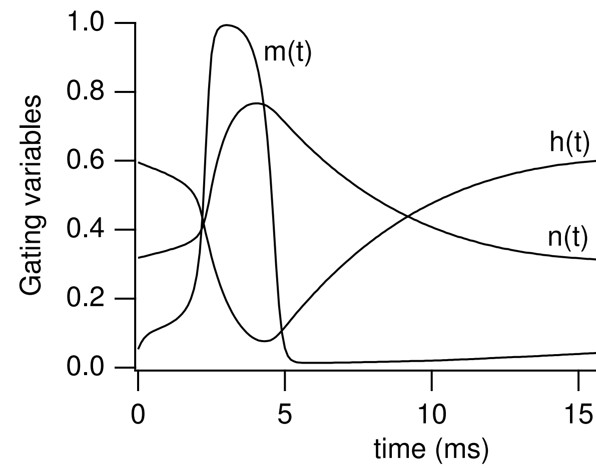
and

$$\frac{dm}{dt} = \alpha_m(1 - m) - \beta_m m, \quad \text{(-11)}$$

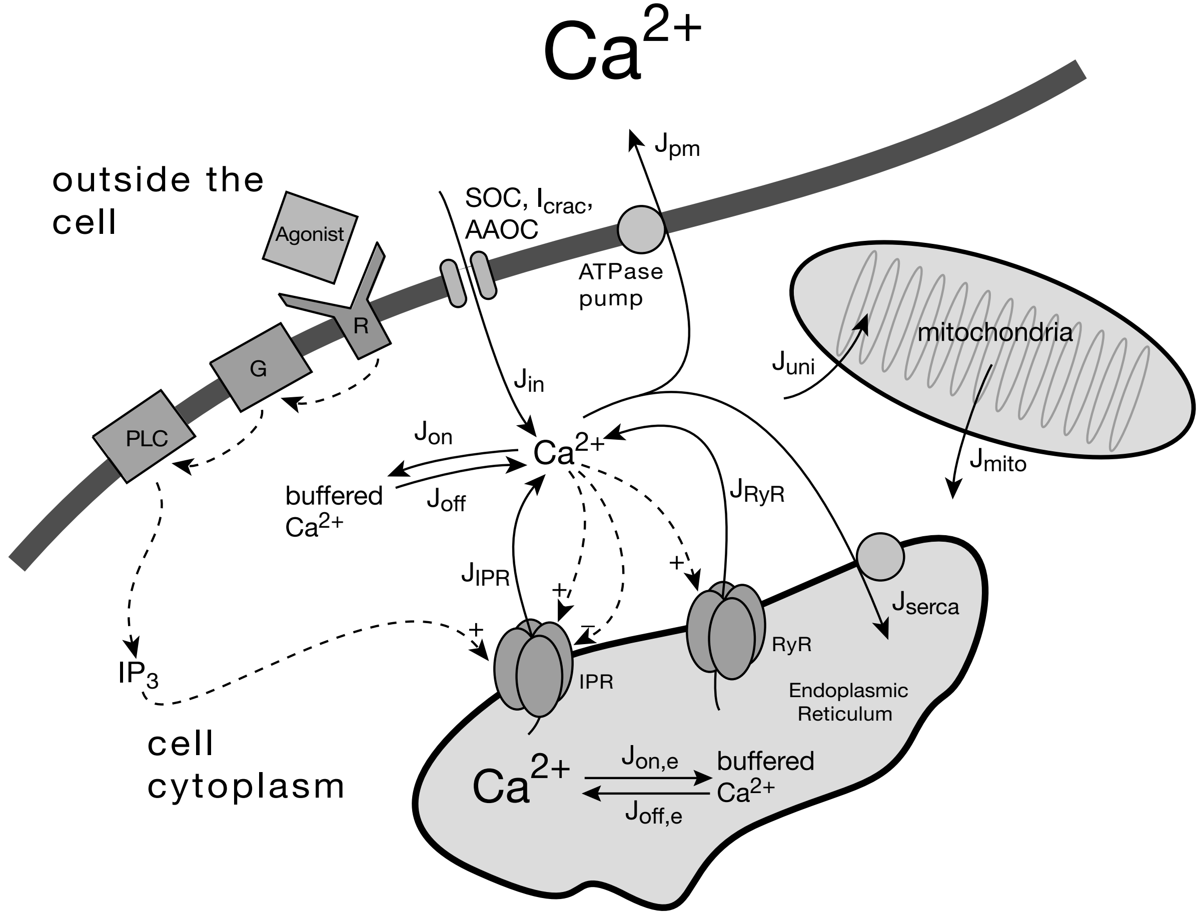
Equations for m, n, h can be written as

$$\tau_n(v)\frac{dw}{dt} = w_\infty(v) - w, w = m, n, \mathrm{or} h \qquad \text{(-13)}$$



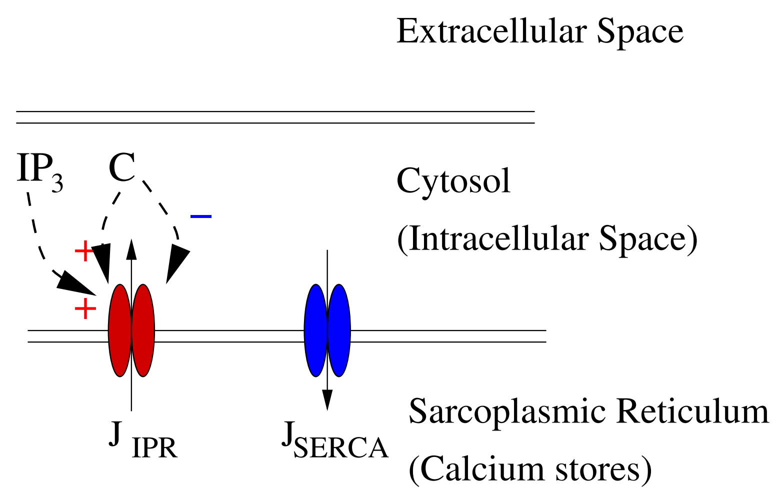


Calcium Handling



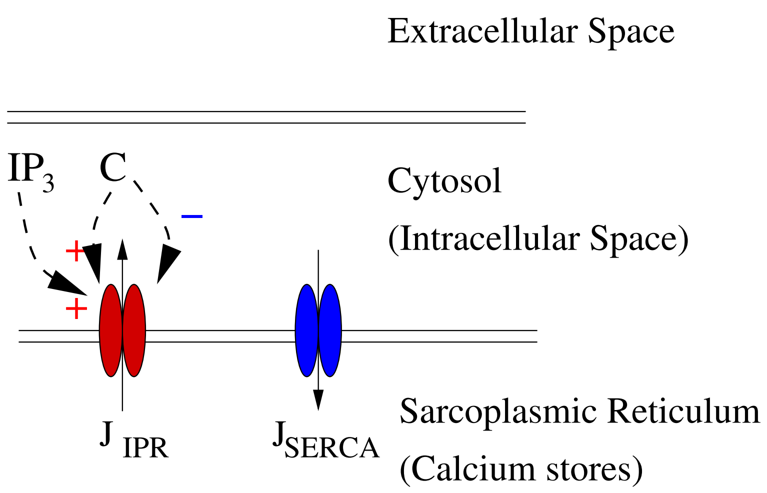
Basic Calcium Model

$$\frac{dc}{dt} = J_{IPR} - J_{SERCA}$$



Basic Calcium Model

$$\frac{dc}{dt} = \boxed{J_{IPR}} - J_{SERCA}$$

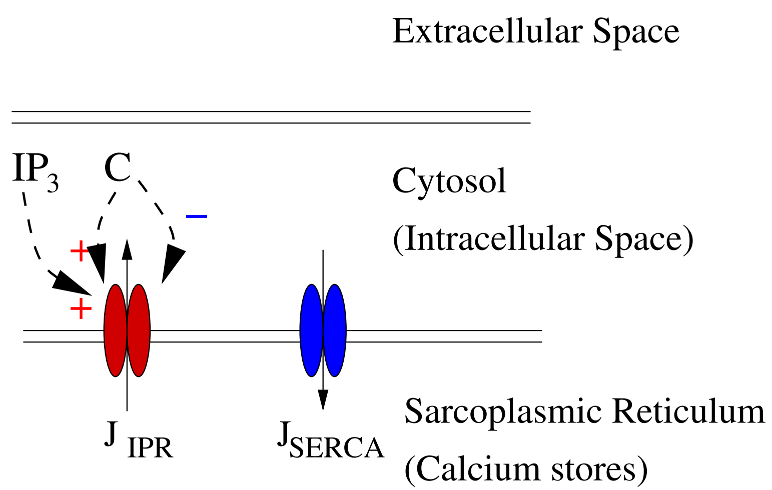


with

J_{IPR} IP$_3$ Receptor - IP$_3$ and calcium regulated calcium channel,

Basic Calcium Model

$$\frac{dc}{dt} = \boxed{J_{IPR}} - \boxed{J_{SERCA}}$$



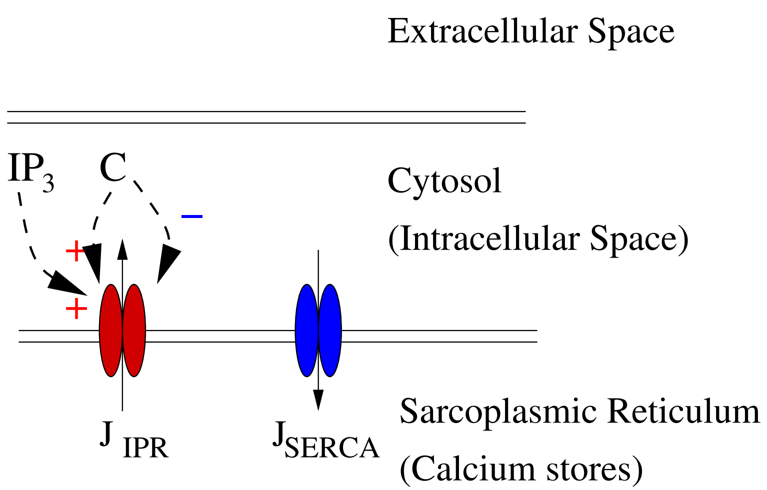
with

J_{IPR} IP$_3$ Receptor - IP$_3$ and calcium regulated calcium channel,

J_{SERCA} Sarco- and Endoplasmic Reticulum Calcium ATPase,

Basic Calcium Model

$$\frac{dc}{dt} = \boxed{J_{IPR}} - \boxed{J_{SERCA}}$$



with

J_{IPR} IP$_3$ Receptor - IP$_3$ and calcium regulated calcium channel,

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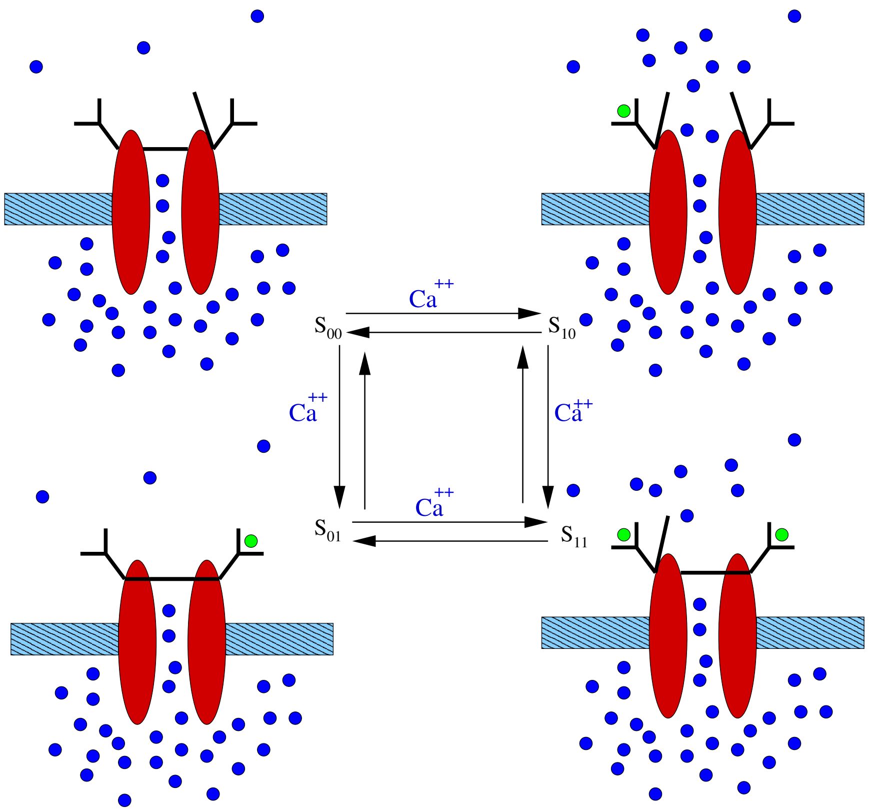
What are the flux terms?

IP_3 Receptors



Flux through IP_3 receptor is diffusive,

$$J_{IPR} = g_{max} P_o (c_{sr} - c)$$

where $P_o = S_{10}^3$ is the open probability.

Calcium Dynamics

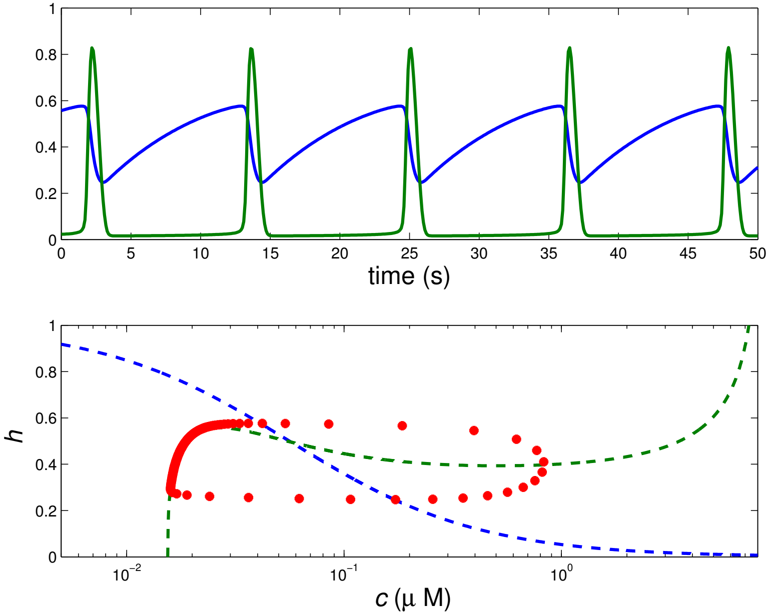
$$\frac{dc}{dt} = (g_{max}P_o + J_{er})(c_{sr} - c) - J_{SERCA},$$

$$\frac{dh}{dt} = \phi_h(c)(1-h) - \psi_h(c)h,$$

where

$$J_{SERCA} = V_{max}\frac{c^2}{K_s^2+c^2},$$

$$P_o = h^3 f(c)$$



Bifurcation Diagram