Math 5110 - Fall 2012
Homework Problem Set 1
Due Sept. 6, 2012

The purpose of this series of exercises is to become familiar with Matlab and to study iterative maps.

The following code can be made into a .m file and modified for the exercises, or you can write your own code.

```
%cobweb.m
% this file is to iterate a nonlinear map

N = 10; %number of iterates
a = 2; %parameter value
x = .5; %initial value

y = [x;x]; %store the x values
z = [0:100]/100;
fz = a*z.*(1-z);
xz = [x];

for j = 1:N
    f = a*x.*(1-x);
    x = f;
    xz = [xz;x];
    y = [y;x;x];
end
figure(1)
plot(y(1:2*N+1),[0;y(3:2*N+2)],z,fz,'r',z,z,'b--')
figure(2)
plot(xz,'*');
```

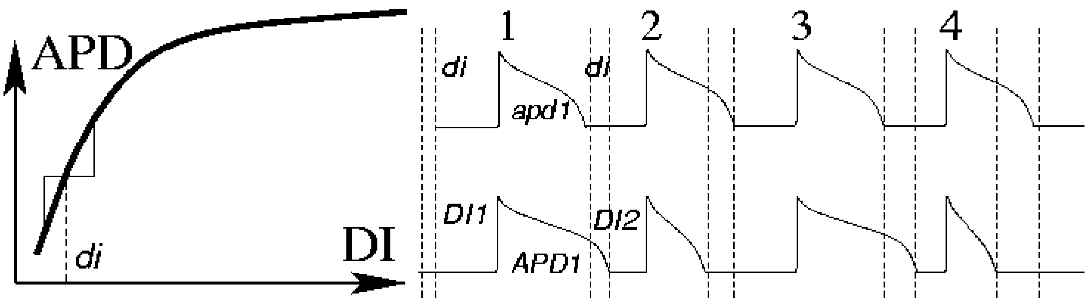

Figure 1: **A**ction **P**otential **D**uration Restitution Curve

1. Read Chapter 1 of Edelstein-Keshet.

2. When a cardiac cell is stimulated, it responds with an electrical event called an action potential, the length of which is called the action potential duration (APD). Suppose that a cardiac cell is stimulated periodically with a basic cycle length BCL. The time period between stimuli consists of two segments, the Action Potential Duration (APD) and the Diastolic Interval (DI) (when the cell is at rest). Thus,

$$APD + DI = BCL. \tag{1}$$

It is observed that the action potential duration depends on the previous diastolic interval, $APD_n = A(DI_{n-1})$. This relationship is called the APD restitution curve, and is depicted in Fig. 1. It follows that on the n^{th} cycle,

$$DI_n = BCL - A(DI_{n-1}), \tag{2}$$

Perform a numerical study of the APD map, with $A(DI) = 100 - \frac{100}{DI-25}$. Determine the following: (Show plots demonstrating your answers.)

   (a) For what BCL values is there a fixed point?

   (b) For what values of BCL is the fixed point stable?

   (c) What is the stable behavior when the fixed point is unstable, and for what values of BCL does this happen?

3. Consider a discrete-time map for the concentration of medication in the bloodstream, M_t

$$M_{t+1} = M_t - f(M_t)M_t + S, \tag{3}$$

where $F(M_t)$ is the fraction absorbed and S is the daily dosage. Suppose that

$$f(M) = \frac{M^n}{K^n + M^n}. \tag{4}$$

Set $K = 2$ and $S = 1$. Find the equilibrium value. For which values of n do solutions oscillate toward the equilibrium? Are there values of n for which the equilibrium is unstable? Make a cobweb diagram for an illuminating case.

4. Newton's method for numerically solving equations can be formalized as a discrete-time dynamical system. Suppose we wish to find $\sqrt{3}$ by finding the roots of $f(x) = x^2 - 3$. Newton's method begins by making a guess, called x_0, and then approximating $f(x)$ with a linear function $\hat{f}(x)$ that is tangent to $f(x)$ at x_0 and then solving $\hat{f}(x) = 0$ to find a new guess for the root x_1.

   Using the function $f(x) = x^2 - 3$ as an example,

   (a) Graph $f(x)$ and $\hat{f}(x)$ at $x_0 = 2$.

   (b) Solve $\hat{f}(x) = 0$ to find x_1;

   (c) Write down the discrete-time dynamical system that gets from one approximate solution to the next.

   (d) Find the equilibrium and its stability for this map.

   (e) Make a cobweb diagram for this map.

   (f) Newton's method converges to the correct solution with great speed. Why?

5. The following model describes a pest population of size N_t that is controlled by the annual release of S sterile males,

$$N_{t+1} = \left(\frac{R}{1 + \frac{(R-1)N_t}{K}} \right) \left(\frac{N_t}{N_t + S} \right) N_t. \tag{5}$$

   The first term represents the per capita egg production of a female as a function of total population size, and the second term gives the probability that the male she mates with is fertile (females mate only once).

   Show that there is a critical value of S above which the population goes extinct. Sketch a bifurcation diagram describing what happens as a function of S. Does the population disappear gradually or catastrophically as S passes a critical value?