

Math 5110 - Fall 2012
Homework Problem Set 5
Due Nov. 5, 2012

1. The differential equation

$$\frac{dN}{dt} = rN \ln\left(\frac{K}{N}\right) \quad (1)$$

is called the Gompertz differential equation. Use the transformation $u = \ln N$ to transform this equation into a linear differential equation and then use this to solve the equation for N .

2. Suppose the population size of some species of organism follows the model

$$\frac{dN}{dt} = \frac{3N^2}{2 + N^2} - N \quad (2)$$

- (a) Find the steady states.
 - (b) Draw the "phase-line" diagram.
 - (c) Which of the equilibria are stable and which are unstable?
 - (d) Interpret these results in biological terms. Why might this population behave as it does for small values?
3. Analyze the behavior of the population model

$$\frac{du}{dt} = ru\left(1 - \frac{u}{K}\right) - \frac{u^2}{1 + u^2} \quad (3)$$

Find the different behavior when there is population size dependent growth (K finite) and when there is no such size dependence ($K \rightarrow \infty$). Explain why the behaviors are different.

4. It is a sad fact that cannibalism is rampant in the natural world. One model describing this is

$$\frac{dJ}{dt} = r(J)A - \mu AJ - \theta J, \quad (4)$$

$$\frac{dA}{dt} = \theta J - \delta A. \quad (5)$$

Suppose that $r(J)$ is a linearly increasing function $r(J) = r_0 + \eta\mu J$ with $\eta < 1$.

- (a) Explain the meaning of each term in the model.
 - (b) Draw the phase plane portrait in all of its glory.
 - (c) Find every equilibrium point and its stability (using the Jacobian matrix).
 - (d) Under what conditions can this population survive?
5. Gonorrhea is a disease for which there is no permanent immunity (an S-I-S disease). Because it is a sexually transmitted disease, a model of this disease needs to account for both male and female populations.

Consider the following model for transmission of gonorrhea.

$$\frac{dI_f}{dt} = \lambda_m I_m (N_f - I_f) - d_f I_f, \quad (6)$$

$$\frac{dI_m}{dt} = \lambda_f I_f (N_m - I_m) - d_m I_m. \quad (7)$$

- (a) Explain the meaning of all the terms in these equations. (Subscript f means female and subscript m means male.) Describe this process in terms of chemical reactions.
- (b) Rescale the equations so that the independent variables are fractions which always lie between 0 and 1.
- (c) Draw the phase plane portrait in all of its glory. How many different cases are there?
- (d) Find every equilibrium point and its stability.
- (e) Under what conditions is the disease endemic?
- (f) Suppose that $d_f = \frac{1}{50}$ per day and $d_m = \frac{1}{10}$ per day. What does this imply about the difference between males and females who get the disease?
- (g) Suppose $\frac{\lambda_f}{d_f} = \frac{\lambda_m}{d_m} = 2$. Suppose also that the total population has $N_f = N_m = 1000$. Is the disease endemic and if so, what is the size of the infected population?
- (h) Suppose one of the populations, but not both, could be halved. Which reduction would be most effective, that is, would result in the smallest total infected population? What would the reduced infected population size be? Describe what happens to each of the individual infected groups under these two possible reduction plans.
- (i) Simulate the stochastic version of this process. What differences do you observe between the stochastic and deterministic models?