6.1 Simple Interest, Sequences

Simple Interest

$I = Prt$

P = principal (original amt invested)

r = rate (annual interest rate, written as decimal)

t = time (in yrs)

I = interest (after t years)

$\Rightarrow$ future value of investment $= P + I = S$
(where S = future value)

$\Rightarrow$ **Future value for account earning simple interest**

$$\boxed{S = P + Prt = P(1+rt)}$$

Ex 1 If \$10,000 is invested for 3 years at an annual interest rate of 8%, how much interest will be received at the end of the 3-year period?

Math1090
(8f)

6.1 (cont)

Ex 2 You borrow \$2000 at an interest rate of 20%, how much interest is due in 39 weeks?

Ex 3 If \$1000 is invested at 4% simple interest, how long will it take to double?

6.1 (cont)

Sequence Function $\Rightarrow$ a function w/ domain of positive integers (counters)

(a_i = i^{th} term of sequence)

outputs $a_1, a_2, a_3, a_4, a_5, \ldots$

ex $\underset{a_1}{2}, \underset{a_2}{5}, \underset{a_3}{8}, \underset{a_4}{11}, \underset{a_5}{14}, \ldots$ $a_n = 3n - 1$

Ex 4 Write first 5 terms of sequence defined by $a_n = \dfrac{(-1)^n}{3n}$

Arithmetic Sequence $\Rightarrow$ a sequence whose consecutive terms differ by same amount (by adding/subtracting)

$a_n = a_{n-1} + d \quad n > 1$

ex $3, 7, 11, 15, 19, \ldots$

$+4 \quad +4 \quad +4 \quad +4$

Ex 5 Are these arithmetic sequences?

(a) $-2, 5, 12, 19, 26, \ldots$

(b) $21, 15, 9, 3, -3, -9, \ldots$

(c) $2, 6, 18, 54, \ldots$

6.1 (cont)

$a_n = a_{n-1} + d$ is recursive formula (it depends on previous terms to get to next term)

Can we get iterative formula? (i.e. a direct formula)

$a_1 = a_1$

$a_2 = a_1 + d$

$a_3 = a_2 + d = a_1 + d + d = a_1 + 2d$

$a_4 = a_3 + d = a_1 + 2d + d = a_1 + 3d$

$a_5 = a_4 + d = a_1 + 3d + d = a_1 + 4d$

$\vdots$

$\boxed{a_n = a_1 + (n-1)d}$ Arithmetic sequence

Ex 6 (a) Find 12^{th} term of arithmetic sequence w/ first term of 5 and a common difference of 3.

(b) If 1^{st} term of arithmetic sequence is -3 and 7^{th} term is 21, find 21^{st} term.

6.1 (cont)

ex Gauss' cool way of adding $1+2+3+\ldots+100$.

Sum of 1st n terms of an arithmetic sequence

$$S_n = \frac{n}{2}(a_1 + a_n)$$

★ proof on pg 93

Ex 7 Find sum of first 12 terms of arithmetic sequence $1/4, 7/12, 11/12, \ldots$

$a_1 = ?$ $d = ?$

$a_{12} = ?$

$\Rightarrow S_{12} =$

6.1 (cont)

Proof that $S_n = \frac{n}{2}(a_1 + a_n)$ for arithmetic sequence

Since $\{a_n\}$ is arithmetic sequence we know

$$a_n = a_1 + (n-1)d \iff a_n - (n-1)d = a_1$$

① $S_n = a_1 + a_2 + a_3 + \cdots + a_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-1)d)$

and also

② $S_n = a_n + a_{n-1} + a_{n-2} + \cdots + a_1 = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_n - (n-1)d)$

Now, add ① and ②

$$\Rightarrow 2S_n = [a_1 + a_1 + \cancel{d} + a_1 + 2\cancel{d} + \cdots + a_1 + \cancel{(n-1)d}] + [a_n + a_n - \cancel{d} + a_n - \cancel{2d} + \cdots + a_n - \cancel{(n-1)d}]$$

$$= \underset{1}{(a_1 + a_n)} + \underset{2}{(a_1 + a_n)} + \underset{3}{(a_1 + a_n)} + \cdots + \underset{n}{(a_1 + a_n)}$$

i.e. $2S_n = n(a_1 + a_n)$

$$\iff S_n = \frac{n}{2}(a_1 + a_n)$$

//

6.2 Compound Interest; Geometric Sequences

Let's look at \$10,000 invested at 10% interest compounded annually.

then $S_1 = 10{,}000(1+0.1(1)) = 10000(1.1) = \$11000$ (after 1st year)

$S_2 = 11000(1+0.1) = \$12100$ (after 2 years)

$S_3 = 12100(1+0.1) = \$13310$ (after 3 yrs)

$\vdots$

$\Rightarrow$ $S_1 = P(1+r)$

$S_2 = [P(1+r)](1+r) = P(1+r)^2$

$S_3 = [P(1+r)^2](1+r) = P(1+r)^3$

$\vdots$

$S_n = P(1+r)^n$ = future value of account after n years, earning compound interest (compounded annually) P = principal, r = annual int. rate

Ex 1 If \$5000 is invested for 5 years at 9% interest, compounded annually, what is the value of account?

6.2 (cont)

Future value of compounded interest account

$$S = P\left(1+\frac{r}{n}\right)^{nt}$$

P = principal

S = future value

t = time (in years)

r = annual interest rate

n = # compoundings per year

Ex 2 What amount must be invested now in order to have \$1,000,000 for retirement in 35 years if money is compounded quarterly at 9%?

What about in 45 years?

6.2 (cont)

☆ look at table 6.2 (pg 419)

Continuous Compounding Future value

$$\boxed{S = Pe^{rt}}$$

Ex 3 Find the future value if \$3,000 is invested for 15 years at 8% interest compounded continuously.

Let $P = \$100$ be invested at 8% interest compounded (a) quarterly or (b) monthly.

(a) $S = 100\left(1 + \frac{0.08}{4}\right)^{4(1)} = 100\,(1.02)^4 = \$108.24$ after 1 year

(b) $S = 100\left(1 + \frac{0.08}{12}\right)^{12(1)} = \$108.30$ after 1 year

6.2 (cont)

⇒ So investing at 8% compounded quarterly is equivalent to investing same money at 8.24% simple interest, and compounded monthly is equivalent to 8.3% simple interest.

⇒ APY = annual percentage yield (or effective annual rate at simple interest)

$$\boxed{\begin{array}{ll} APY = \left(1+\frac{r}{m}\right)^m - 1 & \text{(periodic compounding)} \\ APY = e^r - 1 & \text{(continuous compounding)} \end{array}}$$

Ex 4 Which is better investment deal? (hint: look at APY)

(a) 10% compounded annually

(b) 9.8% " quarterly

(c) 9.65% " continuously

6.2 (cont)

Ex 5 How long does it take $25,000 to double if it's invested at 9% (a) compounded yearly or (b) compounded continuously.

(a)

(b)

Geometric Sequence ⇒ a sequence whose consecutive terms differ by common ratio (i.e. we get from one term to the next by multiplying by same ratio).

$$\underline{\underline{a_n = r a_{n-1} \quad n > 1}}$$

ex $1, 5, 25, 625, \ldots$

$\times 5 \quad \times 5 \quad \times 5$

6.2 (cont)

Ex 6 Write the next 3 terms of geometric sequences

(a) $2, 10, 50, \ldots$

(b) $3, -6, 12, \ldots$

(c) $-2, -2/3, -2/9, \ldots$

$a_n = r a_{n-1}$ is recursive formula

Can we find iterative formula?

$a_1 = a_1$

$a_2 = r a_1$

$a_3 = r a_2 = r(r a_1) = r^2 a_1$

$a_4 = r a_3 = r(r^2 a_1) = r^3 a_1$

$a_5 = r a_4 = r(r^3 a_1) = r^4 a_1$

$\vdots$

$\boxed{a_n = r^{n-1} a_1}$ Geometric sequence direct formula (iterative)

Ex 7 Find 10^{th} term of geometric sequence w/ first term of 2 + common ratio -3.

6.2 (cont)

$$\boxed{\begin{array}{l}\text{Sum of first } n \text{ terms of geometric sequence} \\ S_n = \dfrac{a_1(1-r^n)}{(1-r)} \quad \text{for all } r \neq 1 \\ \text{(proof on pg (101))}\end{array}}$$

Ex 8 Find sum of first 8 terms of geometric sequence

$$\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \ldots$$

6.2 (cont)

Proof Sum of Geometric sequence first n terms

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

where $a_n = r^{n-1} a_1$

$$\Rightarrow S_n = a_1 + ra_1 + r^2 a_1 + r^3 a_1 + \dots + r^{n-1} a_1$$

$$\Rightarrow rS_n = ra_1 + r^2 a_1 + r^3 a_1 + r^4 a_1 + \dots + r^n a_1$$

and $$S_n - rS_n = (a_1 + \cancel{ra_1} + \cancel{r^2 a_1} + \cancel{r^3 a_1} + \dots + \cancel{r^{n-1} a_1})$$
$$- (\cancel{ra_1} + \cancel{r^2 a_1} + \cancel{r^3 a_1} + \cancel{r^4 a_1} + \dots + r^n a_1)$$

$$\Leftrightarrow S_n - rS_n = a_1 - r^n a_1$$

$$S_n(1-r) = a_1 - r^n a_1$$

$$S_n = \frac{a_1 - r^n a_1}{1-r} = \frac{a_1(1-r^n)}{1-r} \quad //$$