HW#11 Graded Problems

5.3 # 14, 18, 23, 26, 34

5.4 # 4, 8, 15 (32 extra credit)

7.5 # 6, 10 (22 extra credit)

5.3 #14

choose 2 +ve integers less than 100

$${}_{99}C_2 = \frac{99(98)}{2} = 99(49) = 4851$$

5.3 #18

8-coin flip

(a) # outcomes $= 2^8 = 256$

(b) # outcomes w/ 3 H $= {}_8C_3 = 56$

(c) # outcomes w/ at least 3 H $= {}_8C_3 + {}_8C_4 + {}_8C_5 + {}_8C_6 + {}_8C_7 + {}_8C_8$

$$= 2^8 - {}_8C_2 - {}_8C_1 - {}_8C_0 = 256 - 28 - 8 - 1 = 219$$

(d) # outcomes w/ same # H & T $= {}_8C_4 = 70$

5.3 #23

8M, 5W in line, no two women stand next to each other

$\wedge_1\ M_1\ \wedge_2\ M_2\ \wedge_3\ M_3\ \wedge_4\ M_4\ \wedge_5\ M_5\ \wedge_6\ M_6\ \wedge_7\ M_7\ \wedge_8\ M_8\ \wedge_9$

places for women must be in between men

there are 8! ways to order men

and 9 positions to place 5 women $= {}_9P_5$

$$\Rightarrow \text{total ways} = 8!\,({}_9P_5) = 8!\,(9\cdot8\cdot7\cdot6\cdot5) = 609{,}638{,}400$$

5.3 #26 | 13 people

(a) $_{13}C_{10} = \frac{13(12)(11)}{3\cdot 2} = 13(22) = 286$

(b) $_{13}P_{10} = \frac{13!}{3!} = 1{,}037{,}836{,}800$

(c) 3W choose 10 w/
10M at least one W

$\# \text{ ways} = {}_3C_1\,{}_{10}C_9 + {}_3C_2\,{}_{10}C_8 + {}_3C_3\,{}_{10}C_7$

$= 286 - {}_3C_0\,{}_{10}C_{10} = 286 - 1 = 285$

5.3 # 34 | 10M form committee w/ 6 members
15W #W > #M

$_{15}C_4\,{}_{10}C_2 + {}_{15}C_5\,{}_{10}C_1 + {}_{15}C_6\,{}_{10}C_0 = \left(\frac{15(14)(13)(12)}{4\cdot 3\cdot 2}\right)\left(\frac{10\cdot 9}{2}\right)$

$+ \left(\frac{15(14)(13)(12)(11)}{5\cdot 4\cdot 3\cdot 2}\right)(10) + \frac{15(14)(13)(12)(11)(10)}{6\cdot 5\cdot 4\cdot 3\cdot 2}(1)$

$= 15(13)(35)(9) + 15(14)(13)(11) + 35(13)(11)$

$= 96460$

5.4 # 4 | coeff of x^5y^8 in $(x+y)^{13}$

$\text{coeff} = \binom{13}{8} = \binom{13}{5} = \frac{13(12)(11)(10)(9)}{5(4)(3)(2)} = 13(11)(9) = 1287$

5.4 # 8 | coeff of x^8y^9 in $(3x+2y)^{17}$

$\binom{17}{9}(3x)^8(2y)^9 = \frac{17(16)(15)(14)(13)(12)(11)(10)}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}(3^8)(2^9)\,x^8y^9$

$= (13)(17)(110)(3^8)(2^9)\,x^8y^9$

coeff = $\boxed{81662929920}$

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5.4 #15 | show $\binom{n}{k} \le 2^n$ $\forall n \in \mathbb{Z}^+$ and all $k \ni 0 \le k \le n$.

We know $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ $\Rightarrow$ $\binom{n}{k} \le 2^n$ i.e. each term of sum is less than the sum

5.4 #32 (extra credit) |

Prove Binomial Thm using induction

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \quad \forall n = 0, 1, 2, 3, \ldots \quad \& \text{ all } x, y \in \mathbb{R}$$

$n=0$ $(x+y)^0 = 1$

$\sum_{j=0}^{0} \binom{0}{j} x^{-j} y^j = 1$ ✓

pf $n=1$, $(x+y)^1 = x+y$

$$\sum_{j=0}^{1} \binom{1}{j} x^{1-j} y^j = \binom{1}{0} x^1 y^0 + \binom{1}{1} x^0 y^1 = x+y \quad ✓$$

Assume true for n.

check n+1 case.

We know $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$

$$(x+y)^{n+1} = (x+y) \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \sum_{j=0}^{n} \binom{n}{j} x^{n+1-j} y^j + \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^{j+1}$$

$$= \sum_{j=0}^{n} \binom{n}{j} x^{n+1-j} y^j + \sum_{j=1}^{n+1} \binom{n}{j-1} x^{n+1-j} y^j$$

$$= \binom{n}{0} x^{n+1} + \binom{n}{n} y^{n+1} + \sum_{j=1}^{n} \binom{n}{j} x^{n+1-j} y^j + \sum_{j=1}^{n} \binom{n}{j-1} x^{n+1-j} y^j$$

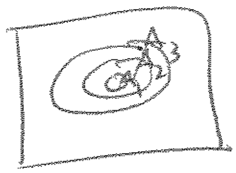
$$= x^{n+1} + y^{n+1} + \sum_{j=1}^{n} \left[\binom{n}{j} + \binom{n}{j-1} \right] x^{n+1-j} y^j$$

$$= (x^{n+1} + y^{n+1}) + \sum_{j=1}^{n} \binom{n+1}{j} x^{n+1-j} y^j = \sum_{j=0}^{n} \binom{n+1}{j} x^{n+1-j} y^j \;//$$

7.5 #6 # of elements in $A_1 \cup A_2 \cup A_3$ if

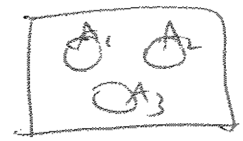
$|A_1| = 100 \quad |A_2| = 1000 \quad |A_3| = 10{,}000$

and (a) $A_1 \subseteq A_2 \subseteq A_3$



$\Rightarrow |A_1 \cup A_2 \cup A_3| = |A_3| = 10{,}000$

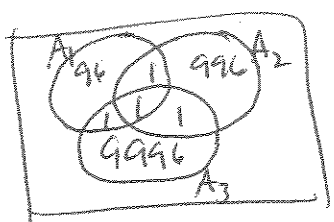
(b) sets are pairwise disjoint



$\Rightarrow |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$

$= 11{,}100$

(c) 2 elements common to each pair + one element in 3-set intersection



$\Rightarrow |A_1 \cup A_2 \cup A_3| = 9996 + 996 + 96 + 4$

$= 11092$

7.5 # 10 # of +ve integers not exceeding 100 $\{1, 2, \ldots, 100\}$
that are divisible by 5 or 7

= # divisible by 5 + # divisible by 7 − # divisible by both 5 & 7

$= \lfloor \frac{100}{5} \rfloor + \lfloor \frac{100}{7} \rfloor - \lfloor \frac{100}{35} \rfloor = 20 + 14 - 2 = 32$

not divisible by 5 or 7 = 68

7.5 # 22 (extra credit) Prove $A_1, \ldots, A_n$ finite sets $n = 2, 3, \ldots$

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j|$$

$$+ \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

using induction

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7.5 #22 (cont)

pf ① $n=2$: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
we already know

② Assume true for n. Check $n+1$ case.
Let $B = A_n \cup A_{n+1}$. Then $A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup B$ is union of n sets.
Then by induction assumption,

(☆) $$|A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup B| = \left(\sum_{1\le i\le n-1} |A_i| + |B|\right) - \left(\sum_{1\le i<j\le n-1} |A_i \cap A_j| + \sum_{i\le n-1} |A_i \cap B|\right) + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_{n-1} \cap B|$$

notice: $\sum |(A_{i_1} \cap \cdots \cap A_{i_m}) \cap B|$

$$= \sum |(A_{i_1} \cap \cdots \cap A_{i_m}) \cap (A_n \cup A_{n+1})|$$

distributive property

$$= \sum |(A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n) \cup (A_{i_1} \cap \cdots \cap A_{i_m} \cap A_{n+1})|$$

$$= \sum |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n| + \sum |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_{n+1}| - \sum |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n \cap A_{n+1}|$$

When we plug this in to every term of (☆), we get exactly what we need. //