5.4 Binomial Coefficients

Thm 1 (Binomial Thm)

$n \in \mathbb{W}$ (constant) $x, y \in \mathbb{R}$ (variables)

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Ex 1 what is coefficient of $x^{11}y^{15}$ in $(3x-4y)^{26}$?

Corollary 1 $n \in \mathbb{W}$. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

i.e. n^{th} row of Pascal's Δ adds to 2^n

Pf use Binomial Thm w/ $x=y=1$ //

Corollary 2 $n \in \mathbb{Z}^+$ $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

Pf use Binomial Thm w/ $x=1$ $y=-1$. //

Corollary 3 $n \in \mathbb{W}$ $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$

Pf use Binomial Thm w/ $x=1, y=2$ //

5.4 (cont)

Ex 2 Give formula for coefficient of x^k in expansion of $(x^2 - \frac{1}{x})^{100}$ $k \in \mathbb{Z}^+$.

$$\boxed{\textbf{Thm 2} \text{ (Pascal's Identity) } n, k \in \mathbb{Z}^+,\ n \geq k. \quad \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}}$$

Pf

5.4 (cont)

Thm 3 (Vandermonde's Identity) $m, n, r \in \mathbb{W}$ $r \le m$, $r \le n$

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Ex 3 Show that if n & j are integers $\ni$ $1 \le j \le n$, then $\binom{n}{j} \le \frac{n^j}{2^{j-1}}$.

Ex 4 Prove $\binom{2n}{n+1} + \binom{2n}{n} = \frac{1}{2}\binom{2n+2}{n+1}$ $\forall$ $n \in \mathbb{Z}^+$.

5.5 Generalized Permutations & Combinations

Thm 1 Number of r-permutations of a set of n objects w/ repitition allowed is n^r.

Thm 2 There are ${}_{n+r-1}C_r$ r-combinations from a set w/ n elements when repitition is allowed.

Ex 1 (a) Suppose a donut shop has 2 different types of donuts. How many different ways can 5 donuts be chosen?

(b) what if there are 3 types and I want 5 donuts?

(c) what if there are 5 types + I want 12 donuts?

5.5 (cont)

Thm 3 The # of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k is

$$\frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

Thm 4 The # of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i=1,2,\ldots,k$, equals

$$\frac{n!}{n_1! n_2! \cdots n_k!}$$

Pf # ways to put n_1 objects in 1st box $= {}_nC_{n_1}$

" " " n_2 " " 2nd box $= {}_{n-n_1}C_{n_2}$

" " " n_3 " " 3rd box $= {}_{n-n_1-n_2}C_{n_3}$

⋮

$\Rightarrow$ use product rule, ${}_nC_{n_1} \, {}_{n-n_1}C_{n_2} \, {}_{n-n_1-n_2}C_{n_3} \cdots {}_{n-n_1-n_2-\cdots-n_{k-1}}C_{n_k}$

$$= \frac{n!}{\cancel{(n-n_1)!}\, n_1!} \cdot \frac{\cancel{(n-n_1)!}}{\cancel{(n-n_1-n_2)!}\, n_2!} \cdot \frac{\cancel{(n-n_1-n_2)!}}{\cancel{(n-n_1-n_2-n_3)!}\, n_3!} \cdots \frac{\cancel{(n-n_1-n_2-\cdots-n_{k-1})!}}{(n-n_1-n_2-\cdots-n_k)!\, n_k!}$$

(notice: $n-n_1-n_2-\cdots-n_k = 0$)

$$= \frac{n!}{n_1! n_2! \cdots n_k!}$$

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5.5 (cont)

Ex2 How many ways are there to select 6 unordered elements from a set w/ 4 elements when repetition is allowed?

Ex3 How many different combinations of pennies, nickels, dimes, quarters and half dollars can a piggy bank contain if it has 20 coins in it?

Ex4 How many solutions are there to the eqn
$x_1 + x_2 + x_3 + x_4 + x_5 = 29$ all $x_i \in \mathbb{W}$.

5.5 (cont)

Ex5 How many different strings can be made from the letters in AARDVARK, using all the letters?

Ex6 We have 9 balls and 6 bins. How many ways can we distribute balls into bins, if

(a) balls and bins are distinguishable, and we want 3 balls in one bin, 2 in another bin and 1 in all other bins.

(b) balls are indistinguishable, bins are distinguishable

5.5 (cont)

Ex 6 (c) balls are distinguishable, bins are indistinguishable, and we want 3 balls in one bin, 2 in another bin + 1 in all other bins

(d) balls and bins are indistinguishable