Math2200 Final Exam Fall, 2010 Kelly MacArthur

Name Key Date

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. You can use one 8.5x11 inch piece of paper for reference, and a calculator.

1. Construct a truth table for each of these statements.

(a) $(-p)\vee(q\rightarrow p)$

p	q	$\neg p$	$q\rightarrow p$	$\neg p\vee(q\rightarrow p)$
T	T	F	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(b) $(p\vee q)\rightarrow(p\wedge -r)$

p	q	r	$\neg r$	$p\vee q$	$p\wedge\neg r$	$(p\vee q)\rightarrow(p\wedge\neg r)$
T	T	T	F	T	F	F
T	T	F	T	T	T	T
T	F	T	F	T	F	F
T	F	F	T	T	T	T
F	T	T	F	T	F	F
F	T	F	T	T	F	F
F	F	T	F	F	F	F
F	F	F	T	F	F	F

2. Prove that if x is irrational and nonnegative, then $\sqrt{x}$ is also irrational.

given x irrational and $x \geq 0$.

Pf Assume $\sqrt{x}$ is rational.

Then $\exists$ $p, q \in \mathbb{Z}^+$, $q \neq 0$, $\gcd(p,q) = 1$

$\ni$ $\sqrt{x} = \frac{p}{q}$

$\Rightarrow$ $x = \frac{p^2}{q^2}$ but since $p, q \in \mathbb{Z}^+$, then $p^2, q^2 \in \mathbb{Z}^+$ also

which means x is a rational #.

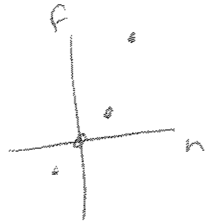
This is a contradiction.

$\Rightarrow$ $\sqrt{x}$ is irrational.

3. Give an example of a function $f: \mathbf{Z} \to \mathbf{Z}$ which is
(a) one-to-one but not onto.
(b) onto but not one-to-one.
(c) one-to-one and onto.
(d) neither one-to-one or onto.

(a) $f(n) = n^3$

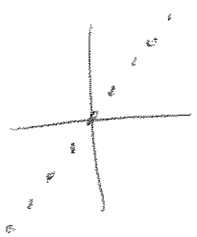
for each output, only one input exists
but not all integers get mapped to.



(b) $f(n) = \begin{cases} n & n \geq 0 \\ n+4 & n < 0 \end{cases}$

notice it takes on all integer values as output
but $f(1) = 1 = f(-3)$
$\Rightarrow f$ not 1-1

(c) $f(n) = n$ is the easiest example but any linear fn of n will do (w/ integer coefficients)



(d) $f(n) = n^2$ (or $f(n) = |n|$)

4. Prove that if n is an odd integer, then $\left\lceil \frac{n^2}{4} \right\rceil = \frac{(n^2+3)}{4}$.

pf If n odd, then $\exists\ m \in \mathbb{Z}^+ \ni$

$n = 2m+1$.

$$\Rightarrow \frac{n^2}{4} = \frac{(2m+1)^2}{4} = \frac{4m^2+4m+1}{4} = m^2+m+\frac{1}{4}$$

$$\Rightarrow \left\lceil \frac{n^2}{4} \right\rceil = \left\lceil m^2+m+\frac{1}{4} \right\rceil = m^2+m+1$$

since m^2 and m are both integers, then adding $\frac{1}{4}$ makes up round up to 1 for the ceiling fn.

but $m^2+m+1 = \left(m^2+m+\frac{1}{4}\right) + \frac{3}{4} = \frac{n^2}{4} + \frac{3}{4} = \frac{n^2+3}{4}$

$$\Leftrightarrow \left\lceil \frac{n^2}{4} \right\rceil = \frac{n^2+3}{4} \quad //$$

5. (a) Solve the congruence. $4x \equiv 5 \pmod 9$

universe of $4 \pmod 9$?

$4c \equiv 1 \pmod 9$

$4 \cdot 7 \equiv 28 \equiv 1 \pmod 9$

$\Rightarrow c = 7 \bmod (9)$

$\Rightarrow \quad 4x \equiv 5 \pmod 9$

$\Leftrightarrow \quad 7(4x) \equiv 7(5) \pmod 9$

$x \equiv 35 \pmod 9 \equiv \boxed{8 \bmod 9}$

(b) Use the Euclidean Algorithm to find the GCD for 10,220 and 33,341.

$GCD(10220, 33341) = ?$

$33341 = 3(10220) + 2681$

$10220 = 3(2681) + 2177$

$2681 = 1(2177) + 504$

$2177 = 4(504) + 161$

$504 = 3(161) + 21$

$161 = 7(21) + 14$

$21 = 1(14) + 7$

$14 = 2(7) \qquad \Rightarrow GCD = 7$

6. Prove $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

(By Induction)

pf ① if $n=1$ $\sum_{k=1}^{1} k^3 = 1^3 = 1$

$$\frac{n^2(n+1)^2}{4} = \frac{1\,(4)}{4} = 1 \quad \checkmark$$

② Assume true for some $n \in \mathbb{Z}^+$, $n > 1$.

i.e. $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

want $\sum_{k=1}^{n+1} k^3 = \frac{(n+1)^2(n+2)^2}{4}$

Then $\sum_{k=1}^{n+1} k^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3$

$$= (n+1)^2\left(\frac{n^2}{4} + n+1\right)$$

$$= \frac{(n+1)^2}{4}(n^2+4n+4)$$

$$= \frac{(n+1)^2(n+2)^2}{4} \quad //$$

7. For f given recursively by $f(0)=0$, $f(n)=f(n-1)+2n+1$ for all n = 1, 2, ... find an explicit formula for $f(n)$ and prove your formula is valid.

n	$f(n)$		
0	0	$=1-1$	$=1^2-1$
1	3	$=4-1$	$=2^2-1$
2	$3+4+1=8$	$=9-1$	$=3^2-1$
3	$8+6+1=15$	$=16-1$	$=4^2-1$
4	$15+8+1=24$	$=25-1$	$=5^2-1$
5	$24+10+1=35$	$=36-1$	$=6^2-1$
$\vdots$			
n	$(n+1)^2-1 = n^2+2n$		

Claim $f(n)=n^2+2n, \quad \forall n=0,1,2,\ldots$

pf ① $n=0$ $f(0)=0$ ✓

and $0^2+2(0)=0$

② Assume true for some $n \in \mathbb{Z}^+$.

i.e. $f(n)=n^2+2n$

check $n+1$ case.

$f(n+1)=f(n)+2(n+1)+1$ from recursive defn

$=f(n)+2n+3$

$=n^2+2n+2n+3$ by induction hypothesis

$=n^2+4n+3$

$=(n^2+2n+1)+(2n+2)$

$=(n+1)^2+2(n+1)$ //

8. An ice cream parlor has 30 different flavors, 5 sauces and 10 toppings.
(a) How many ways are there to get 3 scoops of ice cream with one sauce and two toppings in a cup? (Assume you want all different types of ice cream, no repeats.)
(b) How many ways are there to get three scoops of ice cream with one topping on a cone?

(a) in a cup, order does not matter

$$\underbrace{\binom{30}{3}}_{\text{ice cream}} \quad \underbrace{\binom{5}{1}}_{\text{sauce}} \quad \underbrace{\binom{10}{2}}_{\text{toppings}}$$

cups of ice cream $= \dfrac{30!}{27!\,3!} \cdot \dfrac{5!}{4!\,1!} \cdot \dfrac{10!}{8!\,2!} = \dfrac{\cancel{30}^{5}(29)(28)}{\cancel{3}\cdot\cancel{2}\cdot 1}(5)\dfrac{(\cancel{10}^{5})(9)}{\cancel{2}}$

$= 29(28)(9)(125)$

$= 913{,}500$

(b) in a cone, order matters (permutation)

${}_{30}P_3 = \dfrac{30!}{27!} = 30(29)(28)$ = # ice cream orders

cones $= 30(29)(28)\binom{10}{}$ $= 243600$

toppings

9. How many positive integers less than 800
   (a) have exactly 3 digits?
   (b) have at least one digit equal to 7?
   (c) have no odd digits?
   (d) are palindromes (i.e. the same reading from left to right or right to left)?

(a) $\underline{7} \cdot \underline{10} \cdot \underline{10} = \boxed{700}$

↑ can't have 0 as 1st digit (or 8, or 9)

(b)

3-digit #s

$\underline{7} \; \underline{9} \cdot \underline{9}$ (7 fixed) — 81 choices w/ 7 only in 1st digit

$\underline{6} \; \underline{7} \; \underline{9}$ (7 fixed) — 54 choices w/ 7 only in 2nd digit

$\underline{6} \; \underline{9} \; \underline{7}$ (7 fixed) — 54 choices w/ 7 only in 3rd digit

77 _ → 9
7 _ 7 → 9
_ 7 7 → 6
777 → +1
= 25

25 choices for 2 or 3 7s in a 3-digit #

1-digit & 2-digit #s: 7, 17, 27, ..., 67, 70, 71, ..., 79, 87, 97 — 19

81 + 54 + 54 + 25 + 19 = 233

233 total

(c) $\underset{0,2,4,6}{\underline{4}} \cdot \underset{0,2,4,6,8}{\underline{5}} \cdot \underset{0,2,4,6,8}{\underline{5}} = 5^2 \cdot 4 = \boxed{100}$

this includes 1, 2 or 3 digit #s

(d) $\underline{7} \cdot \underline{10} \cdot \underline{1} = 70$ 3-digit palindromes

$\underline{9} \cdot \underline{1} = 9$ 2-digit "

$\underline{9} = 9$ 1-digit "

$\Rightarrow \boxed{88}$

10. The probability of event E is $\frac{1}{3}$. The probability of event G is $\frac{3}{4}$. Assuming that $S = E \cup G$, where S is the sample space, find these values.

(a) $P(E \cap G)$

(b) $P(E|G)$

(c) $P(G|E)$

$P(E) = \frac{1}{3}$ $\quad P(G) = \frac{3}{4}$

(a)

$$P(E \cup G) = P(S) = 1 = P(E) + P(G) - P(E \cap G)$$

$$\Leftrightarrow P(E \cap G) = P(E) + P(G) - 1$$

$$= \frac{1}{3} + \frac{3}{4} - 1$$

$$= \frac{13}{12} - 1 = \boxed{\frac{1}{12}}$$

(b) $$P(E|G) = \frac{P(E \cap G)}{P(G)} = \frac{1/12}{3/4} = \boxed{\frac{1}{9}}$$

(c) $$P(G|E) = \frac{P(E \cap G)}{P(E)} = \frac{1/12}{1/3} = \boxed{\frac{1}{4}}$$

11. A loaded coin is flipped 15 times. The probability of getting a head with this coin is 0.6. What is the probability that

(a) exactly 9 heads appear?

(b) at most 10 tails appear?

(c) there are exactly 6 heads, given that the first three tosses are tails?

$P_H = 0.6$

$P_T = 0.4$

(a) $P(\text{exactly } 9\ H) = \boxed{\binom{15}{9}(0.6)^9(0.4)^6} = 5005(0.6)^9(0.4)^6 \simeq 0.2066$

(b) $P(\text{at most } 10 \text{ tails}) = 1 - P(11 \text{ tails}) - P(12 \text{ tails}) - P(13 \text{ tails}) - P(14 \text{ tails}) - P(15 \text{ tails})$

$= \boxed{1 - \sum_{i=11}^{15} \binom{15}{i} (0.4)^i (0.6)^{15-i}}$

(c) $P(\text{exactly } 6\ H \mid 1^{st}\ 3 \text{ tosses were tails})$

$= P(\text{exactly } 6\ H \text{ out of } 12 \text{ tosses})$

$= \boxed{\binom{12}{6} 0.6^6\, 0.4^6}$

Extra Credit: Find a formula for the coefficient of any x^n term in the expansion of $\left(\frac{2}{x^3}+x^4\right)^{40}$.

$$\left(\frac{2}{x^3}+x^4\right)^{40} = \sum_{i=0}^{40}\binom{40}{i}\left(\frac{2}{x^3}\right)^i (x^4)^{40-i}$$

$$= \sum_{i=0}^{40}\binom{40}{i} 2^i x^{-3i} x^{160-4i}$$

$$= \sum_{i=0}^{40}\binom{40}{i} 2^i x^{160-7i}$$

i^{th} term $\forall\ i=0,1,\dots,40$

$n = 160-7i$

$7i = 160-n$

$i = \frac{160-n}{7}$

$$= \binom{40}{i} 2^i x^{160-7i}$$

$$= \binom{40}{\frac{160-n}{7}} 2^{\frac{160-n}{7}} x^n$$

$0 \le i \le 40$

$\Rightarrow -120 \le n \le 160$

$$\Rightarrow \text{coeff of } x^n = \begin{cases} \binom{40}{\frac{160-n}{7}} 2^{\frac{160-n}{7}} & \text{when } \frac{160-n}{7} \in \{0,1,\dots,40\} \\ 0 & \text{otherwise} \end{cases}$$