Ex5 (pg (84)) Claim $\displaystyle\sum_{j=1}^{n} \cos(jx) = \cos\left[\tfrac{1}{2}x(n+1)\right] \frac{\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}$

$\forall\, n \in \mathbb{Z}^+$, $\sin\left(\frac{x}{2}\right) \neq 0$.

Pf ① check $n=1$ case.

LHS: $\displaystyle\sum_{j=1}^{1} \cos(jx) = \cos x$

RHS: $\displaystyle\cos\left[\tfrac{1}{2}x(2)\right] \frac{\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} = \cos(x)$ ✓

② Assume true $\forall\ k = 1, \dots, n$. Check $n+1$ case.

we know (from induction assumption)

$$\sum_{j=1}^{n} \cos(jx) = \cos\left[\tfrac{1}{2}x(n+1)\right] \frac{\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

$$\Leftrightarrow \quad \sum_{j=1}^{n+1} \cos(jx) = \cos\left[\tfrac{1}{2}x(n+1)\right] \frac{\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)} + \cos((n+1)x)$$

$\boxed{\star}$

$$= \frac{1}{2}\left(\sin\left(\frac{2n+1}{2}x\right) - \sin\left(\frac{x}{2}\right) + \sin\left(\frac{2n+3}{2}x\right) - \sin\left(\frac{2n+1}{2}x\right)\right)\left(\frac{1}{\sin\left(\frac{x}{2}\right)}\right)$$

(using $\cos a \sin b = \frac{1}{2}(\sin(a+b) - \sin(a-b))$ twice w/ $a_1 = \frac{n+1}{2}$, $b_1 = \frac{nx}{2}$ and $a_2 = (n+1)x$, $b_2 = \frac{x}{2}$)

$$= \frac{1}{2}\left(\sin\left(\frac{2n+3}{2}x\right) - \sin\left(\frac{x}{2}\right)\right)\left(\frac{1}{\sin\left(\frac{x}{2}\right)}\right)$$

$$= \frac{\cos\left(\frac{n+2}{2}x\right) \sin\left(\frac{n+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)} \quad //$$

(using trig identity again w/ $a = \frac{n+2}{2}x$, $b = \frac{n+1}{2}x$

$\boxed{\star}$ I skipped writing a step here.

$$= \left(\cos\left[\tfrac{1}{2}x(n+1)\right] \sin\left(\frac{nx}{2}\right) + \cos((n+1)x)\sin\left(\frac{x}{2}\right)\right)\frac{1}{\sin\left(\frac{x}{2}\right)}$$