5.7 Oblique Triangles

Law of Cosines

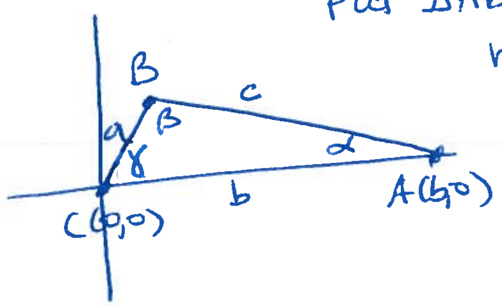
In any ΔABC,

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

Pf



Put ΔABC on rect. coord. axes with C at (0,0).

pt $B = (a\cos\gamma, a\sin\gamma)$

find distance between A & B.

$$c^2 = (a\cos\gamma - b)^2 + (a\sin\gamma - 0)^2$$

$$= a^2\cos^2\gamma - 2ab\cos\gamma + b^2 + a^2\sin^2\gamma$$

$$= a^2(\cos^2\gamma + \sin^2\gamma) - 2ab\cos\gamma + b^2$$

$$= a^2 + b^2 - 2ab\cos\gamma \ \#$$

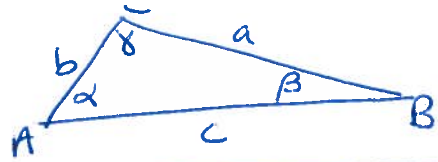
use for these cases:

① SSS

② SAS

③ SSA *

*: can produce one, two or no Δs that solve this case

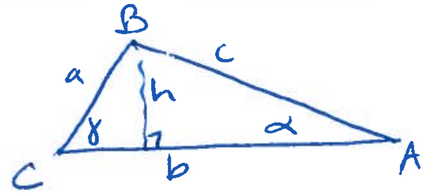


Law of Sines

In any ΔABC,

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Pf



$\sin\gamma = \frac{h}{a}$, $\sin\alpha = \frac{h}{c}$

$\Rightarrow h = a\sin\gamma$ and $h = c\sin\alpha$

$\Rightarrow a\sin\gamma = c\sin\alpha$

$\Rightarrow \frac{\sin\gamma}{c} = \frac{\sin\alpha}{a}$

(and we can likewise draw height from A or C to get the other piece of law of sines)

Use for these cases:

① ASA or AAS

Note: For AAA, there is no unique solution to the triangle.

5.7 (Cont)

Ex 1 Solve the Δ.

(a) $a = 30$
$\beta = 50°$
$\gamma = 100°$

SSS: ① if sum of two smaller sides is $\leq$ length of largest side, N.S.
② if sum of two smaller sides $>$ length of largest side, use Law of Cosines.

SAS: ① Given angle $\geq 180°$, N.S.
② Given angle $< 180°$, use Law of Cosines.

SSA: ① Given angle $\geq 180°$, N.S.
② Given angle $< 180°$, use Law of Cosines to see if there is no, one or 2 solns (depending on quadratic formula results).

ASA (or AAS):
① sum of given angles $\geq 180°$, N.S.
② " " " $< 180°$, use Law of Sines

AAA:
No unique Δ exists because of similarity.

(b) $a = 123$, $b = 225$, $c = 351$

5.7 (cont)

Ex 2 Solve these Δs.

(a) $b = 5.2$, $c = 3.4$, $\alpha = 54.6°$

(b) $a = 5.0$, $b = 4.0$, $\alpha = 125°$

5.7 (cont)

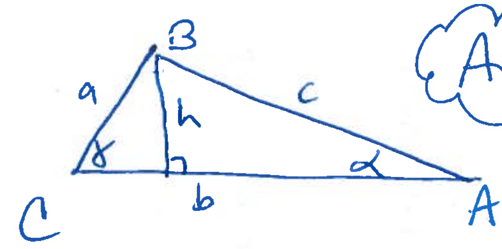
Ex 3 Find area of these triangles.

(a) $b=14$, $c=12$, $\alpha=82^\circ$

(b) $a=b=c=5$

(c) $a=10.2$, $b=11.8$, $\alpha=47^\circ$

Area



Area $= \frac{1}{2}bh$

but $h = a\sin\gamma$ (or $h = c\sin\alpha$)

$\Rightarrow$ $\boxed{\text{Area} = \frac{1}{2}b(a\sin\gamma)}$ (1)

or $\text{Area} = \frac{1}{2}bc\sin\alpha$

or $\text{Area} = \frac{1}{2}ac\sin\beta$

but $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \Leftrightarrow b = \frac{a\sin\beta}{\sin\alpha}$

$\Rightarrow$ $\text{Area} = \frac{1}{2}a\left(\frac{a\sin\beta}{\sin\alpha}\right)\sin\gamma$ (from (1) above)

$$A = \frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin\alpha}$$

or $$A = \frac{1}{2}b^2\frac{\sin\alpha\sin\gamma}{\sin\beta}$$

or $$A = \frac{1}{2}c^2\frac{\sin\alpha\sin\beta}{\sin\gamma}$$

or if we know SSS:

Heron's formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$

6.1 Trigonometric Equations

Ex 1
Solve these eqns.

(a) $\cos x = \frac{-1}{2}$

(b) $\cos x = \frac{-1}{2}$ given $x \in [0, 2\pi)$

Ex 2 Solve $\sin(3x) = \frac{-\sqrt{2}}{2}$

6.1 (cont)

Ex 3 Solve $\sqrt{2}\cos x \sin x = \sin x$

Ex 4 Solve $\cos^2 x = \frac{1}{2}$

6.1 (cont)

Ex 5 Solve $\cos(3x)+1=\sqrt{2}$

Ex 6 Solve $\sin^2 x - \sin x - 2 = 0$

Ex 7 Solve $\sin^2(4x) + \sin(4x) = 1 - \cos^2(4x)$