6.5 De Moivre's Thm

Ex Solve $x^3+1=0$

method 1

$x^3 = -1$

$\sqrt[3]{x^3} = \sqrt[3]{-1}$

$x = -1$

one soln

method 2

$x^3+1=0$ (sum of cubes can factor)

$(x+1)(x^2-x+1)=0$

$x+1=0$ or $x^2-x+1=0$

$x=-1$

$x = \dfrac{1 \pm \sqrt{1-4(1)}}{2}$

$x = \dfrac{1 \pm \sqrt{3}\, i}{2}$

3 solns

goal: We want a way to get <u>all</u> solns, not just the $\mathbb{R}$ solns.

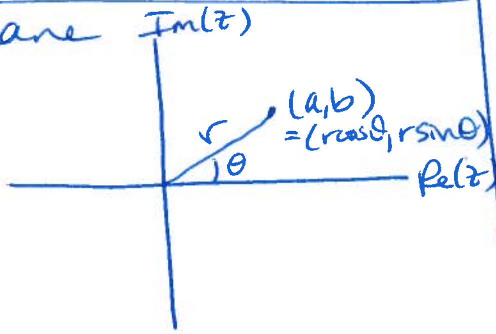
Background:

If $z = a+bi \in \mathbb{C}$, then

$|z| = \sqrt{a^2+b^2}$ (called <u>modulus</u> or <u>absolute value</u> of z)

complex plane

$z = a+bi$



Trigonometric form of complex #:

$z = a+bi = r(\cos\theta + i\sin\theta)$

$=$ "$r \operatorname{cis} \theta$"

$r = \sqrt{a^2+b^2}$

$\tan\theta = \dfrac{b}{a}$ $(a \neq 0)$

(unique for all z, except $z=0$)

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6.5 (cont)

Ex 1 Plot $-5+4i$ and find modulus.

Ex 2 Change to trigonometric form

(a) $-4i$

(b) $-1-\sqrt{3}\,i$

Ex 3 Plot and change to rectangular form.

$5(\cos 30^\circ + i\sin 30^\circ)$

6.5 (cont)

Products & Quotients

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$(z_1, z_2 \neq 0)$ Then $z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

and $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$

Pf (Quotient)

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\left(\frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}\right)$$

$$= \frac{r_1}{r_2}\left(\frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}\right)$$

$$= \frac{r_1}{r_2}\left((\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\right)$$

$$= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$$

De Moivre's Thm

$n \in \mathbb{N}$ $\quad (r\cos\theta + ir\sin\theta)^n = r^n(\cos(n\theta) + i\sin(n\theta))$

n^{th} Root Thm

$n \in \mathbb{N}$ Then $\sqrt[n]{r(\cos\theta + i\sin\theta)} = \sqrt[n]{r}\left(\cos\left(\dfrac{\theta+2\pi k}{n}\right) + i\sin\left(\dfrac{\theta+2\pi k}{n}\right)\right)$

$k = 0, 1, 2, \ldots, n-1$

6.5 (cont)

Ex 4 $4(\cos 65^\circ + i\sin 65^\circ)\cdot 12(\cos(87^\circ) + i\sin 87^\circ)$

Ex 5 $(\cos 210^\circ + i\sin 210^\circ)^5$

Ex 6 Find all square roots of $\frac{25}{2} - \frac{25\sqrt{3}}{2}i$

Conic Sections (Chp 7)

In general, conic sections (parabolas, hyperbolas, ellipses) are all given by form
(and circles)

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$A, B, C, D, E, F \in \mathbb{R}$

(If at least one of A, B or C is not zero)

☆ see cool picture on pg 430 of your book)

7.1 Parabolas

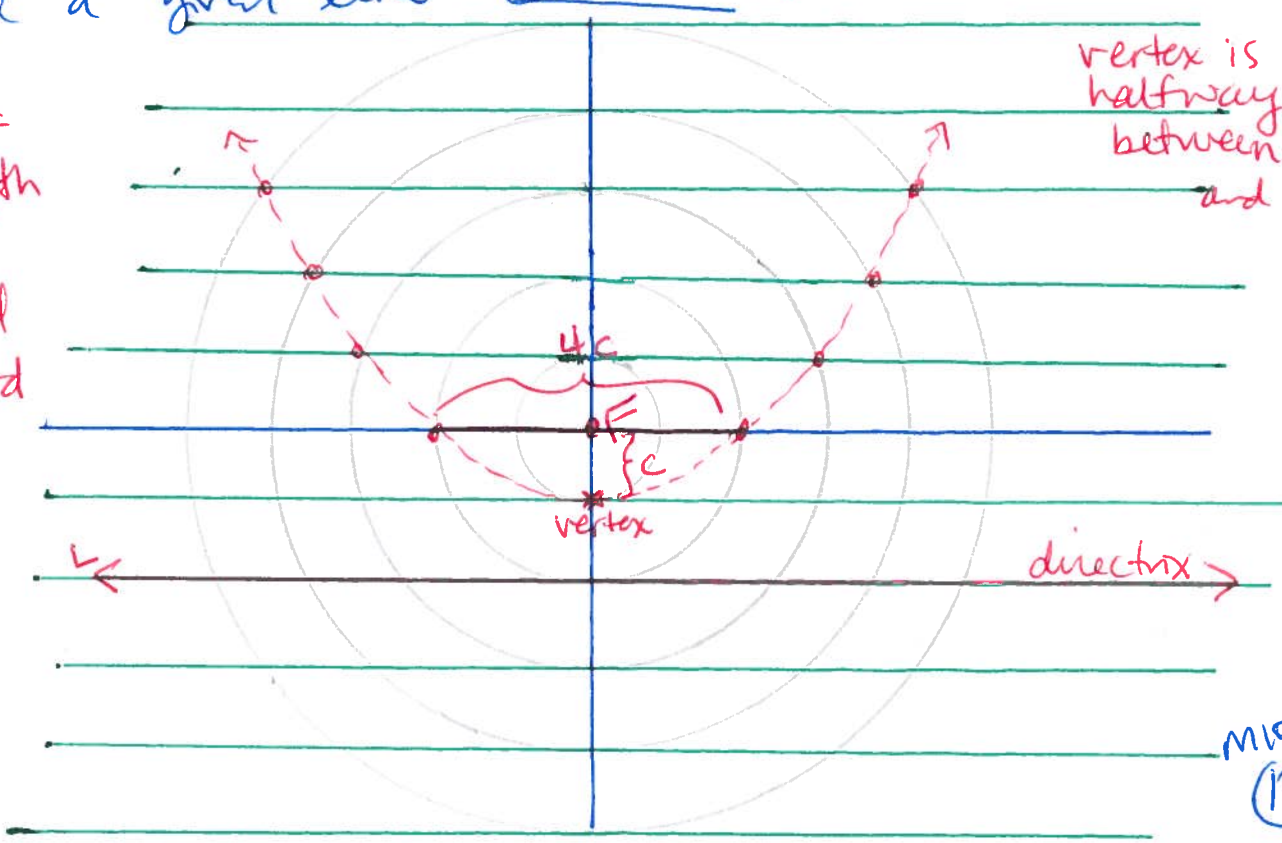
Parabolas are of form $Ax^2 + Dx + Ey + F = 0$ $(B, C = 0)$

or $Cy^2 + Dx + Ey + F = 0$ $(A, B = 0)$

Defn

Parabola: the set of pts in plane that are equidistant from a given pt (called focus) F and a given line (directrix) L

4c = length of focal chord



vertex is halfway between F and L

7.1 (cont)

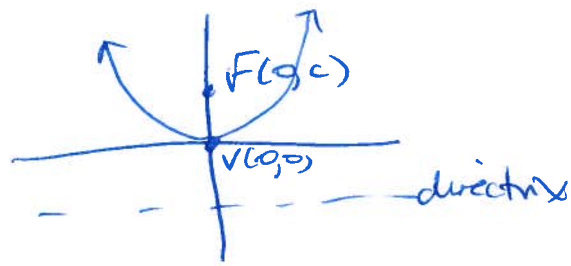
let (x,y) be any pt on parabola.

let F be at $(0,c)$, directrix at $y=-c$, $c \in \mathbb{R}^+$.

$\Rightarrow$ the parabola has vertex at $(0,0)$ and opens upward

distance from (x,y) to F = distance from (x,y) to directrix

$$\sqrt{(x-0)^2+(y-c)^2} = |y+c|$$



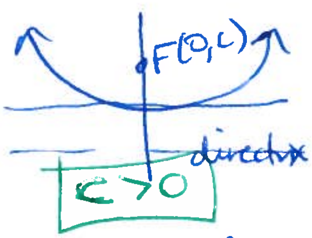
$$\left(\sqrt{x^2+(y-c)^2}\right)^2 = (y+c)^2$$

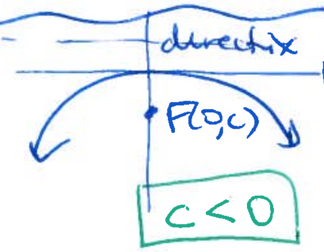
$$x^2+(y-c)^2=(y+c)^2$$

$$x^2+\cancel{y^2}-2cy+\cancel{c^2} = \cancel{y^2}+2cy+\cancel{c^2}$$

$$\boxed{x^2=4cy}$$

Standard Form of Parabolas w/ vertex (0,0)

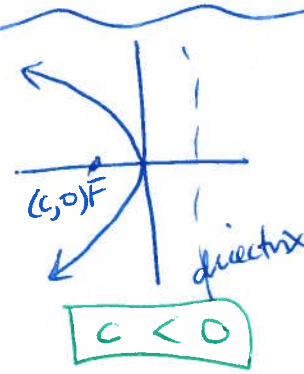


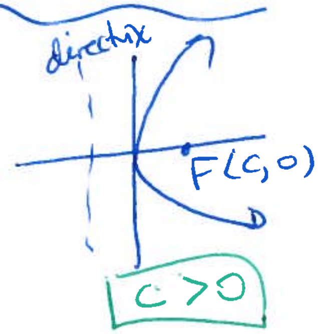


eqn: $x^2=4cy$

$F(0,c)$ focus

directrix $y=-c$





eqn: $y^2=4cx$

$F(c,0)$ focus

directrix $x=-c$

7.1 (cont)

Ex 1 Graph $2x^2 = -4y$

Ex 2 Graph $3y^2 - 12x = 0$

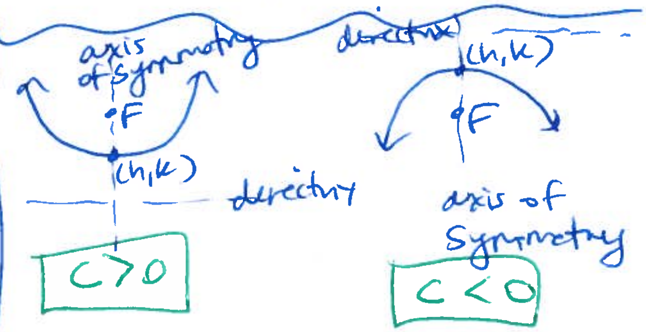
7.1 (cont)

Ex 3 Graph $(x+2)^2 = 2(y-1)$

Ex 4 Graph $y^2 + 4x - 3y + 1 = 0$

(hint: you need to complete the square.)

Standard Form of Parabola w/ vertex (h, k)

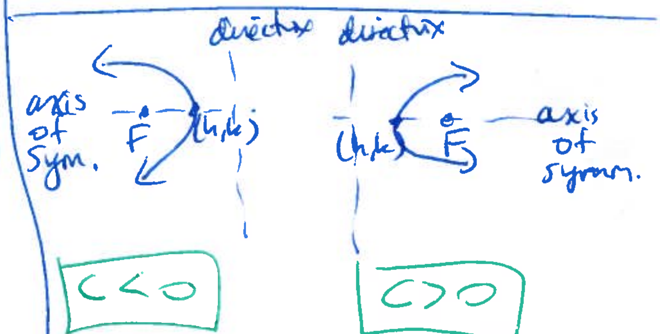


eqn: $(x-h)^2 = 4c(y-k)$

$F(h, k+c)$ focus

directrix: $y = k - c$

axis of symmetry: $x = h$



eqn: $(y-k)^2 = 4c(x-h)$

$F(h+c, k)$ focus

directrix $x = h - c$

axis of symmetry: $y = k$

7.1 (cont)

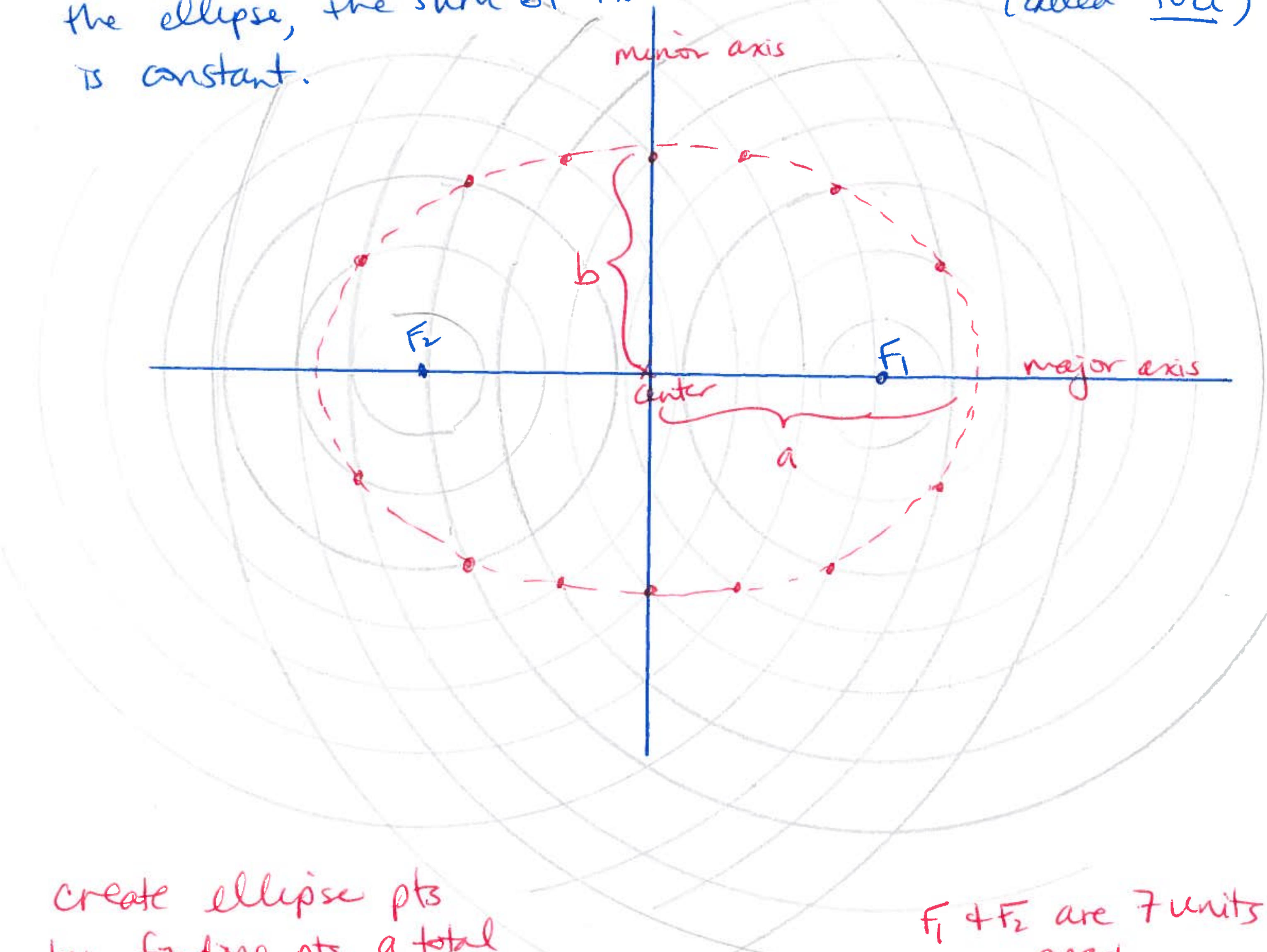
Ex 5 Find eqn of parabola.

(a) directrix at $y = -4$, vertex at $(4, -1)$

(b) vertex at $(4, 2)$ passing through $(-3, -4)$;
axis parallel to x-axis

7.2 Ellipses

<u>ellipse</u>: set of all pts in plane such that, for each pt on the ellipse, the sum of its distances from 2 fixed pts (called <u>foci</u>) is constant.



create ellipse pts by finding pts a total of 10 units (as a sum) from F_1 & F_2

F_1 + F_2 are 7 units apart

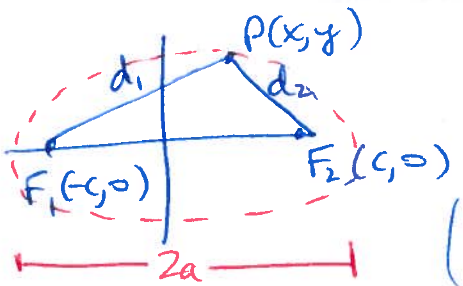
center is midpt of $\overline{F_1F_2}$

length of major axis = 2a
" " minor axis = 2b

From general conic section, ellipse has form
$$Ax^2+Cy^2+Dx+Ey+F=0 \qquad (B=0)$$

7.2 (cont)

standard form of ellipse w/ center $(0,0)$.



$$d_1 + d_2 = 2a$$

$$\sqrt{(x+c)^2+(y-0)^2} + \sqrt{(x-c)^2+(y-0)^2} = 2a$$

$$\left(\sqrt{(x+c)^2+y^2}\right)^2 = \left(2a - \sqrt{(x-c)^2+y^2}\right)^2$$

$$(x+c)^2 + \cancel{y^2} = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2 + \cancel{y^2}$$

$$\cancel{x^2} + 2xc + \cancel{c^2} - 4a^2 - (\cancel{x^2} - 2cx + \cancel{c^2}) = -4a\sqrt{(x-c)^2+y^2}$$

$$\frac{4cx - 4a^2}{-4a} = \frac{-4a\sqrt{(x-c)^2+y^2}}{-4a}$$

$$\left(a - \frac{c}{a}x\right)^2 = \left(\sqrt{(x-c)^2+y^2}\right)^2$$

$$a^2 - \cancel{2cx} + \frac{c^2}{a^2}x^2 = x^2 - \cancel{2cx} + c^2 + y^2$$

$$a^2 - c^2 = x^2\left(1 - \frac{c^2}{a^2}\right) + y^2$$

$$a^2 - c^2 = \left(\frac{a^2-c^2}{a^2}\right)x^2 + y^2 \qquad \text{let } b^2 = a^2 - c^2$$

$$\Rightarrow \quad b^2 = \frac{b^2}{a^2}x^2 + y^2$$

$$1 = \frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 \quad \Leftrightarrow \quad \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

standard eqn of ellipse w/ center at $(0,0)$

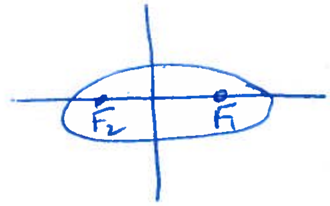
goes thru pts $(\pm a, 0)$ and $(0, \pm b)$

7.2 (cont)

Ex 1 Sketch the curves.

(a) $4x^2+9y^2=36$

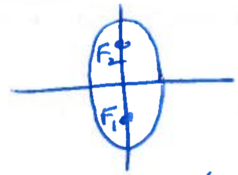
(b) $\dfrac{(x-3)^2}{16}+\dfrac{(y-2)^2}{9}=1$



$c^2=a^2-b^2$

$F(h\pm c, k)$

"horizontal" if $\boxed{a>b}$



$c^2=b^2-a^2$

$F(h, k\pm c)$

"vertical" if $\boxed{a<b}$

(And if $a=b$, then it's a circle.)

Standard Ellipses w/ center at (h,k)

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Ex 2 Graph the curve & find eqn of ellipse.

(a) "Set of pts 3 units from pt $(-2, 3)$."

(b) Set of pts such that sum of distances from $(-4, 1)$ and $(2, 1)$ is 10.

(c) Ellipse w/ vertices at $(-6, 3)$ and $(4, 3)$ and foci at $(-4, 3)$ + $(2, 3)$.

7.2 (cont)

Ex 3 Graph this curve.

$$x^2+9y^2-4x-18y-14=0$$

Eccentricity: a number that measures how "flat" the ellipse is, notation: ε.

case 1 $a>b$

$$\varepsilon=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{1-\left(\frac{b}{a}\right)^2}$$

but $b<a \Rightarrow \frac{b}{a}<1$

$\Rightarrow \varepsilon\in[0,1)$

case 2 $a<b$

$$\varepsilon=\frac{c}{b}=\frac{\sqrt{b^2-a^2}}{b}=\sqrt{\frac{b^2-a^2}{b^2}}=\sqrt{1-\left(\frac{a}{b}\right)^2}$$

$a<b \Rightarrow \frac{a}{b}<1$

$\Rightarrow \varepsilon\in[0,1)$

☆ if $a=b$, then $\varepsilon=0 \Rightarrow$ we have a circle.