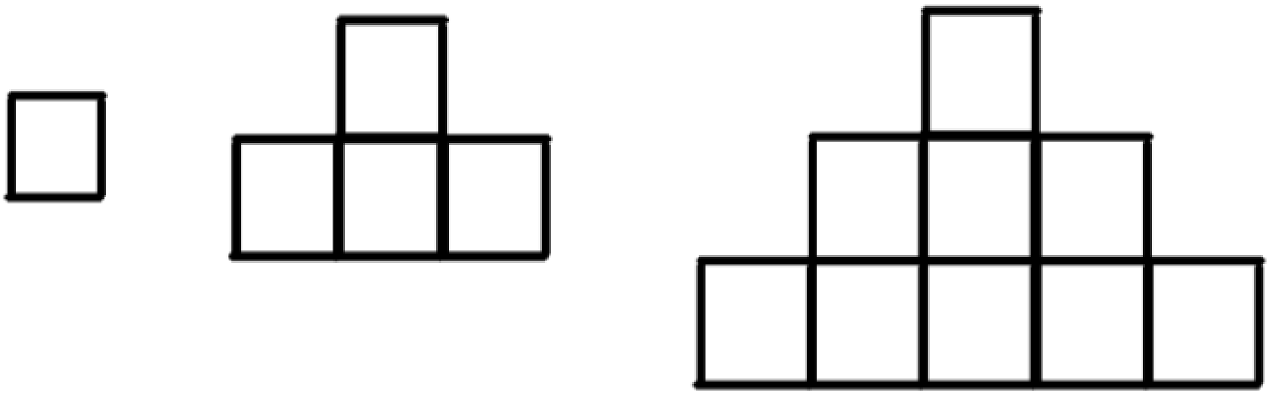
Math5700 Notes
Section 3.3
Problems Involving Real Functions

Consider this pattern of toothpicks.



(a) For counter n, what is s_n, where s_n is the number of squares?
(b) For counter n, what is p_n, where p_n is the outer perimeter?
(c) For counter n, what is t_n, where t_n is the number of toothpicks?

n	s_n	p_n	t_n
1			
2			
3			
4			
5			
6			
n			

What is the difference between a recursive formula and an iterative formula?

Consider this sequence, given in the table.

n	a_n
1	5
2	19
3	59
4	137
5	265
6	455
...	
n	??

Can you find an iterative formula for a_n ?

Another example: Let a_n = sum of the 4th powers of integers from 1 to n. Determine if a polynomial will fit the data. If so, what degree is it? How would you find the polynomial?

Theorem 3.15: Suppose that $f : \mathbb{R} \to \mathbb{R}$.

(a) If f is linear, then f has the following property: There is a real *m* such that $f(x+1) - f(x) = m$ $\forall x \in \mathbb{R}$.

(b) If f has the above property, then the linear function L defined by $L(x) = mx + f(0)$ $\forall x \in \mathbb{R}$ agrees with f at all $x \in \mathbb{Z}^+$.

Corollary: A real sequence $\{f_n\}$ has a constant difference d, i.e. $d = f_{n+1} - f_n$ $\forall n \geq 0$ iff $f_n = dn + f_0$ $\forall n \geq 0$.

Theorem 3.17: Suppose $f : \mathbb{R} \to \mathbb{R}$. If the nth difference function $\Delta^n f$ of f is a nonzero constant, then there exists an nth-degree polynomial that agrees with the function f at every nonnegative integer.

Theorem: If $\{f_n\}$ is a sequence with constant second differences, then there is a quadratic function $Q(n) = an^2 + bn + c$ such that $Q(n) = f(n)$ (notation: $f_n = f(n)$) for all n = 0, 1, 2, ..., where $a, b, c, \in \mathbb{R}$.

Prove:

(1) Find an iterative formula for f_n (the fact that you have constant second differences for $\{f_n\}$ means you have a recursive formula). Let the common second difference be d.

(2) Now define $Q(x)=ax^2+bx+c$ where $a, b, c, \in \mathbb{R}$. Show that $\Delta^2 Q$ is constant $\forall x \in \mathbb{R}$.

(3) We can certainly force infinitely many quadratic functions to go through the two points (0, f(0)) and (1, f(1)). Find a, b, c for $Q(x)$ in order that $Q(0)=f(0)$ and $Q(1)=f(1)$.

(4) By induction, prove that $Q(n)=f(n)$ for all n = 0, 1, 2, ...

Consider this sequence, given in the table.

n	a_n
1	5.5
2	13.2
3	31.7
4	76
5	182.5
6	437.9
...	
n	??

Can you find an iterative formula for a_n ?

Notation: Let $\Theta f(x) = \frac{f(x+1)}{f(x)}$ be the first quotient operator.

Theorem 3.16: Suppose that $f : \mathbb{R} \to \mathbb{R}$.

(a) If f is an exponential function, then f has the following property: There is a real *b* such that $\frac{f(x+1)}{f(x)} = b \quad \forall x \in \mathbb{R}$.

(b) If f has the above property, then the exponential function G defined by $G(x) = f(0) b^x$ $\forall x \in \mathbb{R}$ agrees with f at all $x \in \mathbb{Z}^+$.

Corollary: A real sequence $\{g_n\}$ has a constant ratio r, i.e. $r = \frac{g_{n+1}}{g_n} \quad \forall n \geq 0$ iff $g_n = g_0 r^n \quad \forall n \geq 0$.

What is the "growth rate" of exponential function?