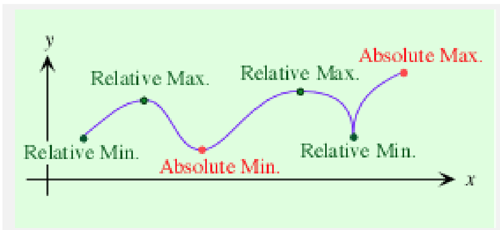
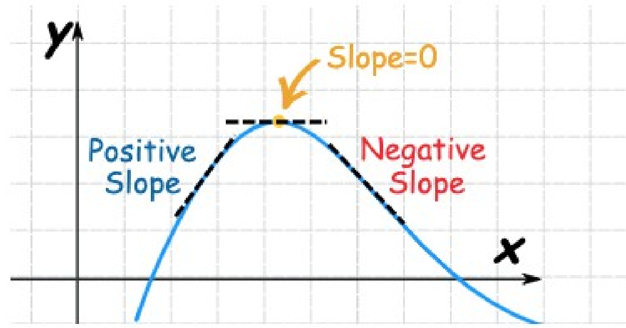
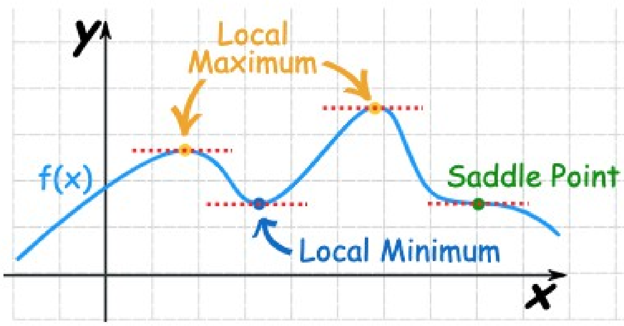
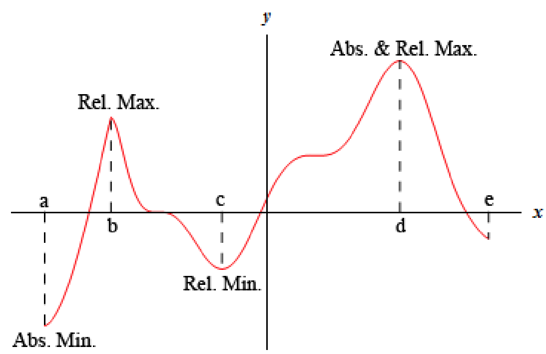
10.1 Relative Maxima and Minima: Curve Sketching

Let's first get an idea of what we mean by "relative min and max points" by looking at some pictures that will help give us an intuitive sense of min/max points.





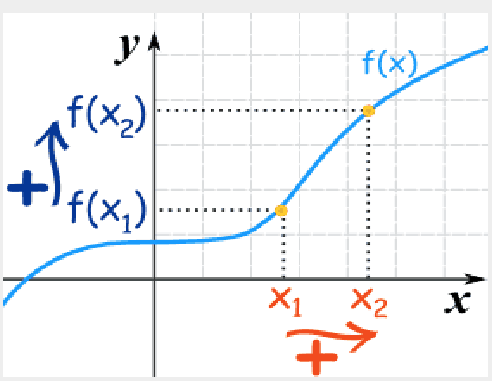




Now for definitions:

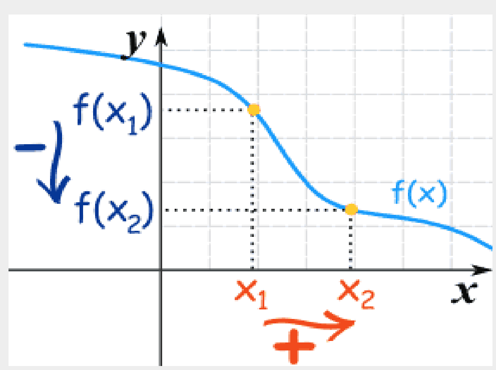
- $f'(x) > 0$ on interval I $\Rightarrow$ $f(x)$ is increasing

<u>increasing</u> means $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$



- $f'(x) < 0$ on interval I $\Rightarrow$ $f(x)$ is decreasing

<u>decreasing</u> means $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$



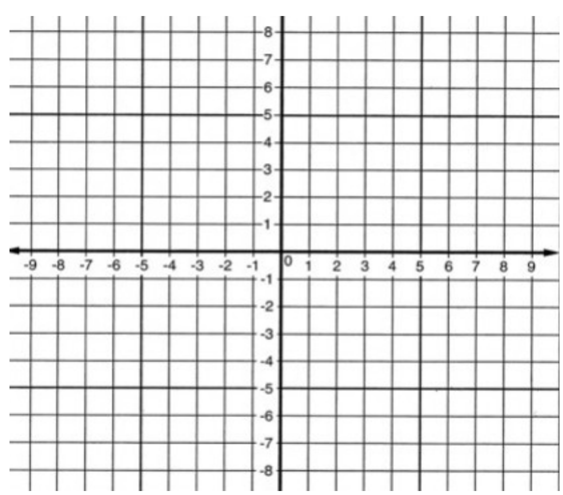
critical point:	Strategy:
A critical point (or x-value), x=c, satisfies $f'(c)=0$ or $f'(c)$ is undefined (i.e. we say there's a critical point on the curve where the slope is either zero or the slope is undefined). Three cases of how the slope can be undefined: 1. There's a moment of vertical slope at that point on the curve. 2. The curve is discontinuous at that point (a discontinuity could either be a (a) jump, (b) hole or (c) VA). 3. The point on the curve is too "pointy" (not smooth).	To find min/max x-values: 1. find where the first derivative is either zero or undefined. 2. create a first derivative sign line. 3. interpret the sign line for possible min/max points. More vocabulary: • relative/local min/max points are determined by the shape of the curve (and nothing else, i.e. is it a peak or a valley?) • absolute/global min/max points are the highest point(s) and lowest point(s) on the curve for some given interval (or possibly over the whole domain of the curve); these points can be found sometimes by the shape but also by the end points. Note: An entire curve may or may not have absolute/global min/max points.

Ex 1: Find all critical points for the curve given by $f(x)=\frac{x^4}{4}-\frac{x^3}{3}-3x^2+8$. Identify the min and max points. (Note: When we ask for "the" min/max points, we're asking for local/relative min/max points.)

10.1 (continued)

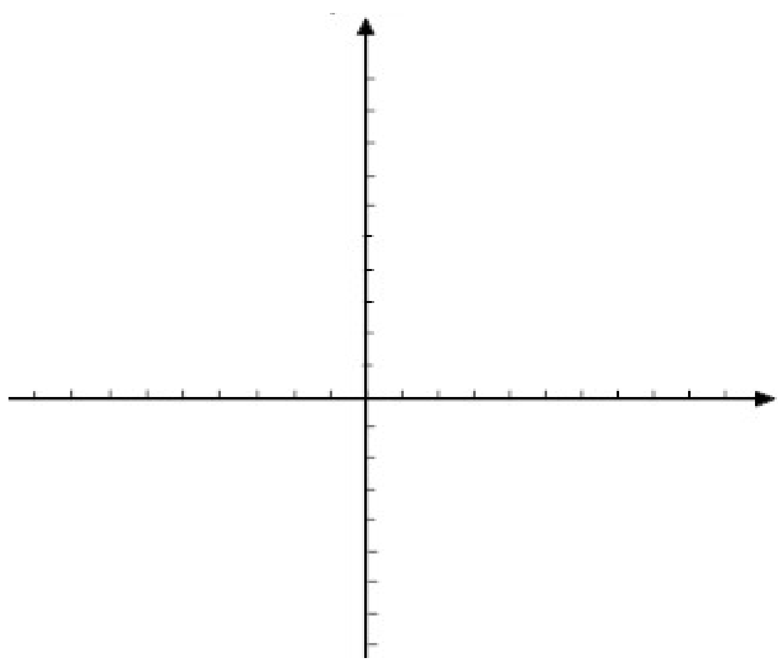
Ex 2: Find all min/max points for $y=(x+2)^{2/3}$.

Ex 3: Sketch a graph of a function that meets the following conditions:
1. f(x) is continuous
2. f(x) is not necessarily differentiable.
3. domain of f(x) is [0, 6]
4. f(x) has a max value of 4 (at x = 3)
5. f(x) has a min value of 2 (at x = 1)
6. f(x) has no stationary points

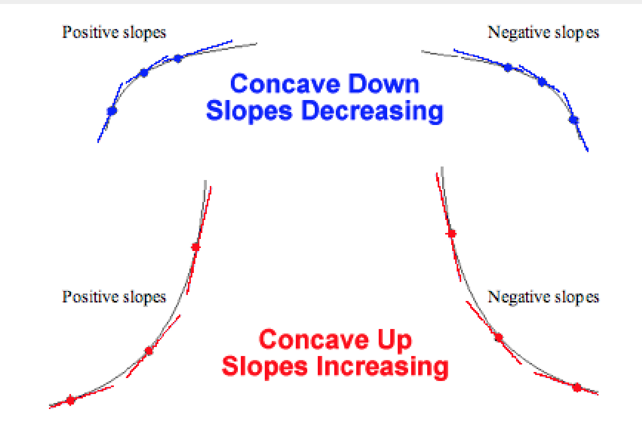


10.1 (continued)

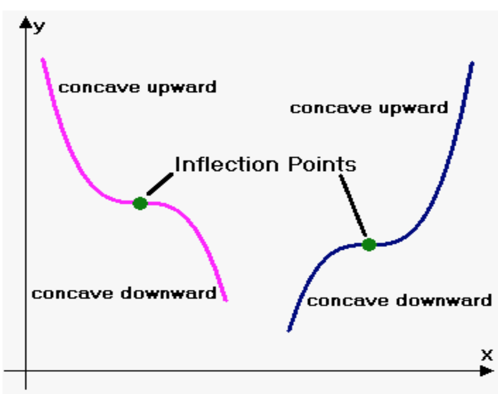
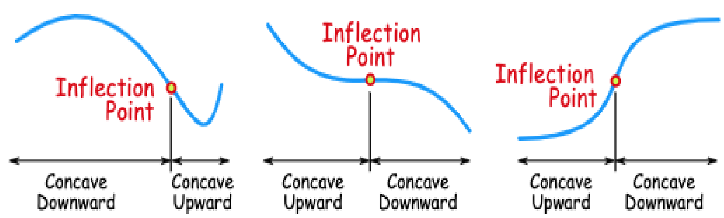
Ex 4: Find all min/max points for $y = x^3(x+5)^2$ and sketch the graph based on that information.



10.2 Concavity: Points of Inflection

concave up means $f'(x)$ is increasing on interval I concave down means $f'(x)$ is decreasing on interval I  	inflection point: where the concavity changes (from up to down or down to up) Recall--To find min/max x-values: 1. find where the first derivative is either zero or undefined. 2. create a first derivative sign line. 3. interpret the sign line for possible min/max points. To find inflection point x-values: 1. find where the second derivative is either zero or undefined. 2. create a second derivative sign line. 3. interpret the sign line for possible inflection points.

Let's see some pictures that can help us see what we mean by inflection points.

Second Derivative Test: If $f'(c)=0$ then (1) $f''(c)<0 \Rightarrow (c, f(c))$ is a max . and (2) $f''(c)>0 \Rightarrow (c, f(c))$ is a min.

10.2 (continued)

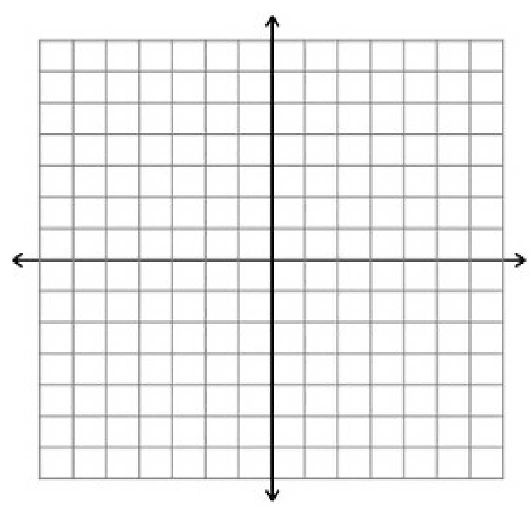
Ex 1: For $f(x)=\frac{1}{2}x^4-x^3+5$, find all min/max points, inflection points, where f(x) is increasing and decreasing and sketch the graph.

Step 1: Find the first derivative and create the first derivative sign line.

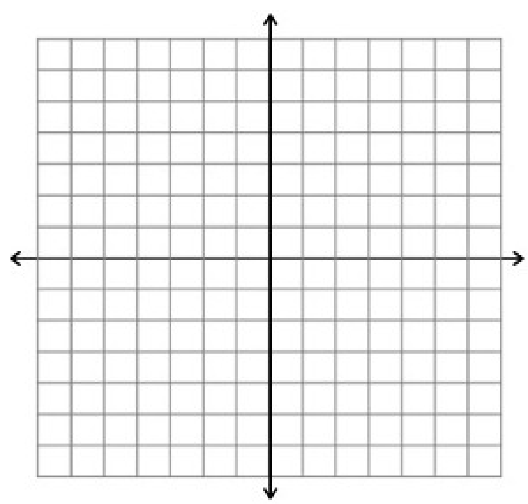
Step 2: Find the second derivative and create the second derivative sign line.

Step 3: Find all min/max and inflection points. (Remember you need two coordinates for every point.)

Step 4: Sketch the graph with all of that information.

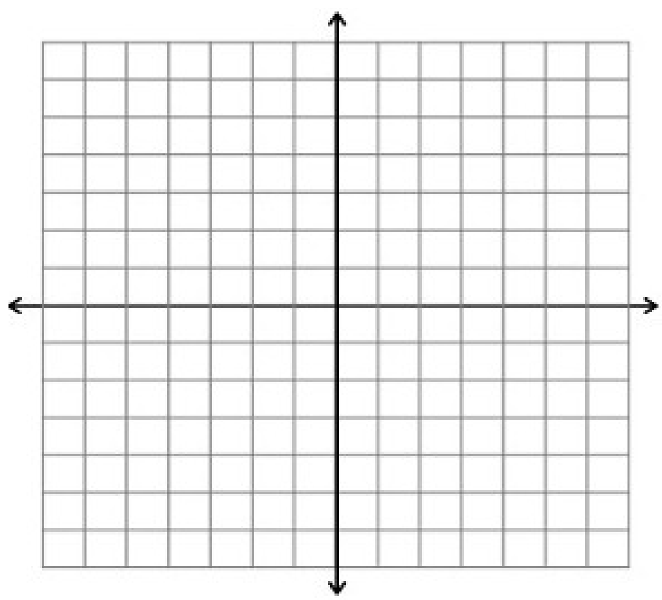


Ex 2: For $f(x)=x^2-\frac{2}{x}$, find all min/max points, inflection points, where f(x) is increasing and decreasing, where it's concave up and concave down, and sketch the graph.

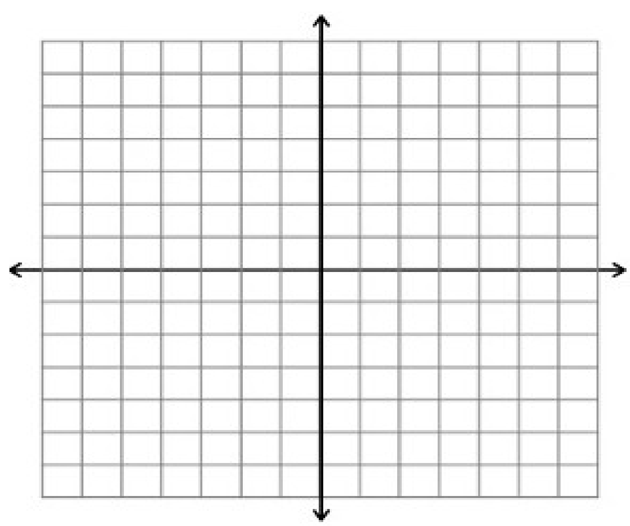


Ex 3: Sketch the graph of a continuous function that satisfies all of the following conditions:

1. f(0) = f(3) = 3
2. f(2) = 4
3. f(4) = 2
4. f(6) = 0
5. $f'(x) > 0$ on (0, 2)
6. $f'(x) < 0$ on (2, 4) and on (4, 5)
7. $f'(2) = f'(4) = 0$
8. $f'(x) = -1$ on (5, 6)
9. $f''(x) < 0$ on $(0,3) \cup (4,5)$
10. $f''(x) > 0$ on $(3,4)$



Ex 4: For $f(x)=x(x+3)^2$, find all min/max points, inflection points, where f(x) is increasing and decreasing, where it's concave up and concave down, and sketch the graph.



10.3 Optimization in Business and Economics

Ex 1: Since my mom started helping me with my cookie business, Hot Chippies, our revenue function is now given by $R(x)=8{,}000x-40x^2-x^3$ where x is the number of cookies sold. If only 50 cookies can be sold per day (my mom has arthritis and this is all she can make in a day...and I'm too busy teaching to devote much time to Hot Chippies), find the number of cookies that must be sold to maximize revenue? And, what is the max revenue? (Hint: First find the realistic domain for the values of x.)	We're guaranteed absolute/global max and min points if (1) the curve is continuous and (2) we have a closed interval. Max and min points in this case will occur: 1. as endpoints of the closed interval OR 2. as stationary points (where f'(x) = 0) OR 3. as singular points (where f'(x) does not exist) Absolute max point: the point that is the "tallest" (i.e. the largest y-value) on the graph in the given interval. Absolute min point: the points that is the "lowest" (i.e. the smallest y-value) on the graph in the given interval.

Average Cost: is given by $\overline{C}(x)=\frac{C(x)}{x}$.

Ex 2: Given a cost function $C(x)=\frac{1}{4}x^2+4x+100$, what number of units will result in a minimum average cost per unit?

In a monopolistic market, demand function is $p=f(x)$ which means total revenue is $R(x)=x\,f(x)$ (i.e. the demand price times the number of units sold at that price).

Ex 3: The monthly demand for a product sold by a monopoly is $p=1300-2x^2$ and the average cost is given by $\overline{C}(x)=540+3x+0.5x^2$ dollars. Find the profit function.

10.4 Applications of Maxima and Minima

This section is about FUN story problems/real life problems that we can solve with derivatives and calculus...the height of fun and useful!

Ex 1: A rectangular dog kennel with total area of 640 square feet is to be constructed. It will have four same-sized kennels inside the overall structure, each rectangular in shape. The cost for the fencing is \$4 per foot for the sides of the kennel and \$1 per foot for the ends and dividers. What dimensions will minimize the cost of this kennel?	<u>Steps to solve min/max story problems</u>: 1. Draw a picture and/or list all the information given. 2. Write down what needs to be optimized, as a function. 3. If you have more than one input variable for the function you're optimizing, find an equation to eliminate one of those input variables. 4. Differentiate your function. 5. Set the derivative to zero or find where it's undefined, i.e. look for singular and stationary points for the function. 6. Check to ensure you have found the min or max (whatever the problem asked for). 7. Answer the stated questions.

Ex 2: Find the volume of the largest open box that can be made from a piece of cardboard that is 24 by 9 inches. Find the dimensions of the box that yields the maximum volume. (You'll form the box by cutting squares from each corner and fold up.)

Ex 3: My mom is taking over more control of Hot Chippies. She says she can only produce at most 48 cookies per day. The total cost function now is $C(x)=500+1500x$ and our total revenue function is $R(x)=1600x-x^2$. How many cookies should she make to maximize profit?

10.5 Rational Functions: More Curve Sketching

Ex 1: Analyze the function and sketch its graph.

$$f(x)=\frac{1}{x-4}+2=\frac{2x-7}{x-4}$$

<u>To analyze a function's graph</u>:

1. Find all VA and HA and its domain. (VA = vertical asymptotes, HA = horizontal asymptotes)

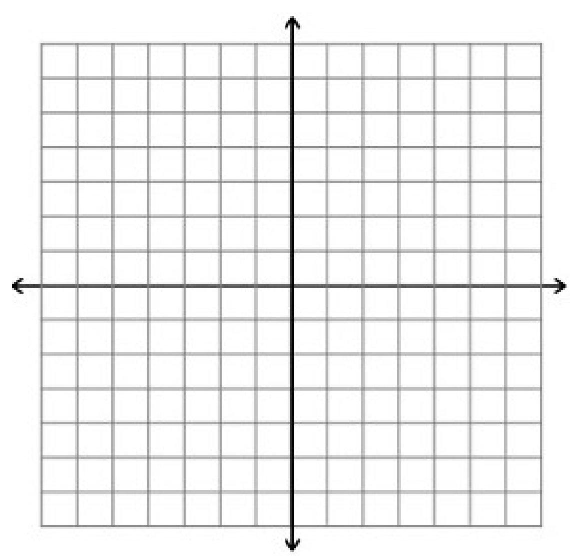
2. Find x-intercepts.

3. Find the derivative and fill in the sign-line; find min/max points.

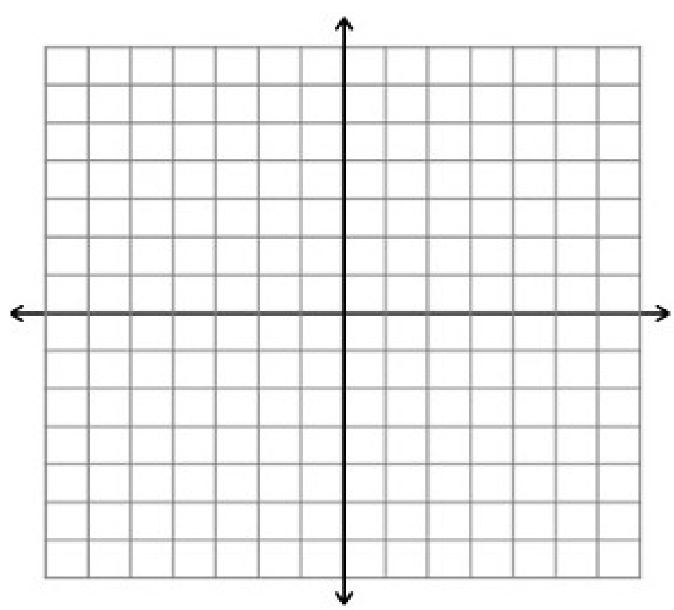
4. Find the second derivative and fill in that sign-line; find inflection points.

5. Start the graph, by plotting
    - x-intercepts
    - all asymptotes
    - min/max points
    - inflection points
    - one or two more points, as needed.

6. Fill in the rest of the graph with knowledge of slope, concavity, etc.



Ex 2: For $f(x)=\frac{x+3}{x-5}$, find all asymptotes, min/max points, inflection points, where f(x) is increasing and decreasing, where it's concave up and concave down, and sketch the graph.



10.5 (continued)

Ex 3: Analyze the function and sketch its graph. $f(x)=\dfrac{x^2+x-6}{x-1}$