9.1 Limits **"Calculus is the study of limits."**

Let's explore what happens to this function as x gets close to 2, $f(x)=\frac{x-2}{x^2-4}$.

x	$y = f(x)$
1	$\frac{1-2}{1-4}=\frac{1}{3}$
1.5	≈ 0.2857
1.9	≈ 0.25641
1.999	≈ 0.250062
...	
2	???
...	
2.01	≈ 0.24994
2.1	≈ 0.244
2.5	≈ 0.222

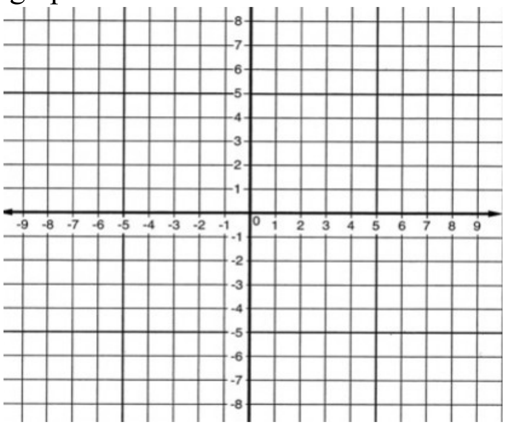
But what if we try to "plug in" x = 2 to the function?

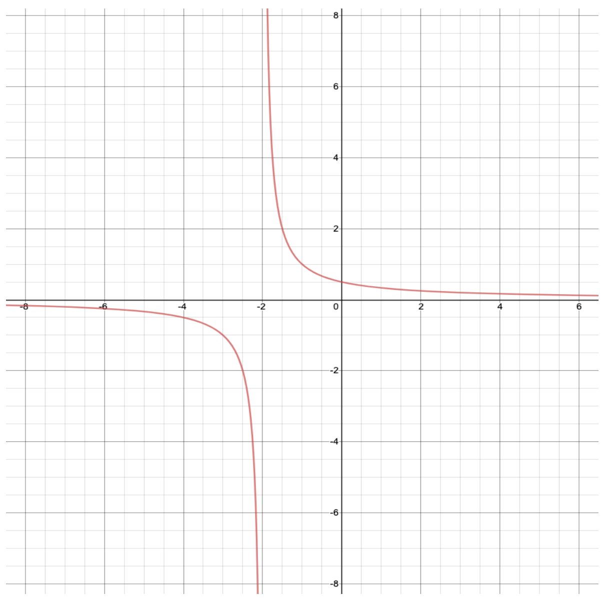
What case do we get?

Note: 0/0 case is called an indeterminate form of a limit. Colloquially, it means it's the fun case where we have more work to do to figure out the limit.

What's going on graphically?

Note: I graphed this with desmos.com and that allows you to see the general shape of the curve. But there is something problematic with this curve.
Let's graph it ourselves.





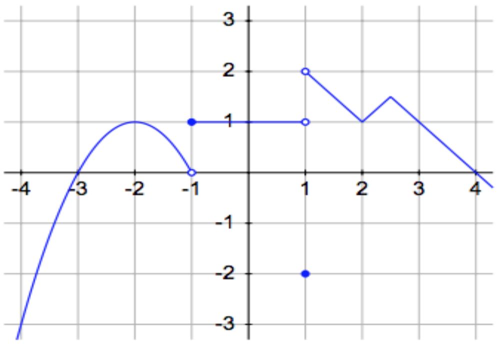
How to formally write this result algebraically.

$$\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{2+2}=\frac{1}{4}$$

Ex 1: Find the limits. (a) $\lim_{x \to 1} \frac{2x+5}{x+1}$	Defn: **Limit** of a function $\lim_{x \to c} f(x) = L$ (with L finite) means as x gets close to c, the function value gets close to L. We would say "the limit of f(x) as x approaches c equals L." Additionally, we can talk about **right and left hand limits**. If f(x) goes to L as x approaches c from the left side of c, then $\lim_{x \to c^-} f(x) = L$. Likewise, if f(x) goes to L as x approaches c from the right side of c, then $\lim_{x \to c^+} f(x) = L$.
(b) $\lim_{x \to 0} \frac{3x^3 + 2x^2 - x^4}{x^2}$	And, $\lim_{x \to c} f(x) = L$ iff $\lim_{x \to c^-} f(x) = L$ AND $\lim_{x \to c^+} f(x) = L$. (In other words, the limit exists and is L only if both the right and left hand limits exist and yield the same result.) Note: iff is not a typo. In mathematics, it means "if and only if", meaning that the implication goes both ways logically.
(c) $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$	

9.1 (continued)

Ex 2:
If this represents the graph of y = f(x), answer the questions below.



(i) $\lim\limits_{x \to -1^-} f(x) =$ ____________

(ii) $\lim\limits_{x \to -1^+} f(x) =$ ____________

(iii) $\lim\limits_{x \to -1} f(x) =$ ____________

(iv) $f(-1) =$ ____________

(v) $\lim\limits_{x \to 1^+} f(x) =$ ____________

(vi) $\lim\limits_{x \to 1^-} f(x) =$ ____________

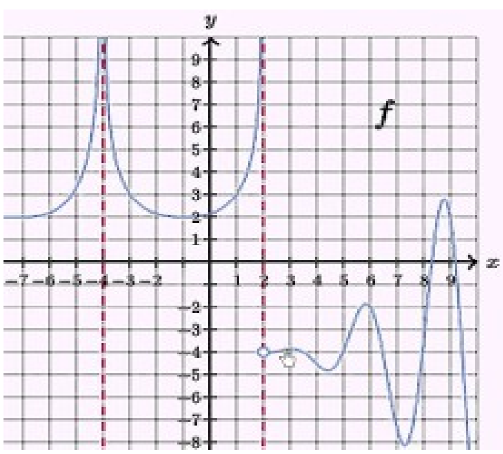
(vii) $\lim\limits_{x \to 1} f(x) =$ ____________

(viii) $f(1) =$ ____________

(ix) $\lim\limits_{x \to -2} f(x) =$ ____________

(x) $f(-2) =$ ____________

Ex 3: Find the following limits, given the graph of y = f(x).



(i) $\lim\limits_{x \to -4^-} f(x) =$ ____________

(ii) $\lim\limits_{x \to -4^+} f(x) =$ ____________

(iii) $\lim\limits_{x \to -4} f(x) =$ ____________

(iv) $f(-4) =$ ____________

(v) $\lim\limits_{x \to 2^+} f(x) =$ ____________

(vi) $\lim\limits_{x \to 2^-} f(x) =$ ____________

(vii) $\lim\limits_{x \to 2} f(x) =$ ____________

(viii) $f(2) =$ ____________

(ix) $\lim\limits_{x \to -1} f(x) =$ ____________

(x) $f(-1) =$ ____________

9.1 (continued)

Ex 4: Find the following limits, given the function $g(x)=\begin{cases} \dfrac{x^3-4}{x-3}, & \text{if } x\leq 2 \\ \dfrac{3-x^2}{x}, & \text{if } x>2 \end{cases}$

(a) $\lim_{x\to 2^-} g(x)$ (b) $\lim_{x\to 2^+} g(x)$ (c) $\lim_{x\to 2} g(x)$

Ex 5: Find these limits.

(a) $\lim_{x\to -3} \dfrac{x^2-14\text{x}-51}{x^2-4\text{x}-21}$

(b) $\lim_{x\to -1} \dfrac{x^2+x}{x^2+1}$

Limit Theorems

(basically limits distribute through everything you'd like them to distribute through as long as the limits of the individual functions exist)

Assume $n\in\mathbb{N}$, k is a constant $f(x)$ and $g(x)$ are functions such that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then,

(1) $\lim_{x\to c} k=k$

(2) $\lim_{x\to c} x=c$

(3) $\lim_{x\to c} kf(x)=k\lim_{x\to c} f(x)$

(4) $\lim_{x\to c} (f(x)\pm g(x))=\lim_{x\to c} f(x)\pm\lim_{x\to c} g(x)$

(5) $\lim_{x\to c} (f(x)\cdot g(x))=\lim_{x\to c} f(x)\cdot\lim_{x\to c} g(x)$

(6) $\lim_{x\to c} \dfrac{f(x)}{g(x)}=\dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ as long as $\lim_{x\to c} g(x)\neq 0$

(7) $\lim_{x\to c} (f(x))^n=(\lim_{x\to c} f(x))^n$

(8) $\lim_{x\to c} \sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to c} f(x)}$ as long as $\lim_{x\to c} f(x)>0$ if n is even.

9.1 (continued)

Let's look at a different sort of limit case now.

<table>
<tr>
<td>
Ex 6: Given this limit $\lim_{x \to 2} \frac{1}{x-2}$
What case is this?

Is there algebra we can do to simplify the expression?

Find the limit.
</td>
<td>
Limit Cases/Strategies: $\lim_{x \to c} f(x)$

1. Try plugging in the x-value. If you get a finite number, you're done.
(Graphically, the function
y = f(x) goes through the point (c, f(c)) and the value you get for the limit is indeed f(c).)

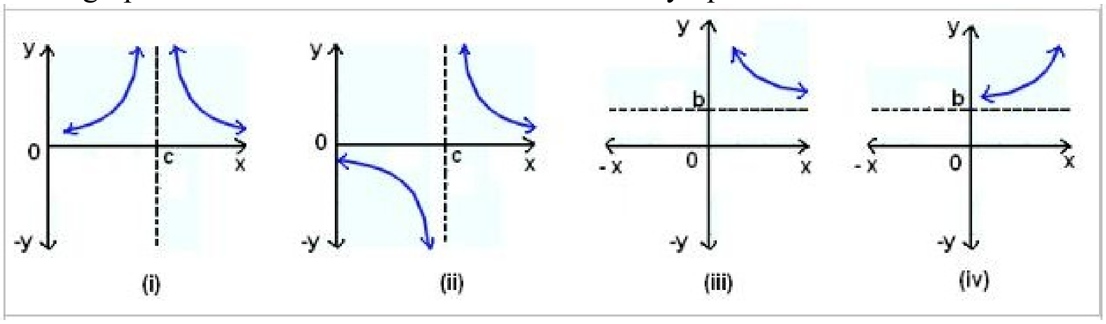
2. If you get the 0/0 case, you have algebra to do to simplify the expression before you can figure out what to do next.
(If this turns into a finite limit, then graphically for y = f(x), the function has a hole at that x-value.)

3. If you get the nonzero/0 case, then the limit DNE. (Graphically for y = f(x), the function has a vertical asymptote, VA, at that x-value.)
</td>
</tr>
</table>

Ex 7: Suppose the cost C, for a steel company, of removing p% of air pollution for a manufacturing plant is given by $C(p) = \frac{6{,}400\,p}{100-p}$. What happens to cost if this company tries to remove 100% of the air pollution from its manufacturing?

9.2 Limits at Infinity; Continuous Functions

Here are some graphs that show us vertical and horizontal asymptotes.

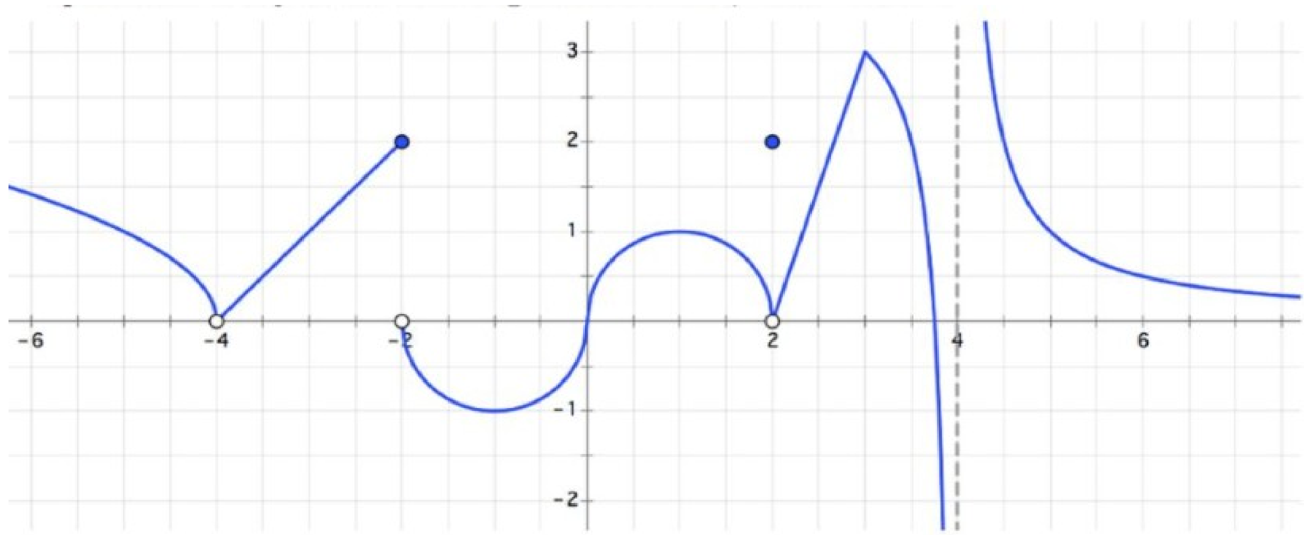


Can you write corresponding limit statements for each of these images?

Ex 1: Find the limits. (a) $\lim_{x\to-\infty} \frac{2x-1}{x+5}$	<u>Properties of Limits as x goes to some sort of infinity:</u> 1. $\lim_{x\to\pm\infty} c=c$ (for c a constant) 2. $\lim_{x\to\pm\infty} \frac{A}{x^k}=0$ (for A and k constants and k > 0)
(b) $\lim_{x\to-\infty} \frac{2x^2-1}{x+5}$	(c) $\lim_{x\to-\infty} \frac{2x-1}{x^2+5}$

9.2 (continued)

Ex 2: For this graph of y=f(x), answer the limit questions.



(a) $\lim_{x\to 2} f(x) =$ ______ (b) $\lim_{x\to 4^-} f(x) =$ ______ (c) $\lim_{x\to 4^+} f(x) =$ ______

(d) $\lim_{x\to -2^+} f(x) =$ ______ (e) $\lim_{x\to -2^-} f(x) =$ ______ (f) $\lim_{x\to -2} f(x) =$ ______

(g) $\lim_{x\to \infty} f(x) =$ ______ (h) $\lim_{x\to -\infty} f(x) =$ ______

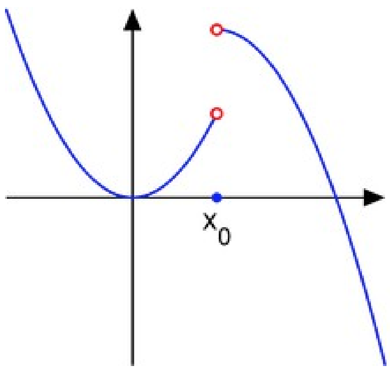
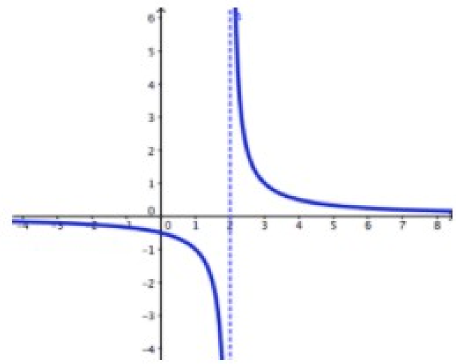
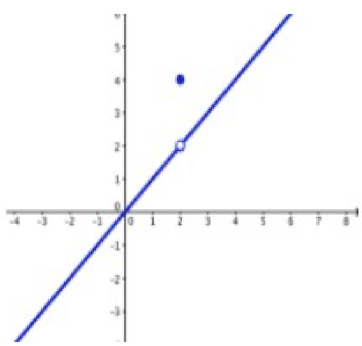
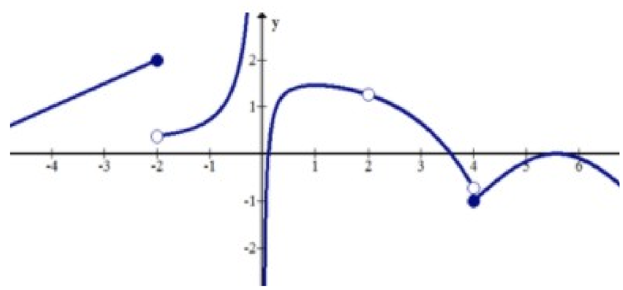
Ex 3: Find the limits.

(a) $\lim_{x\to -\infty} \frac{3\sqrt[3]{x^5}+x-7}{\sqrt[3]{5x}+3x+1}$	(b) $\lim_{x\to \infty} \frac{3\sqrt[3]{x^5}+x-7}{\sqrt[3]{5x}+3x+1}$
(c) $\lim_{x\to \infty} \sqrt[5]{\frac{2x^3-5x+1}{\pi x^3+3}}$	(d) $\lim_{x\to -\infty} \sqrt[5]{\frac{2x^3-5x+1}{\pi x^3+3}}$

Ex 4: (a) Is this function continuous? (b) If not, where is it discontinuous and why does it fail continuity? (c) Can you fix it (patch it) so you remove the discontinuity? If so, do that. $f(x)=\begin{cases} x \text{ , if } x\leq 0 \\ x^2 \text{ , if } 0<x\leq 1 \\ 3-x \text{ , if } x>1 \end{cases}$	**Continuity at x = c:** A function, f(x), is continuous at x = c if (1) $\lim_{x\to c} f(x)$ exists and (2) $f(c)$ exists and (3) $\lim_{x\to c} f(x)=f(c)$ There are three types of discontinuities: 1. Jump 2. Hole 3. Vertical Asymptote
Ex 5: (a) Is this function continuous? (b) If not, where is it discontinuous and why does it fail continuity? (c) Can you fix it (patch it) so you remove the discontinuity? If so, do that. $f(x)=\frac{\sqrt{x}-1}{x-1}$	

9.2 (continued)

Ex 6: For each graph, answer the following questions:
(a) List all the x-values where the discontinuities occur, if there are any.
(b) For each x-value from (a), classify what type of discontinuity it is (Jump, Hole, VA).
(c) List all the intervals where the function is continuous.

(i) 	(ii) 
(iii) 	(iv) 

9.3 Average and Instantaneous Rates of Change: The Derivative

Ex 1: Given the function $f(x)=2x^2+1$, answer the following questions. (a) Find the average rate of change of this function on the interval [0, 1]. (b) Use the definition of the derivative to find the derivative.	Average Rate of Change of a Function: The average rate of change of f(x) on the x-interval [*a, b*] is given by $\text{average value of } f(x)=\frac{f(b)-f(a)}{b-a}$. Derivative definition: Assuming f(x) is a continuous function $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ The derivative is the instantaneous slope at any point on the curve of y = f(x). Note: slope, rate of change, derivative, speed, velocity, marginal something.... all these words should make you think of derivative immediately!! Marginal Revenue/Cost/Profit: Given Revenue, R(x), function, the marginal revenue is denoted as $\overline{MR}$ and is given as the derivative of R(x). (Similarly for profit and cost functions.) Marginal revenue, for example, represents/estimates how much revenue you get from the sale of one more item.

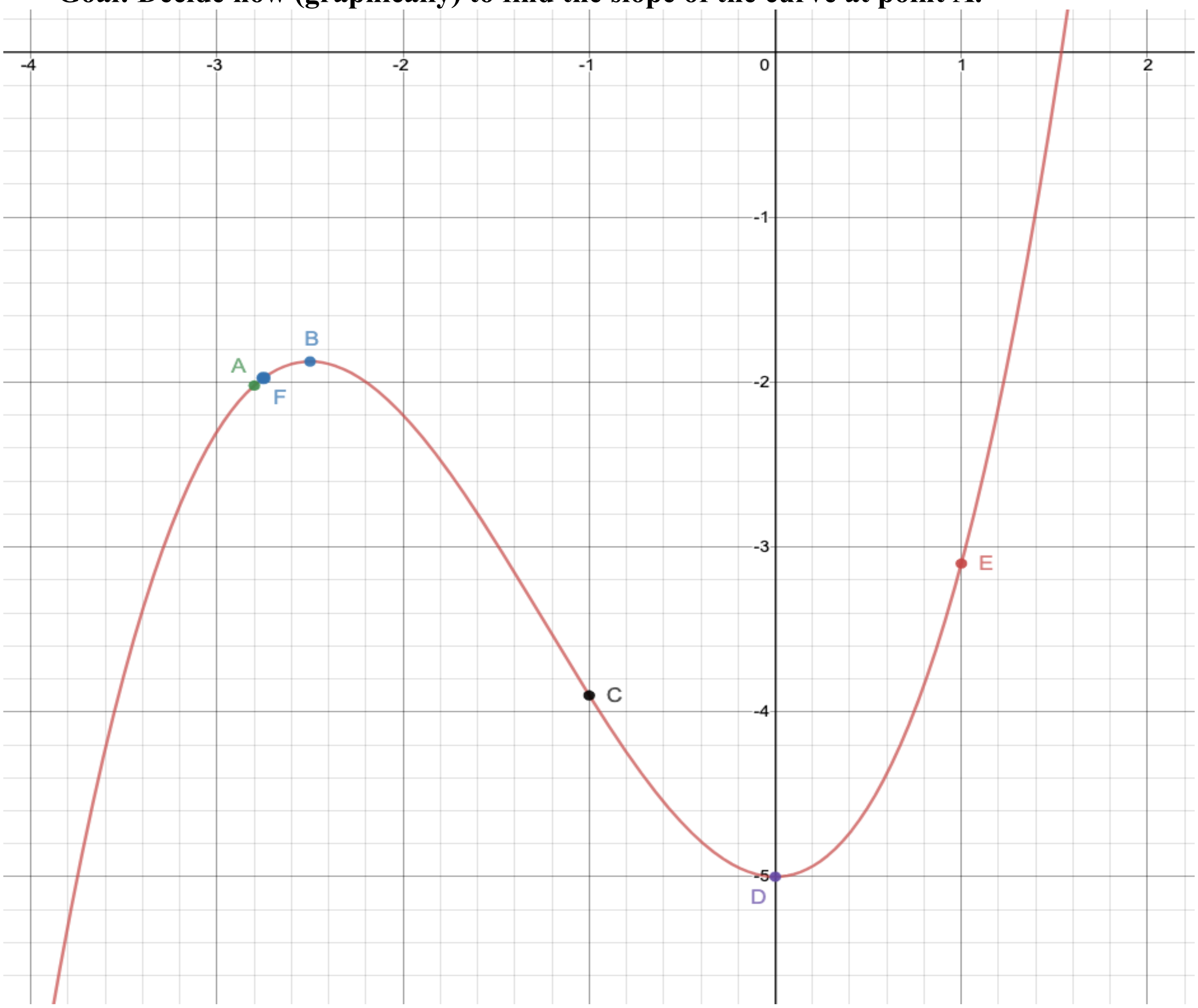
9.3 (continued)

Ex 2: Find the slope formula for $y=\frac{3}{x^2}$. Then, find the equation of the tangent line to the curve at x = -1.

Spaghetti Noodle Derivatives

Let's use this graph and your dry spaghetti noodle to help us understand what a derivative is telling us and why the definition works.

Goal: Decide how (graphically) to find the slope of the curve at point A.



0. Place your spaghetti noodle on the graph to create a tangent line to the curve at point A. The slope of this line is the slope of the curve at that point.

1. put your spaghetti noodle on the graph so it goes through points A and E.
This is called a secant line because it goes through two points on the curve.
Does the slope of this line look like a good approximation for the slope of the curve at point A?

2. What if we look at the lines between A and D, A and C, A and B? Do they produce lines whose slopes are closer to the slope of the curve at point A?

3. Which of the secant lines produces a slope that's closest (or best approximation) to the slope of the curve at point A?

slope between A and F   or   slope between A and E

4. Can you describe in words how we can use what we've learned here to get the slope of the curve at point A?

9.3 (continued)

Ex 4: Suppose the revenue function for my chocolate chip cookies that I bake as a side business is given by $R(x) = 40x - 0.05x^2$ dollars, where x is the number of cookies sold.

(a) Find the marginal revenue function.

(b) What is the marginal revenue when 100 cookies are sold? What does that mean?

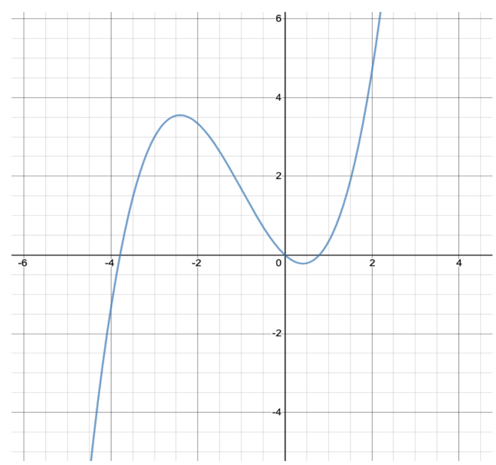
9.4 Derivative Formulas

Ex 1: Use "shortcuts" to find the derivatives for the given functions. (a) $f(x)=x^{12}+5x^{-2}-\pi x^{-10}+\pi^2$	<u>Derivative Rules</u> 1. $D_x(k)=0$ for any constant k 2. $D_x(x^n)=n x^{n-1}$ (Power Rule for integer exponents)
(b) $y=\frac{3}{x^5}-x^{-1}+\frac{e}{x^6}$	3. <u>Derivative is a Linear Operator</u>, which means it satisfies BOTH the following conditions: (a) $D_x(f(x)+g(x))=D_x(f(x))+D_x(g(x))$ (i.e. the derivative operator distributes through addition) AND (b) $D_x(k f(x))=k D_x(f(x))$ for any constant k (i.e. the derivative operator commutes with scalar multiplication or with multiplication by a constant).
(c) $y=(3x^{-2}+2x)(x^5+8)$	Note: this #3 point, in your book, is called the coefficient rule, sum rule and difference rule. One more note: From now on, use the shortcuts for taking derivatives, unless the instructions explicitly state to use the definition of derivative.

Ex 2: For each part of Example 1, find the slope of the function's curve at x = 1.

Ex 3: Find the equation of the tangent line to the curve of $y = x^5 + 2x^2 - 5$ at x = -1.

Ex 4: Find all points on the curve $y = \frac{1}{3}x^3 + x^2 - x$ where the tangent line has a slope of 1.



9.5 The Product Rule and the Quotient Rule

Ex 1: Use "shortcuts" to find the derivatives for the given functions. (a) $y=\dfrac{5x^3-x}{36-x^2}$	More Derivative Rules 4. Product Rule: $(f\cdot g)'=f'\cdot g+g'\cdot f$
(b) $g(x)=(x^4+6)(\pi^2-3x+\dfrac{9}{x})$	5. Quotient Rule: $\left(\dfrac{f}{g}\right)'=\dfrac{g\cdot f'-f\cdot g'}{g^2}$ ("low d-hi minus hi d-low over low-squared")
(c) $f(x)=\dfrac{5x^2+9x-2}{x^{-2}-5}$	(d) $y=(3x^{-2}+2x)(x^5-3x+8)$

Ex 2: Given the function $f(x)=\frac{3\sqrt[3]{x}+1}{x+2}$, find $f'(-1)$. Graphically, how would you interpret the number you get for this answer?

Ex 3: Suppose that the proportion P of voters who recognize a candidate's name t months after the start of the campaign is given by $P=\frac{13t}{t^2+100}+0.18$.

(a) Find the rate of change of P when t = 6. What does this number mean?

(b) Find the rate of change of P when t = 12. What does this number mean?

(c) One month prior to the election, is it better for P'(t) to be positive or negative? Explain.

9.6 The Chain Rule and the (Generalized) Power Rule

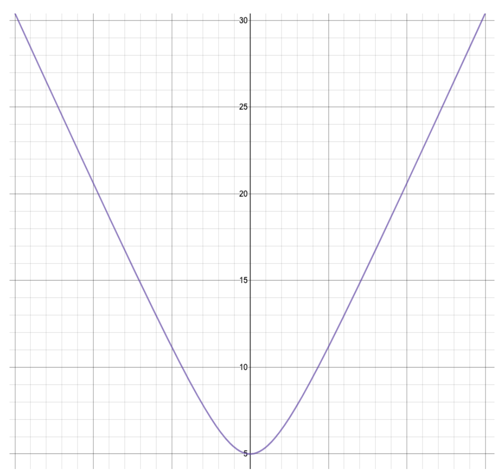
<table>
<tr>
<td>Ex 1: Find the derivative of y.
$y=\left(2x^{7/2}-4x^2\right)^3+\dfrac{1}{(3x-1)^2}$</td>
<td>Chain Rule (work from the outside in)
$\left(f\left(g(x)\right)\right)'=f'\left(g(x)\right)\cdot g'(x)$

Another way to write this, if we have $y=f(u)$ and $u=g(x)$ (so overall we have $y=f\left(g(x)\right)$) is:
$\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$</td>
</tr>
<tr>
<td colspan="2">Ex 2: Find the derivative for this function.
$D_x\left(\left(x+\left(x^2+7x\right)^3\right)^5+2x\right)$</td>
</tr>
<tr>
<td colspan="2">Ex 3: Find the derivative of y. $y=\left(\dfrac{7^3+x^3}{\sqrt{x}-\dfrac{1}{x^2}}\right)^4$</td>
</tr>
</table>

9.6 (continued)

Ex 4: Find $D_x\left(F\left(x^2 - \frac{1}{x^2}\right)\right)$ if F(x) is a differentiable function.

Ex 5: Find the equation of the tangent line to the function $y = \sqrt{4x^2 + 25}$ when x = -1.



9.7 Using Derivative Formulas

Let's first practice putting all the derivative rules together.

Ex 1: Find the derivative for the following functions.

(a) $y=((x^3+5x)(x+9))^{30}+4x$	(b) $y=\frac{1}{4}x^2(x^3+6)^{40}$
(c) $f(x)=\left(\frac{5-x^2}{x+8}\right)^{-2}$	(d) $g(x)=\frac{3}{(2-x)^3}$

And now for a fun story problem to see an application.

Ex 2: If the national consumption function is given by $C(y)=2\sqrt{y+1}+0.4y+4$, find $\frac{dC}{dy}$ which represents the marginal propensity to consume.

9.8 Higher-Order Derivatives

Ex 1: Find $\frac{d^3(x^{-3})}{dx^3}$

Ex 2: Find $D_x^{14}(96x^{14}-81x^9+\pi)$

Notation: For y = f(x), the following notations all "work" for the prescribed derivatives.

1. First derivative

$$D_x(f(x))=f'(x)=\frac{dy}{dx}=\frac{d(f(x))}{dx}=y'=D_x(y)$$

2. Second derivative

$$D_x^2(f(x))=f''(x)=\frac{d^2y}{dx^2}=\frac{d^2(f(x))}{dx^2}=y''=D_x^2(y)$$

3. Third derivative

$$D_x^3(f(x))=f'''(x)=\frac{d^3y}{dx^3}=\frac{d^3(f(x))}{dx^3}=y'''=D_x^3(y)$$

4. nth derivative (for any n = 4, 5, 6, ...)

$$D_x^{(n)}(f(x))=f^{(n)}(x)=\frac{d^ny}{dx^n}=\frac{d^n(f(x))}{dx^n}=y^{(n)}=D_x^{(n)}(y)$$

Ex 3: Find $f''(2)$ for $f(x)=\frac{(x+1)^2}{x-1}$.

9.8 (continued)

Ex 4: If $s(t)=\frac{1}{10}(t^4-14t^3+60t^2)$, find the velocity of the moving object when its acceleration is zero.

Ex 5: Fill in the table (find the pattern) to establish the formula for the nth derivative of the following functions.

(a) $f(x)=\frac{4}{x}$

n	$f^{(n)}(x)$
1	
2	
3	
4	
5	
n	

(b) $y=\frac{3}{(x-2)^3}$

n	$y^{(n)}$
1	
2	
3	
4	
5	
n	

9.9 Applications of Derivatives in Business and Economics

Ex 1: Suppose the cost function for a commodity is $C(x) = 300 + 6x + \frac{1}{20}x^2$.

(a) Find the marginal cost at 8 units, i.e. $\overline{MC}(8)$.

(b) Calculus C(9) - C(8) to find the actual change in cost.

Ex 2: For the profit function given by $P(x) = 30x - x^2 - 200$, answer the following questions.

(a) Find the marginal profit function.

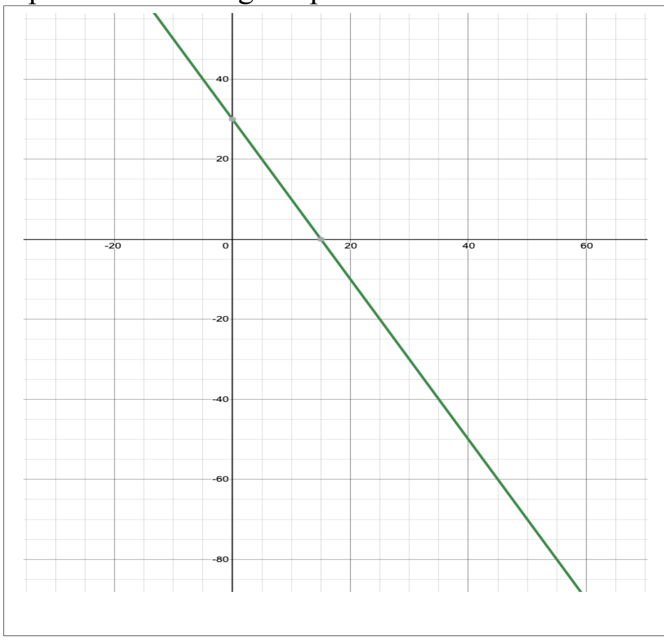
(b) What level of production gives a marginal profit of zero?

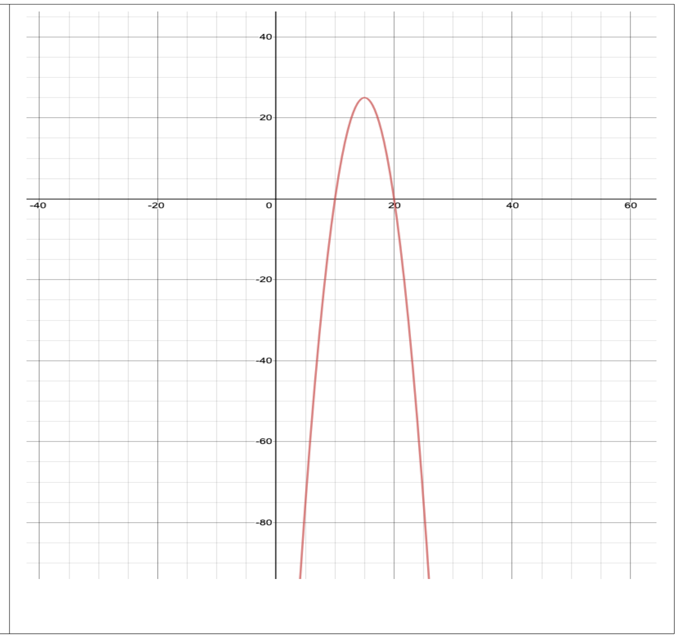
(c) At what level will profit be a maximum?

(d) What is the maximum profit?

9.9 (continued)

Ex 2 (continued):
(e) On the two graphs given, identify which graph represents the profit function and which graph represents the marginal profit function.





Ex 3: Suppose that I revised my business plan for selling my chocolate chip cookies. My marketing agent decided to name it Hot Chippies...I thought it was catchy. With some analysis, we've found that the total revenue Hot Chippies is $R(x)=32x$ and the cost function is $C(x)=200+2x+x^2$ (I had to hire my mom to help...and she had to be trained.) where x is the number of cookies produced and sold.

(a) Find the profit function.

(b) Find the profit from sales of 20 cookies.

(c) Find the marginal profit function.

(d) Find $\overline{MP}(20)$. Explain what this number means.