1.1 System of Equations

Defn: A **linear equation** in variables $x_1, x_2, \dots, x_n$ is an eqn of form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

constants/coefficients (real or complex #s)

Intuition: no powers of x_i higher than 1

linear eqn ex

1) $3x_1 + 7x_2 + 9 = x_2$

2) $x_3 = x_2 - \pi x_1 + (\sqrt{2} + \sqrt{6}) x_2$

can be rearranged

1) $3x_1 + 6x_2 = -9$

2) $\pi x_1 - (1 + \sqrt{2} + \sqrt{6}) x_2 + x_3 = 0$

non-examples: (i.e. these are NOT linear)

1) $x_1 + x_2 x_3 = 4$

can't multiply variables

2) $\frac{1}{x_2} + \sqrt{x_1} = 5$

can't have variable in sq. root

can't have variable in denominator

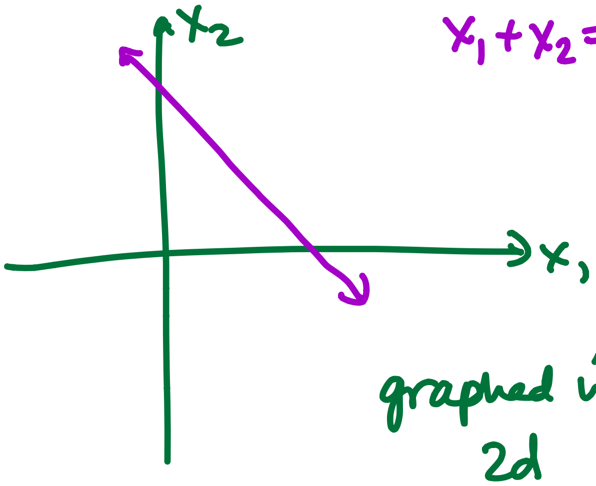
Defn: A **system of linear eqns** (or **linear system**) is a collection of one or more linear eqns.

(we can solve such a system for solutions that satisfy all eqns simultaneously)

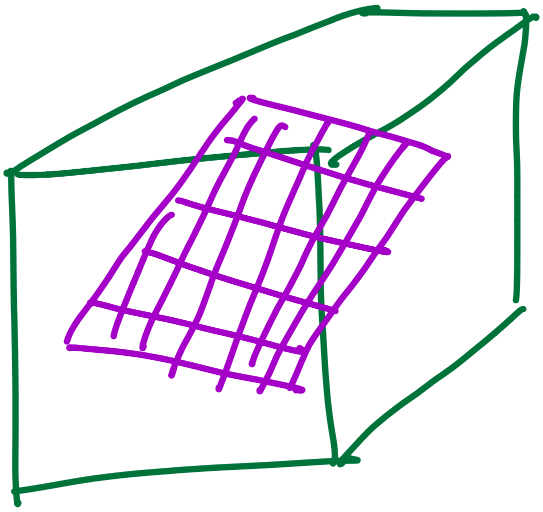
Defn: A **solution** of the system is a list $(s_1, s_2, \ldots, s_n)$ of #s that make each eqn true when $(s_1, s_2, \ldots, s_n)$ are substituted in for $(x_1, x_2, \ldots, x_n)$.

Pictures: linear eqn represents a "flat thing".

$x_1 + x_2 = 3$



graphed in 2d but a line is 1-d object



$x_1 + x_2 + 2x_3 = 4$

graphed in 3d but a plane is a 2-d object

in n variables, $x_1 + 2x_2 + \cdots + nx_n = 5$, is an (n-1)-dim. "hyperplane" (i.e. flat thing)

What is the graphical representation of the solution set of a system of eqns?

Ex 1 Solve these systems, and graph.

A) $x_1 - 3x_2 = -3$

$2x_1 + x_2 = 8$

B) $x_1 + 4x_2 = 9$

$3x_1 = 20 - 12x_2$

C) $5x_1 + 2x_2 = 3$

$6 - 4x_2 - 10x_1 = 0$

A system of linear eqns has:

1) no solution

or 2) exactly one solution

or 3) infinitely many solutions

VOCAB:

"inconsistent system"

"consistent systems"

Matrix Notation:

- systems of linear eqns can be made into matrices

ex

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 6 \\ 4x_1 \qquad + x_3 &= 2 \\ 2x_2 + 5x_3 &= -8 \end{aligned}$$

coefficient matrix

$$\begin{bmatrix} 1 & 3 & -1 \\ 4 & 0 & 1 \\ 0 & 2 & 5 \end{bmatrix}$$

3x3 matrix

augmented matrix

$$\left[\begin{array}{ccc:c} 1 & 3 & -1 & 6 \\ 4 & 0 & 1 & 2 \\ 0 & 2 & 5 & -8 \end{array}\right]$$

3x4 matrix

Defn: The size of a matrix tells how many rows & columns it has. An <u>m×n matrix</u> (read "m by n") has m rows and n columns.

Solving a System of linear Eqns

Strategy: • replace system by an equivalent system.

- observe what this corresponds to in augmented matrix.

Qn: for a 3x3 system, what might solutions look like?

Ex2 Solve this system.

① $x_1 + 3x_2 - x_3 = 6$

② $4x_1 + x_3 = 2$

③ $2x_2 + 5x_3 = -8$

step 1: keep x_1 in first eqn, but eliminate x_1 from 2nd & 3rd eqns.

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 6 \\ 4 & 0 & 1 & 2 \\ 0 & 2 & 5 & -8 \end{array}\right]$$

-4(eqn 1)
+ eqn 2
new eqn 2

$$\begin{array}{r} -4x_1 - 12x_2 + 4x_3 = -24 \\ +\ 4x_1 \qquad\qquad + x_3 = 2 \\ \hline -12x_2 + 5x_3 = -22 \end{array}$$

new system:

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 6 \\ -12x_2 + 5x_3 &= -22 \\ 2x_2 + 5x_3 &= -8 \end{aligned}$$

You finish it:

Check solution:

Elementary Row Operations (ERO)

1) (replacement or elimination) replace one row by sum of itself and a multiple of another row

2) (interchange or swap) interchange two rows.

3) (scaling) multiply all entries of one row by a non-zero constant.

note: EROs are reversible!

Defn: Two matrices are called row equivalent if you can get from one to the other by a series of EROs.

note: If augmented systems of 2 linear systems are row equivalent, then the systems have same solution set.

Existence & Uniqueness

Fundamental Qns:

1) Is the system consistent? (i.e. it has at least one solution)

2) If a solution exists, is it the only one?

Qn: what do these words mean?

Ex 3 Determine if these systems, given by their augmented matrices, are consistent.

A) $\begin{bmatrix} 1 & 3 & 2 & 7 \\ 0 & 4 & 8 & -5 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

B) $\begin{bmatrix} 1 & 3 & 2 & 7 \\ 0 & 4 & 8 & -5 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

1.2 Row Reduction and Echelon Forms

Idea: We'll continue solving linear systems using elementary row ops, like we did in 1.1, but we'll explicitly lay out a process for doing so. We'll introduce row echelon form (REF) and reduced REF.

Defn: • A matrix is in row echelon form (REF) if

1) non-zero rows are above rows of all zeros
2) leading entry of a row is to the right of the leading entry of the row above it.
3) Entries below a leading entry are zero.

[intuition: It's "upper triangular".

ex

leading entries

$$\begin{bmatrix} 1 & 4 & 3 & -5 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

ex

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\blacksquare$ = non-zero #

$*$ = any #

Defn (continued):

- A matrix is in reduced row echelon form (RREF) if in addition,

   4) the leading entry in each row is 1.

   5) each leading 1 is the only non-zero entry in its column.

ex $\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ ex $\begin{bmatrix} 1 & 0 & 4 & 9 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Note: Both REF & RREF are called "echelon forms".

Abuse of notation/lingo: I might use RREF as a verb.
ex "RREF this matrix" means use elem. row ops to put this matrix into RREF.

Thm 1: Each matrix is row equivalent to exactly one RREF matrix (i.e. RREF is unique).

Pivot Positions

"pivot position" = matrix entry corresponding to a leading 1 in RREF.

"pivot column" = column containing a pivot position.

Note: leading entries in REF are in same position as those in RREF ⇒ we can figure out pivot positions & columns from either REF or RREF matrices.

$$\text{ex}\quad \begin{array}{c} \begin{array}{ccccc}\swarrow & \swarrow & & \swarrow & \text{pivot columns}\end{array} \\ \left[\begin{array}{ccccc} \boxed{1} & 2 & 5 & 3 & -8 \\ 0 & \boxed{3} & 0 & 1 & -6 \\ 0 & 0 & 0 & \boxed{-4} & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \uparrow \quad \uparrow \quad \uparrow \\ \text{pivot positions} \end{array}$$

note: the #s 1, 3, −4 are NOT pivot positions … the place they are in are the pivot positions (i.e. the 1,1 position, 2,2 and 3,4 positions are pivot positions)

Ex 1 Apply elem. row ops to put these matrices first in REF & then in RREF.

label pivot columns and positions

A) $$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$$

B) $$\begin{bmatrix} 1 & 1 & -5 & 3 \\ 2 & 4 & 0 & 9 \end{bmatrix}$$

Solutions of Linear Systems

RREFing an augmented matrix for a linear system solves the system.

Ex2: Solve the linear systems.
(Hint: notice relationship of this problem w/ Ex1.)

A) $x_1 - 7x_2 + 6x_4 = 5$
$x_3 - 2x_4 = -3$
$-x_1 + 7x_2 - 4x_3 + 2x_4 = 7$

B) $x_1 + x_2 - 5x_3 = 3$
$2x_1 + 4x_2 = 9$

---

* variables corresponding to pivot columns are called "basic variables".

* other variables are called "free variables"

* The general solution gives basic variables in terms of free variables.

1.3 Vector Equations

Changing Gears: In this section, we introduce the notion of vectors in $\mathbb{R}^n$ (real Euclidean space). These will eventually give us another way to describe systems of linear eqns.

Defn: A matrix with only one column (i.e. an $n \times 1$ matrix) is a vector in $\mathbb{R}^n$.
↑ read "Rn"

ex $\vec{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ vector in $\mathbb{R}^2$

ex $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \end{bmatrix}$ vector in $\mathbb{R}^4$

note: to distinguish between scalars and vectors, your book uses bold facing for vectors. In hand writing, I'll write vectors w/ → above them.)

- Scalar Multiplication

ex $\vec{u} = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$

$3\vec{u} = \begin{bmatrix} 9 \\ -24 \end{bmatrix}$ $-\vec{u} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$

Addition of vectors

ex $\vec{w} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$\vec{u} + \vec{w} = \begin{bmatrix} 3+5 \\ -8+0 \end{bmatrix} = \begin{bmatrix} 8 \\ -8 \end{bmatrix}$

note: we can only add vectors of same size.

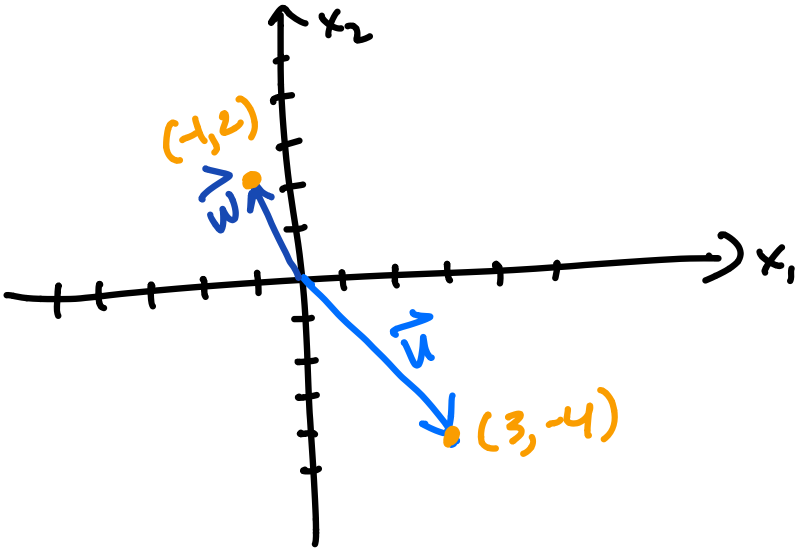
Geometric description of vectors

We can associate a vector in $\mathbb{R}^n$ w/ a point in $\mathbb{R}^n$.

ex $\vec{u} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \rightsquigarrow$ pt $(3,-4) \in \mathbb{R}^2$

element of (or "in")

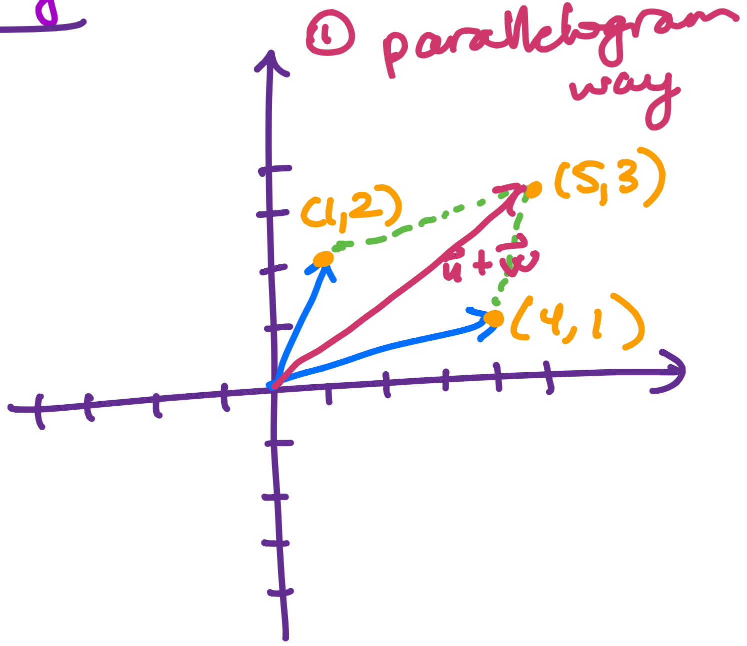
$\vec{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \rightsquigarrow$ pt $(-1,2) \in \mathbb{R}^2$



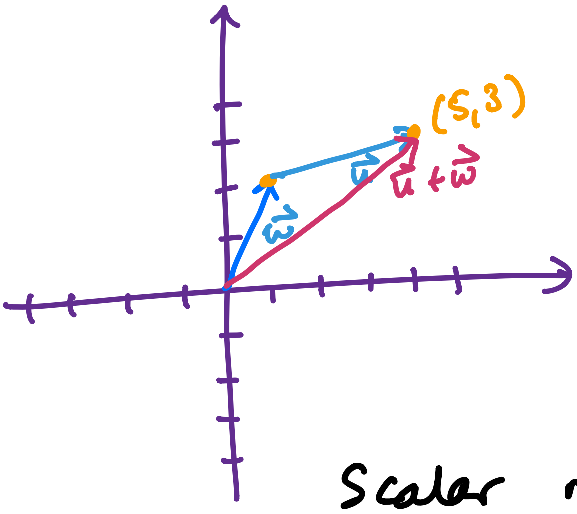
vector addition (geometrically)

ex $\vec{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

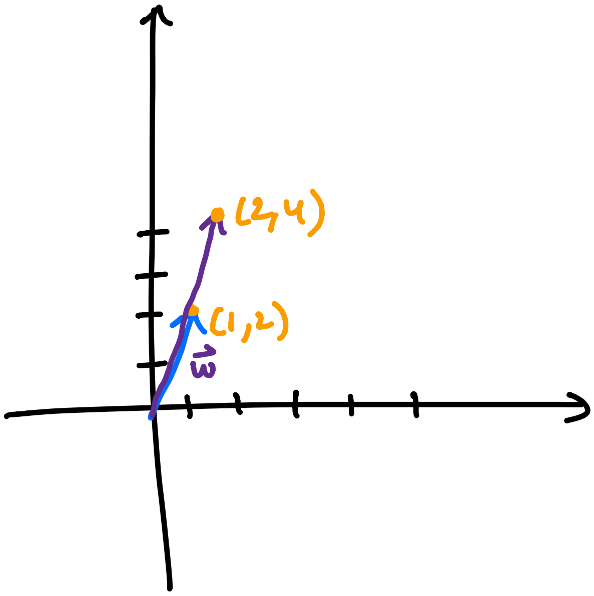
$\vec{u} + \vec{w} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$


① parallelogram way

or ② tail-to-end



Scalar multiplication (geometrically)



$$\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$2\vec{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Algebraic Properties of vectors:

$$\overbrace{\vec{u}, \vec{v}, \vec{w}}^{\text{vectors}} \in \mathbb{R}^n, \ \overbrace{c, d}^{\text{scalars}} \in \mathbb{R}$$

(i) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ (commutativity of addition)

(ii) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$ (associativity)

(iii) $\vec{u}+\vec{0} = \vec{0}+\vec{u} = \vec{u}$ (additive identity)

(iv) $\vec{u} + (-\vec{u}) = -\vec{u}+\vec{u} = \vec{0}$ (additive inverse)

(v) $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$ (scalar mult. is distributive)

(vi) $(c+d)\vec{u} = c\vec{u}+d\vec{u}$ (different sort of distributivity)

(vii) $c(d\vec{u}) = (cd)\vec{u}$ (associativity of scalar mult.)

(viii) $1\vec{u} = \vec{u}$ (scalar multiplicative identity)

Linear Combinations

Defn: Given vectors $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$ and scalars $c_1, c_2, \dots, c_p$, the vector

$$\boxed{\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n}$$

is called a linear combination of $\vec{v}_1, \dots, \vec{v}_p$ w/ weights $c_1, \dots, c_p$.

ex 1) $5v_1 + 2\vec{v}_2$ is linear combo of $\vec{v}_1 + \vec{v}_2$

2) $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2$ is " " " "

Ex1: let $\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix}$. Is $\vec{b} = \begin{bmatrix} 3 \\ -19 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ & $\vec{v}_2$?

In other words, does there exist scalars x_1 & x_2 s.t. $x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{b}$?

such that

setup:

$$x_1 \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -19 \\ 0 \end{bmatrix}$$

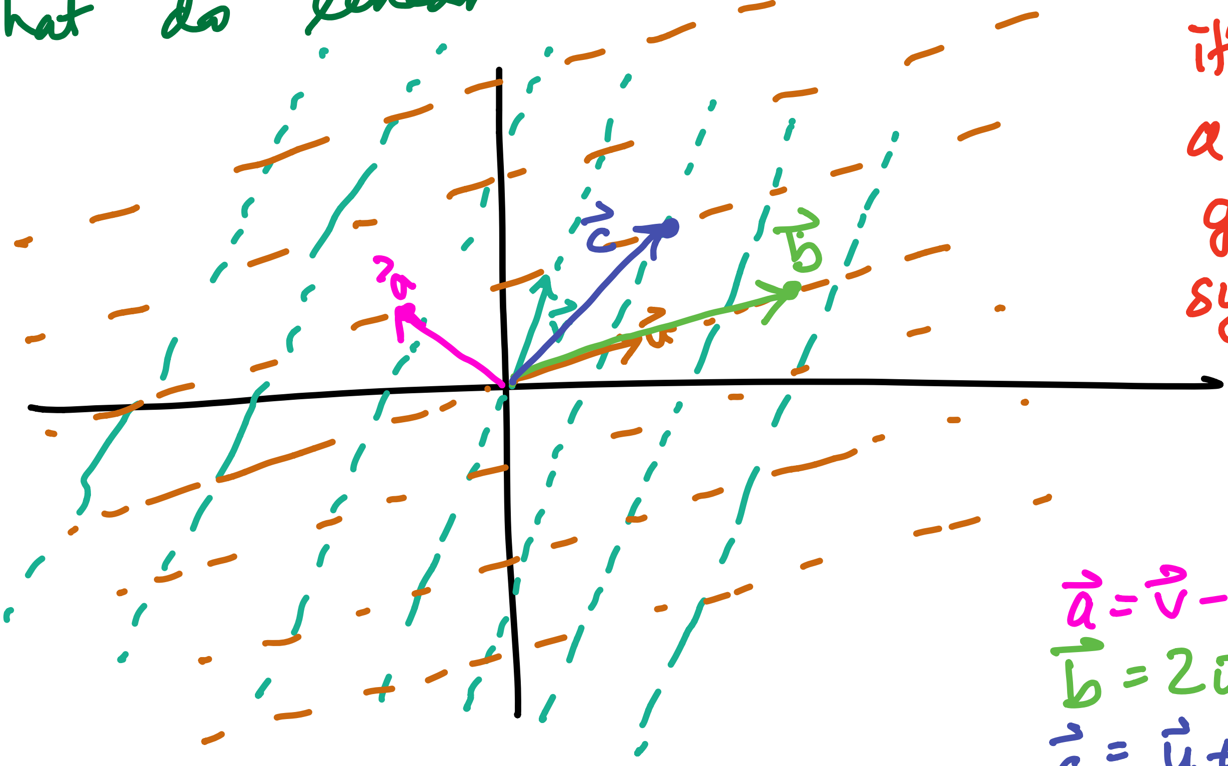
system of linear eqns

$$\begin{bmatrix} x_1 \\ -3x_1 + 5x_2 \\ 4x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -19 \\ 0 \end{bmatrix} \iff \begin{cases} x_1 = 3 \\ -3x_1 + 5x_2 = -19 \\ 4x_1 + 6x_2 = 0 \end{cases}$$

FACT: The vector eqn $x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n = \vec{b}$ has same soln as augmented matrix

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n & \vec{b} \end{bmatrix}.$$

What do linear combinations look like?



It's like a new grid system

$\vec{a} = \vec{v} - \vec{u}$

$\vec{b} = 2\vec{u}$

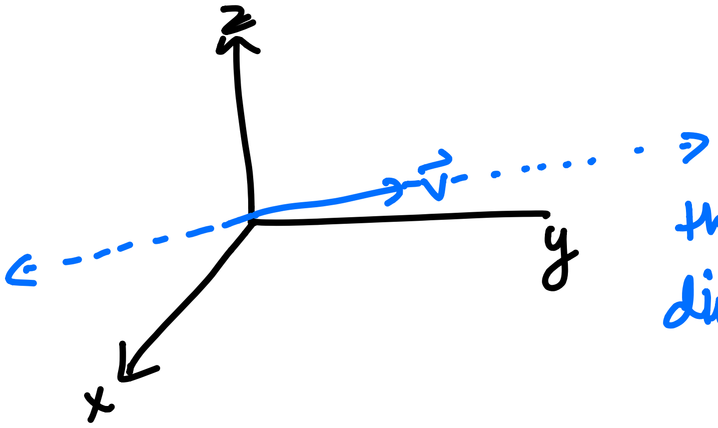
$\vec{c} = \vec{u} + \vec{v}$

Defn $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$. The set of all linear combinations of $\vec{v}_1, \dots, \vec{v}_p$ is called a span of $\vec{v}_1, \dots, \vec{v}_p$ (or the subset of $\mathbb{R}^n$ spanned by $\vec{v}_1, \dots, \vec{v}_p$) and is denoted by

$$\boxed{\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}}$$

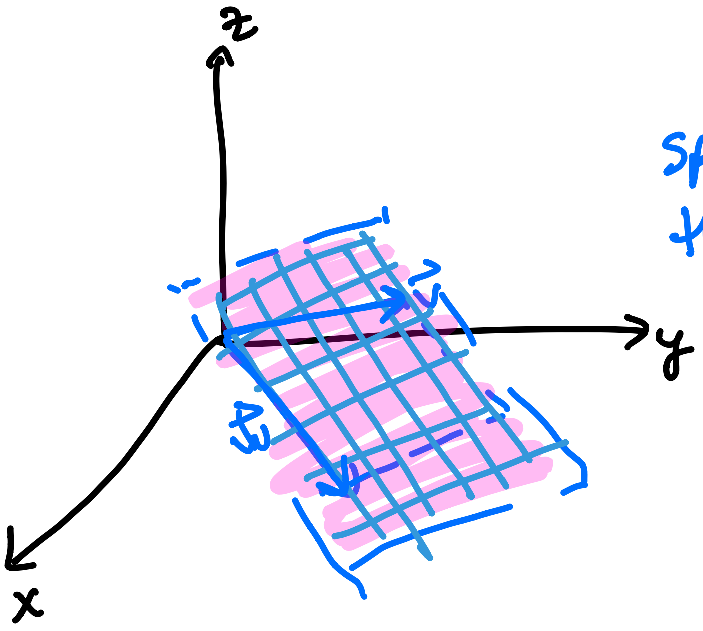
Geometric description of span

- span$\{\vec{v}\}$ for $\vec{v} \in \mathbb{R}^3$



span$\{\vec{v}\}$ is a line through the origin in direction of $\vec{v}$

- span$\{\vec{v}, \vec{w}\}$ for $\vec{v}, \vec{w} \in \mathbb{R}^3$



span$\{\vec{v}, \vec{w}\}$ is a plane through the origin that contains vectors $\vec{v}$ and $\vec{w}$

1.4 The Matrix Eqn $A\vec{x}=\vec{b}$

Another way to interpret systems of linear eqns is using matrix eqns. In this section, we learn several equivalent ways to answer the same question we did in section 1.2.

Multiplying a matrix by a vector (computational practice):

- A is $\underset{\substack{\nearrow \\ m \text{ rows}}}{m} \times \underset{\substack{\nwarrow \\ n \text{ columns}}}{n}$ matrix
- $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ vector

can think of columns of A as vectors

$$\Rightarrow A\vec{x} = \begin{bmatrix} | & | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \cdots & \vec{a}_n \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \underbrace{x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n}_{\text{linear combo of columns of } A \text{ with entries of } \vec{x} \text{ as weights}}$$

Properties of matrix-vector product $A\vec{x}$

Qn: Does this look familiar?

A is $m \times n$ matrix, $\vec{u}, \vec{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$,

(a) $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$

(b) $A(c\vec{u}) = c(A\vec{u})$

Ex 1: $$A\vec{x} = \underbrace{\begin{bmatrix}2 & 5\\ 1 & 0\\ -3 & 4\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}1\\ -6\end{bmatrix}}_{\vec{x}} = 1\begin{bmatrix}2\\ 1\\ -3\end{bmatrix} + -6\begin{bmatrix}5\\ 0\\ 4\end{bmatrix} = \begin{bmatrix}2+-30\\ 1+0\\ -3+-24\end{bmatrix} \quad ①$$

OR

two ways to think of $A\vec{x}$

$$= \begin{bmatrix}[2\ 5]\begin{bmatrix}1\\ -6\end{bmatrix}\\ [1\ 0]\begin{bmatrix}1\\ -6\end{bmatrix}\\ [-3\ 4]\begin{bmatrix}1\\ -6\end{bmatrix}\end{bmatrix} = \begin{bmatrix}2+-30\\ 1+0\\ -3+-24\end{bmatrix} \quad ②$$

Ex 2: Write the system

$$\begin{aligned}3x_1 + x_2 - 5x_3 &= 9\\ x_2 + 4x_3 &= 0\end{aligned}$$

as ① a vector eqn and ② a matrix eqn.

Three ways of viewing the same problem:

① $\left\{ \text{The solution set to matrix eqn } A\vec{x} = \vec{b} \right\}$ $\equiv$ ② $\left\{ \text{The solution set to vector eqn } x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{b} \right\}$ $\equiv$ ③ $\left\{ \text{solution set of the system of linear eqns whose augmented matrix is } \begin{bmatrix}\vdots & \vdots & & \vdots & \vdots\\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{b}\\ \vdots & \vdots & & \vdots & \vdots\end{bmatrix} \right\}$

In ①: A: m×n matrix; $\vec{x}$: vector in $\mathbb{R}^n$; $\vec{b}$: vector in $\mathbb{R}^m$

In ②: $x_1, x_2, \dots, x_n$: scalars; $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n, \vec{b}$: vectors in $\mathbb{R}^m$

In ③: $m \times (n+1)$ matrix

Ex 3 What are 3 ways we can solve the linear system?

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + 3x_2 + x_3 = 3 \\ 4x_2 - 2x_3 = 0 \end{cases}$$

① RREF $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 3 & 1 & 3 \\ 0 & 4 & -2 & 0 \end{bmatrix}$

② ?

③ ?

Existence of Solutions:

Thm Let A be an $m \times n$ matrix. The following statements are logically equivalent:

(a) For each $\vec{b} \in \mathbb{R}^m$, the eqn $A\vec{x} = \vec{b}$ has a solution.

(b) Each $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.
(i.e. every $\vec{b} \in \mathbb{R}^m$ is in $\text{span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$)

(d) A has a pivot position in every row.

↑ warning: this is about the coefficient matrix A, NOT the augmented matrix $[A \ b]$.

Reminder: $\vec{b} \in \text{span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$ means $\vec{b}$ can be written as linear combo of $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$.

Ex 4 For $B = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}$ Do the columns of B span $\mathbb{R}^3$?

Equivalent questions:

- For each $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$, does eqn $B\vec{x} = \vec{b}$ have a solution?
- Is each $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$ a linear combo of $\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$?

easiest to answer → - Does B have a pivot position in each row?

To answer the qn, we need to REF B.

1.5 Solution Sets of Linear Systems

We've learned a few ways of solving systems of linear eqns. In this section, we analyze the solution sets themselves, using our new vector notation.

Defn: A linear system is homogeneous if it can be written in the form $\underset{m\times n \text{ matrix}}{A}\vec{x} = \underset{\text{zero vector in } \mathbb{R}^m}{\vec{0}}$.

Note: A homogeneous system always has a solution of $\vec{x} = \vec{0}$. called the "trivial solution"

Qn: When does a homogeneous system have a non-trivial solution?

i.e. when is there $\vec{x} \neq \vec{0}$ s.t. $A\vec{x} = \vec{0}$? ← such that

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Leftrightarrow$ when is there non-trivial solution to

$$\underbrace{\begin{bmatrix} A & \vec{0} \end{bmatrix}}_{\text{augmented matrix}}$$

$\boxed{\text{DISCUSS}}$

FACT: The homogeneous eqn $A\vec{x}=\vec{0}$ has a nontrivial solution iff the eqn has at least one free variable.

"if and only if" $\Longleftrightarrow$

Ex 1 Determine if homogeneous system has a nontrivial solution. Then describe the sdn set.

(a) $\begin{cases} x_1-3x_2+7x_3=0 \\ -2x_1+x_2-4x_3=0 \\ x_1+2x_2+9x_3=0 \end{cases}$

Plan: REF the augmented matrix. Find pivot columns. See if there are any free variables.

$$\begin{array}{c} (-1)\ (2) \\ \\ \\ \end{array}\begin{bmatrix} 1 & -3 & 7 & 0 \\ -2 & 1 & -4 & 0 \\ 1 & 2 & 9 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -3 & 7 & 0 \\ 0 & -4 & 10 & 0 \\ 0 & 5 & 2 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{array}{c} \\ (-5) \\ \\ \end{array}\begin{bmatrix} 1 & -3 & 7 & 0 \\ 0 & 1 & 12 & 0 \\ 0 & 5 & 2 & 0 \end{bmatrix} \rightsquigarrow \begin{array}{c} \\ \\ (-\frac{1}{58}) \end{array}\begin{bmatrix} 1 & -3 & 7 & 0 \\ 0 & 1 & 12 & 0 \\ 0 & 0 & -58 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} \textcircled{1} & -3 & 7 & 0 \\ 0 & \textcircled{1} & 12 & 0 \\ 0 & 0 & \textcircled{1} & 0 \end{bmatrix}$$

pivot entries

no free variables

$\Rightarrow$ NO nontrivial solutn for this homogeneous system

(b) $\begin{cases} x_1 + 3x_2 - 5x_3 = 0 \\ x_1 + 4x_2 - 8x_3 = 0 \\ -3x_1 - 7x_2 + 9x_3 = 0 \end{cases}$ You finish this one.

$$\begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To Do: Write your answer (1) parametrically and (2) as a vector eqn $\vec{x} = \dots$

Note: $\vec{x} = x_3\begin{bmatrix}-4\\3\\1\end{bmatrix}$ is a parametric vector eqn.

$\iff \vec{x} = x_3\vec{v}$ s.t. $\vec{v} = \begin{bmatrix}-4\\3\\1\end{bmatrix}$ ↙ const. vector

- Every solution of $A\vec{x}=\vec{b}$ is a multiple of $\vec{v}$.
- All multiples of $\vec{v}$ are solutions.
- The solution set is a line. (and notice this is a line through the origin)

We can describe solutions to non-homogeneous systems in parametric vector form as well.

Ex 2: Describe all solutions of $A\vec{x}=\vec{b}$, where

$$A = \begin{bmatrix}1 & 3 & -5\\ 1 & 4 & -8\\ -3 & -7 & 9\end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix}4\\0\\-20\end{bmatrix}$$

note: this is same A from last example.

augmented matrix $\begin{bmatrix}1 & 3 & -5 & 4\\ 1 & 4 & -8 & 0\\ -3 & -7 & 9 & 5\end{bmatrix}$

$$\xrightarrow{\text{RREF}} \begin{bmatrix}1 & 0 & 4 & 16\\ 0 & 1 & -3 & -4\\ 0 & 0 & 0 & 0\end{bmatrix} \begin{array}{l}\Rightarrow x_1 = -4x_3 + 16\\ \Rightarrow x_2 = 3x_3 - 4\\ \Rightarrow x_3 \text{ free}\end{array}$$

Solutions in parametric vector form:

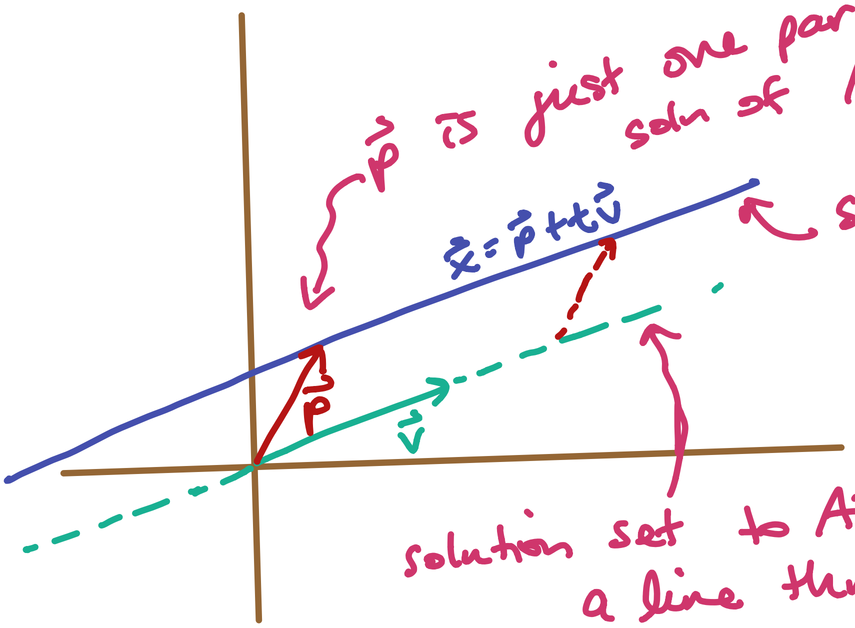
$\vec{x} =$

Notice:

- For a non-homogeneous system $A\vec{x}=\vec{b}$, $\vec{b}\neq\vec{0}$, the vector eqn $\vec{x}=\vec{p}+t\vec{v}$ describes soln set
- For a homogeneous system, $A\vec{x}=\vec{0}$, with same A, the vector eqn $\vec{x}=t\vec{v}$ describes soln set.

these only differ by $\vec{p}$.

· What does that look like geometrically?



$\vec{p}$ is just one particular soln of $A\vec{x}=\vec{b}$

solutn set to $A\vec{x}=\vec{b}$ is a line parallel to line thru origin but translated by $\vec{p}$

solution set to $A\vec{x}=\vec{0}$ is a line through the origin

Thm: Suppose $A\vec{x}=\vec{b}$ is consistent for some given $\vec{b}$, and $\vec{p}$ is a solution. Then the solutn set of $A\vec{x}=\vec{b}$ is the set of all vectors of the form $\boxed{\vec{w}=\vec{p}+t\vec{v}_h}$ where $\vec{v}_h$ is any solutn to the homogeneous system $A\vec{x}=\vec{0}$.

Intuition: The solutn set of $A\vec{x}=\vec{b}$ is just solutn set of $A\vec{x}=\vec{0}$ translated by any particular solution $\vec{p}$ of $A\vec{x}=\vec{b}$.