1.6 Applications of Linear Systems

Goal: You might expect real life problems to have one solution. These two applications show you that multiple solutions can arise naturally.

Ex1: Balancing Chemical Eqns (#5 from book)

$$B_2S_3 + H_2O \longrightarrow H_3BO_3 + H_2S$$ (unbalanced)

Boron sulfide reacts violently with water to form boric acid and hydrogen sulfide gas. The unbalanced eqn is given above. Balance the chemical eqn.

(need same # of atoms on right as on left side; atoms can be neither destroyed nor created)

Think of molecules as vectors: $\begin{bmatrix} B \\ H \\ O \\ S \end{bmatrix}$

B_2S_3 given by $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}$, H_2O as $\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

H_3BO_3 as $\begin{bmatrix} 1 \\ 3 \\ 3 \\ 0 \end{bmatrix}$ and H_2S as $\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

balanced chemical eqn: (as vector eqn)

$$x_1\begin{bmatrix}2\\0\\0\\3\end{bmatrix}+x_2\begin{bmatrix}0\\2\\1\\0\end{bmatrix}=x_3\begin{bmatrix}1\\3\\3\\0\end{bmatrix}+x_4\begin{bmatrix}0\\2\\0\\1\end{bmatrix}$$

$$x_1\begin{bmatrix}2\\0\\0\\3\end{bmatrix}+x_2\begin{bmatrix}0\\2\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-1\\-3\\-3\\0\end{bmatrix}+x_4\begin{bmatrix}0\\-2\\0\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\Longleftrightarrow \underbrace{\begin{bmatrix}2&0&-1&0\\0&2&-3&-2\\0&1&-3&0\\3&0&0&-1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}}_{\vec{x}}=\underbrace{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}}_{=\vec{0}} \xrightarrow{\text{RREF}} \begin{cases}x_1 = x_4 - x_3\\ x_2 = 2x_4\\ x_3 = \frac{2}{3}x_4\\ x_4 \text{ free}\end{cases}$$

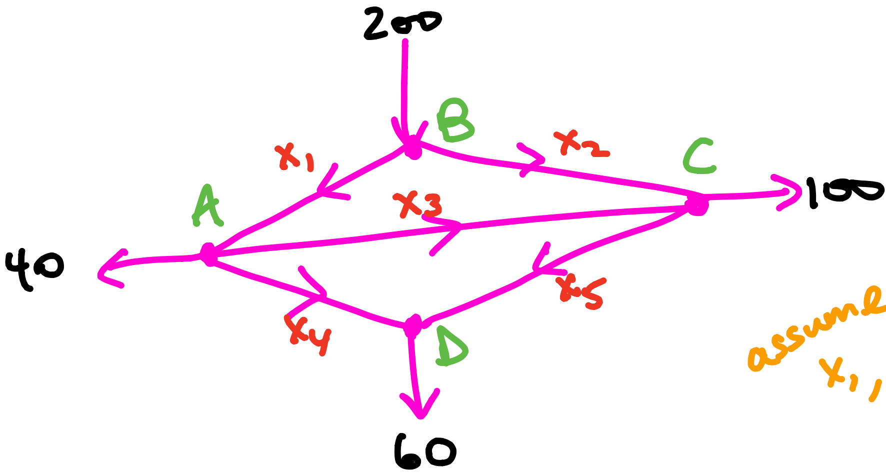
given that we need whole number of atoms, find one solution to balance this chemical eqn.

Ex2: Network Flow

(#12 in book)

Flow of some quantity through a network

(a) Find general traffic pattern in freeway network. (flow rates are in cars/minute.)



key idea:

flow in = flow out

assume $x_1, x_2, x_3, x_4, x_5 \geq 0$

plan: apply "flow in = flow out" at each junction

A: $400 + x_4 = x_1$

B:

C:

D:

total system:

augmented matrix:

(RREF and solve)

1.4

(b) Describe general traffic pattern when the road whose flow is x_4 is closed.

(c) When $x_4 = 0$, what is minimum value of x_1?

1.7 Linear Independence

In this section, we introduce a key idea in linear algebra, namely linear independence. We define the concept, then work through some strategies of determining the linear independence of a set of vectors through computation and through inspection.

Last sections: $A\vec{x} = \vec{0}$ homogeneous eqn

Qn: when does this have non-trivial solns?

This section: Translate to a homogeneous vector eqn $\vec{a}_1 x_1 + \vec{a}_2 x_2 + \dots + \vec{a}_n x_n = \vec{0}$ and answer same qn

Defn: Vectors $\{\vec{v}_1, \dots, \vec{v}_p\} \in \mathbb{R}^n$ are **linearly independent** if the eqn

$$(*) \quad x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n = \vec{0}$$

has only the trivial soln.

- if the eqn $(*)$ has a non-trivial soln, i.e. $\exists\, c_1, c_2, \dots, c_p \in \mathbb{R}$ not all zero s.t. $c_1\vec{v}_1 + \dots + c_p\vec{v}_p = \vec{0}$, then the vectors are **linearly dependent**.

there exists

Ex 1 $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}$ $\quad \vec{v}_3 = \begin{bmatrix} 2 \\ -5 \\ 23 \end{bmatrix}$

(a) Determine if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

(b) Find a linear dependence relation among $\vec{v}_1, \vec{v}_2, \vec{v}_3$ if possible.

plan: determine if $(*)$ has non-trivial soln. If so, find relationship between $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$.

(a) $$x_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -5 \\ 23 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

does this have non-trivial soln?

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & -3 & -5 & 0 \\ 4 & 5 & 23 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

basic variables

free variable

you finish

Note for (b): There are infinitely many dependence relationships.

We could have done this for columns of a matrix. 1.7

Fact: The columns of A are linearly independent IFF the eqn $A\vec{x}=\vec{0}$ has only the trivial soln.

Linear Independence of one or two vectors

- When is a set containing a single vector $\vec{v}$ lin. indep?

What can go wrong? I.e. when does the eqn $x_1\vec{v}_1=\vec{0}$ have a non-trivial soln?

- When is a set containing two vectors lin. indep.?

What could go wrong? I.e. when does the eqn $x_1\vec{v}_1+x_2\vec{v}_2=\vec{0}$ have a non-trivial soln?

Take-away: we can determine by inspection (calculating "in our head") whether sets of 1 or 2 vectors are lin. indep.

Ex 2: Are $\{\vec{v}_1, \vec{v}_2\}$ lin. indep.?

(a) $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 8 \\ 24 \end{bmatrix}$

(b) $\vec{v}_1 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Thm: $\{\vec{v}_1, \dots, \vec{v}_p\}$ is lin. dependent iff at least one $\vec{v}_i$, $1 \le i \le p$, is a lin. combination of the other vectors.

This develops a relationship between linear independence and linear combinations, and span.

Ex 3: Given $\vec{u} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}$

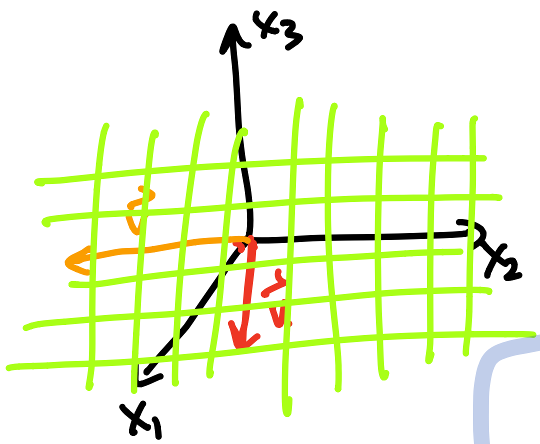
(a) describe span$\{\vec{u}, \vec{v}\}$

(b) Show that $\vec{w} \in$ span$\{\vec{u}, \vec{v}\}$ iff $\{\vec{u}, \vec{v}, \vec{w}\}$ are linearly dependent.

(a) we can tell by inspection that $\vec{u}$ + $\vec{v}$ are lin. indep. Why?

$\Rightarrow$ they span a plane in $\mathbb{R}^3$.

(the $x_1 x_2$ plane)



Fact: Two lin. indep. vectors in $\mathbb{R}^n$ span a plane in $\mathbb{R}^n$ (i.e. a 2d space).

Another Fact: n lin. indep. vectors in $\mathbb{R}^m$ span an n-dim. subspace of $\mathbb{R}^m$.

(b) (our first proof - yay!)

($\Longrightarrow$) Let $\vec{w} \in$ span$\{\vec{u}, \vec{v}\}$. Then we can write $\vec{w}$ as $\vec{w} = c_1\vec{u} + c_2\vec{v}$.

we 1st prove claim going this direction

$\Longleftrightarrow$ $c_1\vec{u} + c_2\vec{v} - \vec{w} = \vec{0}$ has nontrivial soln.

$\iff \{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent. [.]

($\Leftarrow$) Assume $\{\vec{u}, \vec{v}, \vec{w}\}$ are lin. dep.

Then by the previous thm, we know at least one of the vectors is a lin. combo of the other vectors.

<u>Case 1</u>: $\vec{w}$ is lin. combo of $\vec{u}$ + $\vec{v}$.

$\Rightarrow \vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$ #.

this is my personal symbol for "the proof is done."

<u>Case 2</u>: If $\vec{v}$ or $\vec{u}$ is lin. combo of other 2 vectors, then we have

$\vec{v} = c_1\vec{u} + c_2\vec{w}$ (or $\vec{u} = d_1\vec{v} + d_2\vec{w}$)

where c_1 + c_2 are not both 0.

$\Rightarrow \vec{w} = \frac{1}{c_2}\vec{v} - \frac{c_1}{c_2}\vec{u}$ (or $\vec{w} = \frac{1}{d_2}\vec{u} - \frac{d_1}{d_2}\vec{v}$)

what if $c_2 = 0$? Then we have $\vec{v} = c_1\vec{u}$ but we already showed that's not true (in (a)).

$\Rightarrow \vec{w}$ is lin. combo of $\vec{v}$ + $\vec{u}$

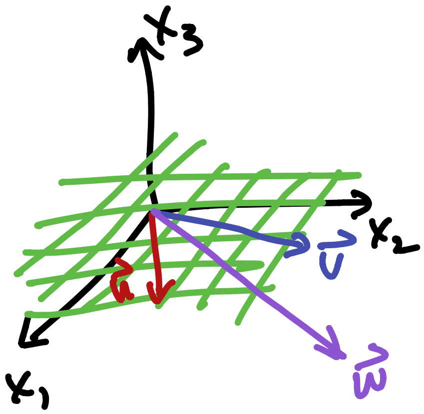
$\Rightarrow \vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$ #

Follow-up: what did we show?

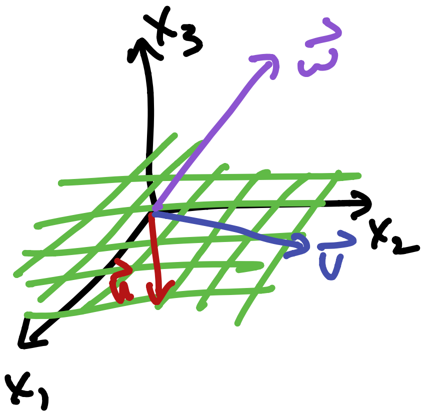
$$\boxed{\vec{w} \in \operatorname{span}\{\vec{u}, \vec{v}\} \iff \{\vec{u}, \vec{v}, \vec{w}\} \text{ lin. dep.}}$$

relationship between span and linear dependence

$\Rightarrow$ allows us to visualize lin. independence.



if $\{\vec{u}, \vec{v}, \vec{w}\}$ lin. dep., $\vec{w}$ is in the plane spanned by $\vec{u}$ and $\vec{v}$.



if $\{\vec{u}, \vec{v}, \vec{w}\}$ lin. indep. $\vec{w}$ is not on the plane spanned by $\vec{u}$ and $\vec{v}$.

Sometimes linear dependence is automatic:

Thm: If a set contains more vectors than entries in each vector, then the set is linearly dependent.

Ex4: Are $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

lin. indep? i.e. does $x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

have non-trivial soln?

$$\begin{bmatrix} 2 & 5 & 3 & 0 \\ 3 & -1 & 6 & 0 \end{bmatrix}$$

Thm: If a set $\{\vec{v}_1, \dots, \vec{v}_p\}$ contains the zero vector, then it is linearly dependent set.

Why? If $\vec{v}_i = \vec{0}$, then the eqn

$$0\vec{v}_1 + 0\vec{v}_2 + \dots + 0\vec{v}_{i-1} + 1\vec{v}_i + 0\vec{v}_{i+1} + \dots + 0\vec{v}_p = \vec{0}$$

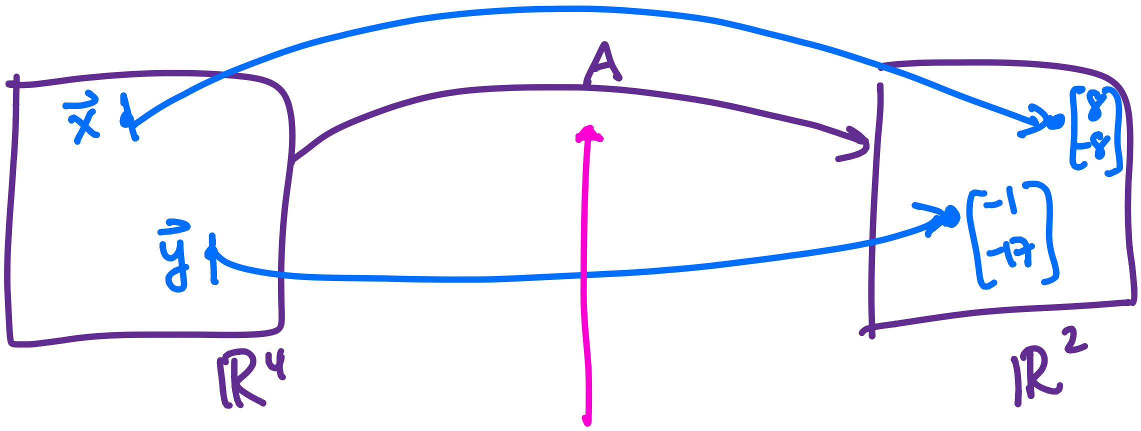
is true (i.e. there's a nontrivial soln).

$\Rightarrow$ they are lin. dep.

1.8 Introduction to Linear Transformations

Important Fact: Matrices are Functions!!!!

Ex 1 $A = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 0 & -5 & -2 & 1 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix}$



A takes vectors in $\mathbb{R}^4$ to vectors in $\mathbb{R}^2$

$$A\vec{x} = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 0 & -5 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -8 \end{bmatrix} \qquad A\vec{y} = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 0 & -5 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -17 \end{bmatrix}$$

Function Notation:

domain

codomain

function/mapping/transformation

$$A : \mathbb{R}^n \to \mathbb{R}^m$$

$$\vec{x} \mapsto A\vec{x}$$

the image of $\vec{x}$

an element of the domain

Note: The set of all images $A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$ is the range.

Notice: The domain and codomain tell us size of A. A is $m \times n$

[range tells us # rows
domain " " " columns]

Ex 2: $A = \begin{bmatrix} 2 & 0 \\ -2 & 5 \\ 1 & 3 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 10 \\ -6 \end{bmatrix}$ $\vec{c} = \begin{bmatrix} -5 \\ 10 \\ -6 \end{bmatrix}$

$A: \mathbb{R}^2 \to \mathbb{R}^3$ is a transformation

$\vec{x} \longmapsto A\vec{x}$

(a) Find the image of $\vec{u}$ under A (i.e. find $A\vec{u}$).

(b) Find an $\vec{x} \in \mathbb{R}^2$ whose image under A is $\vec{b}$, i.e. solve $A\vec{x} = \vec{b}$. Does more than one $\vec{x} \in \mathbb{R}^2$ get mapped to $\vec{b}$?

(c) Is $\vec{c}$ in the range of A?

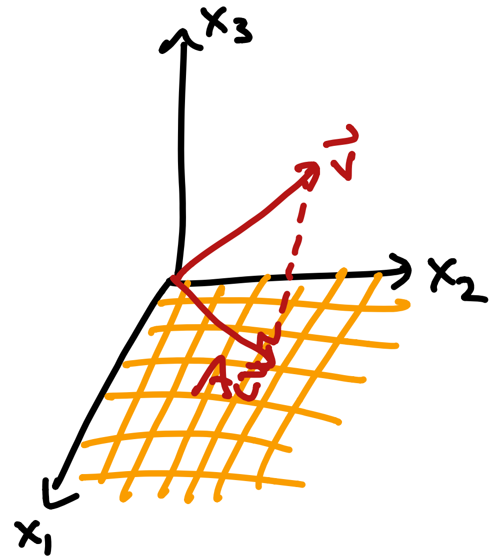
(a)

What do matrix transformations look like?

Ex A: $A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ $A:\mathbb{R}^3\to\mathbb{R}^3$, $\vec{x}\mapsto A\vec{x}$

$$A\vec{x}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\0\end{bmatrix}$$

this A projects vectors (or pts) in $\mathbb{R}^3$ onto the x_1x_2 plane.

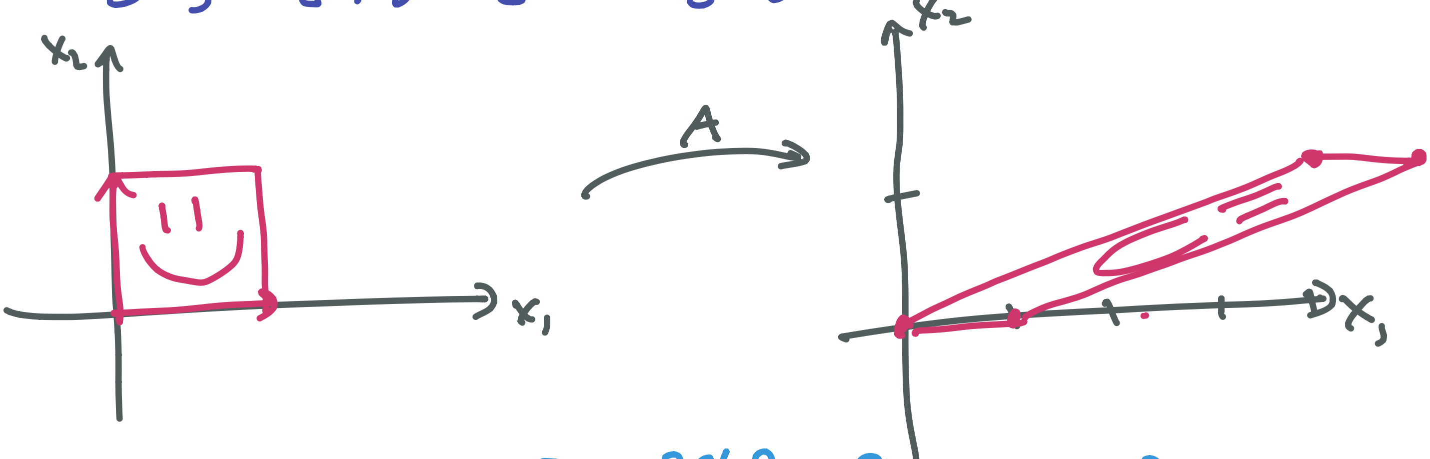


Ex B: $A=\begin{bmatrix}1&4\\0&1\end{bmatrix}$ $A:\mathbb{R}^2\to\mathbb{R}^2$, $\vec{x}\mapsto A\vec{x}$

This is a shear transformation.

To visualize its effect, see what it does to

$\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}$ This will tell us what A does to unit square.



$$\begin{pmatrix}1&4\\0&1\end{pmatrix}\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\quad \begin{bmatrix}1&4\\0&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix},\quad \begin{bmatrix}1&4\\0&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}5\\1\end{bmatrix}$$

etc.

Big Idea: Matrices are linear Transformations.

Defn A transformation T is linear if

(i) $T(u+v) = T(u) + T(v)$ $\forall\, u,v \in$ domain of T

and (ii) $T(cu) = c\,T(u)$ for $c \in \mathbb{R}$, $u \in$ domain of T.

Can we prove matrices fit these criteria?
(i.e. multiplying vectors by matrix A will satisfy these 2 conditions)

Note: If T is linear transformation, then

(a) $T(\vec{0}) = \vec{0}$

(b) $T(c\vec{u} + d\vec{v}) = c\,T(\vec{u}) + d\,T(\vec{v})$

(c) $T(c_1\vec{v}_1 + \dots + c_p\vec{v}_p) = c_1 T(\vec{v}_1) + \dots + c_p T(\vec{v}_p)$.

1.9 The matrix of a linear Transformation

Last Time: matrix transformations are linear.

This Time: linear transformations are matrices.

Ex1: $T: \mathbb{R}^2 \to \mathbb{R}^3$ is a lin. transformation where

s.t. $T(\vec{e}_1) = \begin{bmatrix} 2 \\ -3 \\ -6 \end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}$

$$\underbrace{\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

These are the columns of the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. They are also the unit $\vec{x}$ and $\vec{y}$ vectors in $\mathbb{R}^2$.

Goal: represent T as a matrix.

First, let's observe that

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2$$

for any $\vec{x} \in \mathbb{R}^2$.

Second, since T is linear $T(\vec{x}) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$

$$= x_1 \begin{bmatrix} 2 \\ -3 \\ -6 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ -3x_1 \\ -6x_1 + 8x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ -3 & 0 \\ -6 & 8 \end{bmatrix}}_{\text{A matrix!!!!}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow T(\vec{x}) = A\vec{x}$ where

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 0 \\ -6 & 8 \end{bmatrix} = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$$

Good news: We can ALWAYS do this!

Thm: let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

Then $\exists!$ matrix A s.t.

there exists a unique

$$T(\vec{x}) = A\vec{x} \quad \forall \ \vec{x} \in \mathbb{R}^n$$

for all

A is the matrix $A = \begin{bmatrix} T(\vec{e}_1) & \cdots & T(\vec{e}_n) \end{bmatrix}$

where $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th}$ location $\quad i \in \{1, 2, \ldots, n\}$.

A is called the standard matrix for linear transformation T.

Ex 2: Find standard matrix for the dilation transformation

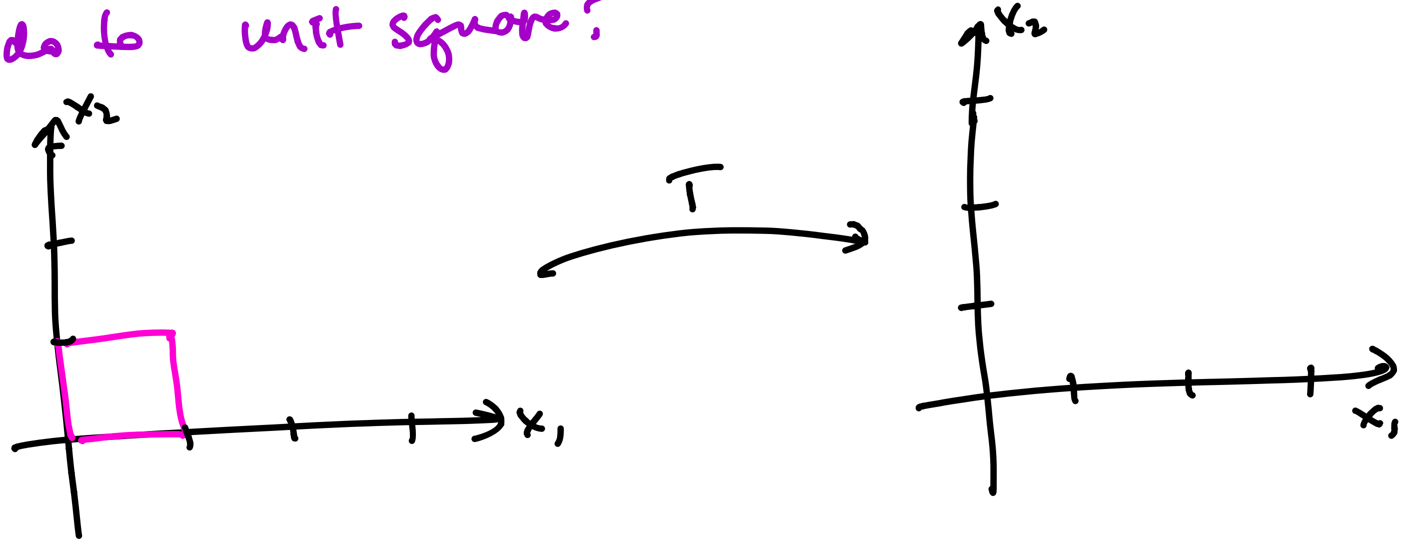
$$T: \mathbb{R}^2 \to \mathbb{R}^2$$
$$x \longmapsto 3\vec{x}$$

See how T acts on $\vec{e}_1$ and $\vec{e}_2$.

$$T(\vec{e}_1) = 3\vec{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad T(\vec{e}_2) = 3\vec{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

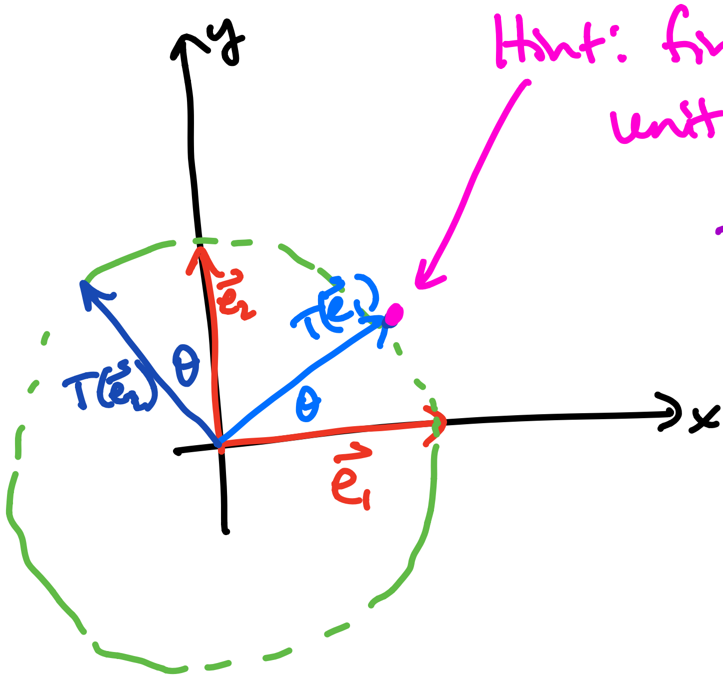
So T acts by $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 3I$

What does this look like, i.e. what does it do to unit square?



Ex 3: $T: \mathbb{R}^2 \to \mathbb{R}^2$ rotates each pt in $\mathbb{R}^2$ about the origin by an angle θ. (θ assume is positive).
Find the standard matrix for T.

Hint: find coords of this pt on unit circle.



$T(\vec{e}_1) = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$

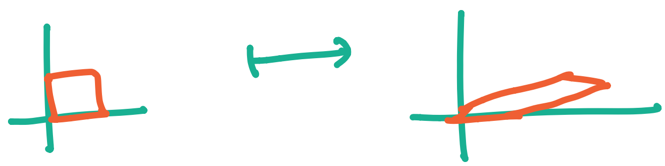
$T(\vec{e}_2) = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$

$\Rightarrow T$ acts by matrix

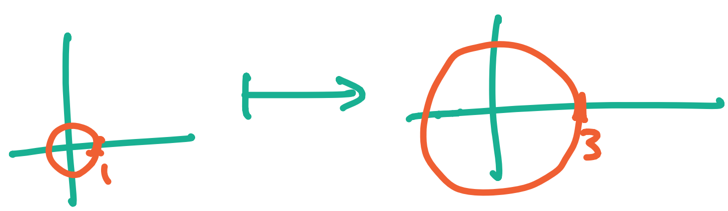
We've seen how several matrices act geometrically.
Let's review.

① $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a projection of pts in $\mathbb{R}^3$ onto x_1x_2 plane

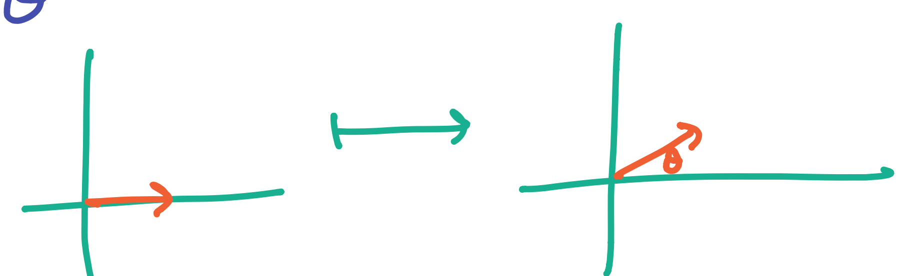
② $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ is a shear transformation



③ $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is a dilation (stretch)



④ $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation (counter-clockwise) by θ



<u>Note:</u> See pages 74-76 of text to see more cool pictures of transformations.

Defn: A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto (surjective) if each $\vec{b} \in \mathbb{R}^m$ is in the image of at least one $\vec{x} \in \mathbb{R}^n$.

Defn: A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one (1-1), injective, if each $\vec{b} \in \mathbb{R}^m$ is the image of at most one $\vec{x} \in \mathbb{R}^n$.

Qn: Matrices are mappings. So when is a matrix transformation 1-1?

Let's look at an example to explore this qn.

Ex 4: let $T: \mathbb{R}^4 \to \mathbb{R}^3$, $\vec{x} \mapsto A\vec{x}$ for $A = \begin{bmatrix} 1 & 3 & -2 & 5 \\ 0 & 4 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(a) Is T onto?

(b) Is T 1-1?

(a) A has a pivot position in each row.

(b) A has a free variable.

Thm: Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with standard matrix A. Then

(i) T is 1-1 $\iff A\vec{x}=\vec{0}$ has only the trivial soln
$\iff$ the columns of A are lin. indep.

(ii) T is onto $\iff$ the columns of A span $\mathbb{R}^m$.

Can we apply this thm to our last example?