We've been working w/ matrices, and we'll work more with them in future.

Section 2.1 Matrix Operations

2.1

ELOs:

- Be able to perform matrix operations (addition and multiplication).
- Be able to describe ways in which matrix properties are similar and different to those of real numbers.
- Be able to find the transpose of a matrix.

This section goes over some algebraic operations w/ matrices.

Matrix Notation:

Recall that we denote an $m \times n$ matrix A where m represents the number of rows and n represents the number of columns in the following ways:

$$A = \begin{bmatrix} \underset{\substack{\uparrow \\ \text{column vector } \vec{a}_1}}{\mathbf{a}_1} & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} \xleftarrow{\text{diagonal entries}} = [a_{ij}].$$

The i^{th} scalar entry of the j^{th} column is denoted a_{ij}. The *main diagonal entries* are $a_{11}, a_{22}, a_{33}, \dots$.

Definition 1 *A diagonal matrix is a square $n \times n$ matrix whose nondiagonal entries are zero.*

Note: An example of a diagonal matrix is the identity matrix, I_n. (must be square)

Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

identity matrices

$$\left\{ \begin{array}{l} I_1 = [1] \\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \right. \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

etc.

Definition 2 *A zero matrix is an $m \times n$ matrix whose entries are all zero.*

Note: A zero matrix is denoted as 0. ★ can be any size at all

Example:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[0]$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2.1

Sums and Scalar Multiples of Matrices

- Two matrices, A and B, are said to be equal if they are the same size, $m \times n$, with the same corresponding entries in each column. That is, $a_{ij} = b_{ij}$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.

Example: $A = B$ if $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

and $B = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

- Two matrices, A and B, of the same size may be added together entrywise. That is, $a_{ij} + b_{ij}$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.

Example: $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 & 10 \\ 9 & -8 & 5 \end{bmatrix}$

Qn: $A + B =$

- A matrix A may be multiplied by a scalar r entrywise. That is, $rA = r\,[a_{ij}]$.

Example: $A = \begin{bmatrix} 1 & 5 & 9 \\ 2 & -4 & -7 \end{bmatrix}$ $r = 5$

Qn: $rA = 5A =$

Theorem 1 Properties of Matrix Addition and Scalar Multiplication *Let A, B, and C be matrices of the same size, $m \times n$, and let r and s be scalars.*

(a) $A + B =$ $B+A$ (additive commutativity) (d) $r(A + B) =$ $rA+rB$ → (distributivity)

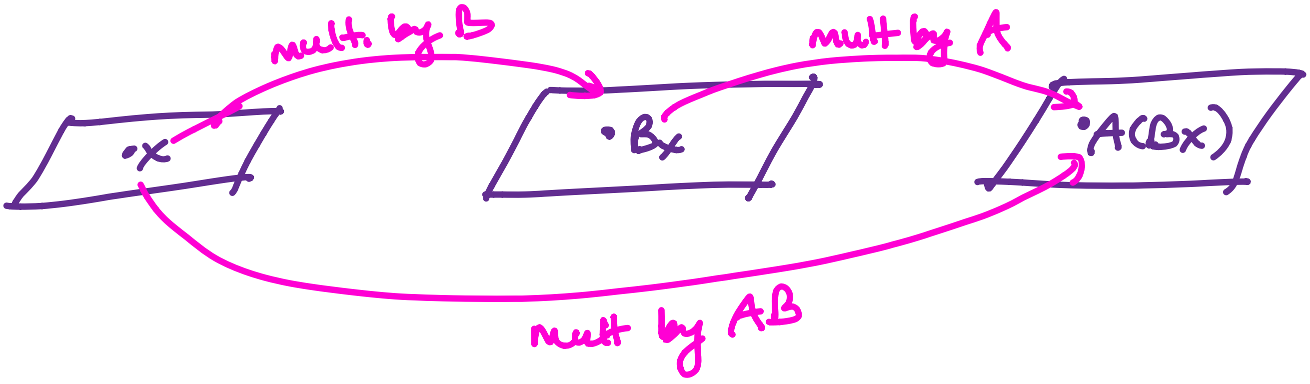
(b) $(A + B) + C =$ $A+(B+C)$ (additive associativity) (e) $(r + s)A =$ $rA+sA$

(c) $A + 0 =$ A (additive identity is 0 matrix of right size) (f) $r(sA) =$ $(rs)A$ associativity

2.1

Matrix Multiplication as Function Composition

Key Idea: Multiplication of matrices corresponds to composition of linear transformations!



ex • B rotates $\vec{x}$ by 45°

• A stretches its input by 5

⇒ $AB\vec{x}$ first rotates $\vec{x}$ by 45° + then stretches that rotated vector by factor of 5.

composit'n of functions is like "layering" of transformations.

Definition 3 *If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$, then the product of AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_p$. That is,*

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_p \end{bmatrix}$$

Thus, $A(B\mathbf{x}) = (AB)\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^p$.

Note: *Each column of AB is a linear combination of the columns of A where the weights are the entries of the corresponding column of B.*

Example: let's practice matrix multiplication!

$$A = \underset{3\times 2}{\begin{bmatrix} 1 & 3 \\ -4 & 5 \\ 0 & 2 \end{bmatrix}} \qquad B = \underset{2\times 4}{\begin{bmatrix} 1 & 2 & 0 & -4 \\ -2 & 3 & 1 & 5 \end{bmatrix}}$$

AB will be $(3\times 2)(2\times 4)$ 3×4 matrix

Qn:

$$\begin{bmatrix} 1 & 3 \\ -4 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & -4 \\ -2 & 3 & 1 & 5 \end{bmatrix}$$

2.1

Row-Column Rule for Computing AB

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. That is,

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

where A is an $m \times n$ matrix. **Note:** $row_i(AB) = row_i(A) \cdot B$

$(2\times2)(2\times3)$

2×3 resulting matrix

Example: $A = \begin{bmatrix} 0 & -4 \\ 9 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 3 & -4 \end{bmatrix}$

$(AB)_{12} = \begin{bmatrix} 0 & -4 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = 0(-1)+-4(3) = -12$

$(AB)_{23} = \begin{bmatrix} 9 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \end{bmatrix} = 18+-8 = 10$

Example: You practice.

$\begin{bmatrix} 1 & -2 \\ 3 & 5 \\ 0 & -6 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ -2 & 1 \end{bmatrix}$

Theorem 2 Properties of Matrix Multiplication *Let A, B, and C be matrices and r be a scalar such that the sums and products below are defined. Then,*

(a) $A(BC) = (AB)C$ associativity of mult.

(b) $A(B + C) = AB+AC$ distributivity

(c) $(B + C)A = BA+CA$

(d) $r(AB) = (rA)B = A(rB) = (Ar)B$ associativity + commutativity of scalar mult.

(e) $I_mA = A$ mult. identity

2.1

WARNINGS:

- Matrix multiplication is NOT commutative. That is, in general, $\boxed{AB \neq BA.}$
- Cancellation laws do NOT hold. That is, $AB = AC$ does NOT imply $B = C$
- Matrices CAN have zero divisors. That is, $AB = 0$ does NOT imply $A = 0$ or $B = 0$.
- We CAN take powers of matrices as long as they are $n \times n$. That is, $A^k = \underbrace{A \cdots A}_{\text{k times}}$ and $A^0 = I_n$.

Example:

$$AB = \begin{bmatrix} 2 & 3 \\ -1 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 & -1 & 6 \\ -2 & \frac{2}{3} & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

notice neither A nor B =0

Example: $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 36 & 50 \end{bmatrix}$

Qn: $BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} =$

2.1

Transpose of a Matrix, A^T

Definition 4 *Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted A^T, whose columns are formed from the corresponding rows of A.*

Example: Give the transpose of each of the following matrices:

Qn:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{pmatrix}$$

Theorem 3 Transpose Properties *Let A and B be matrices whose sizes are appropriate for the following sums and products.*

(a) $(A^T)^T = A$

(b) $(A+B)^T = A^T + B^T$

(c) For any scalar r, $(rA)^T = rA^T$

(d) $(AB)^T = B^T A^T$

Note: *The transpose of a product of matrices equals the product of their transposes in reverse order.*

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -5 \\ 3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -6 & 1 & 2 \\ 4 & 3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 7 & 2 \\ -9 & -15 & 0 \\ -22 & 0 & 6 \end{bmatrix} \Rightarrow (AB)^T = \begin{bmatrix} 2 & -9 & -22 \\ 7 & -15 & 0 \\ 2 & 0 & 6 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} -6 & 4 \\ 1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -9 & -22 \\ 7 & -15 & 0 \\ 2 & 0 & 6 \end{bmatrix}$$ ✓

Matrices are fns. Some fns have inverses ⇒ 2.2

Section 2.2 The Inverse of a Matrix

Some matrices have inverses!!!

ELOs:

- Be able to explain (at least) three ways to identify whether or not a matrix is invertible.
- Be able to use the inverse of a 2×2 matrix to solve a system of equations.
- Be able to use algorithm to find matrix inverses.

Key Idea: A matrix transformation A has an *inverse*, denoted A^{-1}, when A is one-to-one and onto.

Suppose A is an $m \times n$ matrix, such that $A : \mathbb{R}^n \to \mathbb{R}^m$. A is $m \times n$ matrix

(a) Will the matrix A have an inverse if $n > m$? Why or why not?

maybe? A has n columns ⇒ to span $\mathbb{R}^m$, we must have $m \leq n$.

(b) Will the matrix A have an inverse if $n < m$? Why or why not?

but if we have n lin. indep. columns then span of $\mathbb{R}^m = \mathbb{R}^n$

(c) What about $n = m$?

⇒ $n=m$ must be true for A^{-1} to exist.

(d) Based on the previous observations, what must be true about the size of A in order for A^{-1} to exist?

(e) Based on the previous observations, what does RREF(A) look like?

Definition 1 *A square, $n \times n$, matrix is said to be <u>invertible</u> (or nonsingular) if there exists an $n \times n$ matrix, denoted A^{-1}, such that*

$$A^{-1}A = I_n \quad \textit{and} \quad I_n = AA^{-1}$$

.

Note: A^{-1} is called the inverse of A. A matrix that is not invertible is said to be singular.

notice fn analog: $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ if $f^{-1}(x)$ exists.

Example: $A = \begin{bmatrix} 2 & 5 \\ -6 & 6 \end{bmatrix}$ check if $B = \begin{bmatrix} 6 & -5 \\ 6 & 2 \end{bmatrix}$ is A^{-1}.

Qn:

Theorem 4 **2×2 Inverses** *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If* $ad - bc \neq 0$, *then* A *is invertible, and*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, *then* A *is not invertible (is singular).*

The quantity $ad - bc$ *is called the* **determinant** *of* A.

Thus, a 2×2 *matrix is invertible if and only if* $\det A \neq 0$.

Example: Find A^{-1} if $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$.

Qn: 1(a)

Qn:

1(b) Find A^{-1} if $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$.

2.2

Theorem 5 Matrix Equation Solutions and Inverses *If A is an invertible $n \times n$ matrix, then for each $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution, $\mathbf{x} = A^{-1}\mathbf{b}$.*

Example:

Use Thm5 to solve.

$x_1 + 3x_2 = -7$

$-4x_1 - 5x_2 = 7$

Yay! This gives another way to solve a system of linear eqns.

Theorem 6 Properties of Inverses

(a) If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = \_\_\_\_\_\_\_\_\_\_$$

(b) If A and B are $n \times n$ invertible matrices, then AB is also invertible. The inverse of AB is the product of the inverses of A and B in reverse order. That is,

$$(AB)^{-1} = \_\_\_\_\_\_\_\_\_\_$$

(c) If A is an invertible matrix, then A^T is also invertible. The inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = \_\_\_\_\_\_\_\_\_\_$$

Extension: The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of the individual inverses in reverse order.

We can define elementary matrix as a matrix we get from performing one ERO on I_n.

ex $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

these are all "elementary matrices".

$\Rightarrow$ we can perform row ops by multiplying by several elementary matrices.

Facts: 1) Elem. matrices are invertible. 2) E^{-1} is row op that xforms E back to I.

2.2

Theorem 7 *An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n to A^{-1}.*

This gives us the following algorithm.

Algorithm for Finding A^{-1}

1) Row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$.

2) If A is row equivalent to I, then $\begin{bmatrix} A & I \end{bmatrix}$ is equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

3) Otherwise, A does not have an inverse.

Note: "xform" is short for "transform"

Example:

Find A^{-1} for $A = \begin{bmatrix} 13 & 2 & 3 \\ 6 & -1 & 5 \\ -2 & 0 & -1 \end{bmatrix}$

Matrix Inversion as Simultaneously Solving n Linear Systems

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{array} \right\} \equiv A\vec{x} = \vec{b}$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

Today! Relating everything we've learned so far!

2.3

Section 2.3 Characterizations of Invertible Matrices

ELOs:

- Be able to relate the equivalent statements we learned in Chapter 1 to the Invertible Matrix Theorem.
- Be able to apply the Invertible Matrix Theorem to determine whether or not a matrix is invertible.

important point.

Theorem 8 The Invertible Matrix Theorem *Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.*

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ identity *matrix.*

(c) A has n pivot *postions.*

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial *solution.*

(e) The columns of A form a linearly indep. *set.*

(f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

(g) The equation $A\mathbf{x} = \mathbf{b}$ has at least one *solution for each $\mathbf{b}$ in $\mathbb{R}^n$.*

(h) The columns of A span *$\mathbb{R}^n$.*

(i) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto *$\mathbb{R}^n$.*

(j) There is an $n \times n$ matrix C such that $CA =$ I.

(k) There is an $n \times n$ matrix D such that $AD =$ I

(l) A^T is an invertible *matrix.*

Qu: ~~Ex~~ Is $A = \begin{bmatrix} 3 & 0 & 4 & 5 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 9 \end{bmatrix}$ invertible?

2.3

Invertible Linear Transformations

Definition 1 *A linear transformation* $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is said to be invertible if there exists a function* $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *such that*

$$S(T(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

$$T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

where $S = T^{-1}$ *is the inverse of* T.

Theorem 9 *Let* $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a linear transformation and let* A *be the standard matrix for* T. *Then,* T *is invertible if and only if* A *is an invertible matrix, and the linear transformation* S *given by* $S(\mathbf{x}) = A^{-1}\mathbf{x}$ *is the unique function such that*

$$A^{-1}(A\mathbf{x}) = S(T(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

$$A(A^{-1}\mathbf{x}) = T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

where $S = T^{-1}$ *is the inverse of* T.

Proof:

($\Rightarrow$) Assume T^{-1} exists. Then $T(T^{-1}(\vec{x})) = T(T^{-1}(\vec{x})) = \vec{x}$ $\forall$ $x \in \mathbb{R}^n$, i.e. T is onto $\mathbb{R}^n$ (because if $\vec{b}$ is in $\mathbb{R}^n$ and $\vec{x} = T^{-1}(\vec{b})$ then $T(\vec{x}) = T(T^{-1}(\vec{b})) = \vec{b}$ so each $\vec{b}$ is in range of T). Thus, A^{-1} exists.

($\Leftarrow$) Assume A^{-1} exists. Let $S(\vec{x}) = A^{-1}\vec{x}$. Then S is a lin. operator and $S(T(\vec{x})) = S(A\vec{x}) = A^{-1}(A\vec{x}) = \vec{x} \Rightarrow T^{-1}$ exist. #

Sometimes it's useful to treat a matrix in "blocks". It might show up naturally in applications and also might reduce computing power necessary for large matrix

2.4

Section 2.4 Partitioned Matrices

ELOs:

- Be able to multiply partitioned (block) matrices.

Example:

We can express the matrix

$$A = \begin{bmatrix} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ -8 & -6 & 3 & 1 & 7 & -4 \end{bmatrix}$$

as the 2×3 *partitioned* (or *block*) matrix

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

whose entries are submatrices (or "blocks").

- We can multiply partitioned matrices the same way as non-partitioned matrices, provided the product AB makes sense for each block A and B. Find the product AB where

i.e. they're partitioned in "right" sizes

$$A = \begin{bmatrix} 2 & -1 & 1 & 0 & 4 \\ 1 & 5 & -2 & 3 & -1 \\ 0 & -4 & -2 & 7 & -1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

$$A_{11} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 5 & -2 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 0 & 4 \\ 3 & -1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0 & -4 & -2 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 7 & -1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{bmatrix} \quad B_2 = \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix}$$

2.4

- A matrix of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

is called *block upper triangular*. Assuming A is invertible, A_{11} is a $p \times p$ matrix, and A_{22} is a $q \times q$ matrix. Find a formula for A^{-1}. That is, find a matrix B such that $AB = I_{p+q}$.

$$AB = I_{p+q} \iff \begin{array}{cc} & \begin{matrix} p & q \end{matrix} \\ \begin{matrix} p \\ q \end{matrix} & \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \end{array} \begin{array}{cc} & \begin{matrix} p & q \end{matrix} \\ \begin{matrix} p \\ q \end{matrix} & \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \end{array} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}$$

$\Rightarrow$ ① $A_{11}B_{11} + A_{12}B_{21} = I_p$

② $A_{22}B_{21} = 0$

$\Rightarrow A_{22}^{-1}A_{22}B_{21} = A_{22}^{-1}0$

$$\boxed{B_{21} = 0}$$

③ $A_{11}B_{12} + A_{12}B_{22} = 0$

④ $A_{22}B_{22} = I_q$

$\Rightarrow A_{22}^{-1}A_{22}B_{22} = A_{22}^{-1}I_q$

$$\boxed{B_{22} = A_{22}^{-1}}$$

finish:

$\Rightarrow A^{-1} = \begin{bmatrix} \quad \end{bmatrix}$

2.5

Here we'll focus on one type of matrix factorization. (i.e. write matrix A in some factored form)

Section 2.5 Matrix Factorizations

ELOs:

- Be able to identify why an LU factorization of a matrix is useful.
- Be able to compute the LU factorization of a matrix A.

$A=LU$

LU Factorization

Suppose we want to solve the set of equations

$$
\begin{aligned}
A\mathbf{x} &= \mathbf{b_1}\\
A\mathbf{x} &= \mathbf{b_2}\\
&\vdots\\
A\mathbf{x} &= \mathbf{b_p}
\end{aligned}
$$

each of these is a linear system of eqns.

where A is the same matrix in each equation.

Qn:
- If A is invertible, what is the (unique) solution to each equation?

- What if A is not invertible?

  Idea: If we could rewrite A in a clever way such that $A = LU$ where L is a *lower triangular* matrix and U is an *upper triangular* matrix, this would reduce the number of steps needed to solve the problem. If we assume that an $m \times n$ matrix A can be reduced to REF without row exchanges, then we can perform this factorization $A = LU$. Here's an example:

$$
A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0\\ * & 1 & 0 & 0\\ * & * & 1 & 0\\ * & * & * & 1 \end{bmatrix}}_{\text{L}(m\times m)} \underbrace{\begin{bmatrix} * & * & * & * & *\\ 0 & * & * & * & *\\ 0 & 0 & * & * & *\\ 0 & 0 & 0 & * & * \end{bmatrix}}_{\text{U}(m\times n)}
$$

LU factorization frequently used on square matrices

Example:

$$
A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ 5 & 7 & 2 & 6 & 9\\ 1 & -3 & 8 & 0 & -7\\ 0 & 6 & -8 & 9 & 13 \end{bmatrix}
$$

$$
= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0\\ 5 & 1 & 0 & 0\\ 1 & -2 & 1 & 0\\ 0 & 3 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ 0 & 2 & -3 & 1 & 4\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 5 & 1 \end{bmatrix}}_{U} = LU
$$

Why is this helpful?

It is more computationally efficient!

Ex: $A = \begin{bmatrix} 1 & -2 & -2 & -3 \\ 3 & -9 & 0 & -9 \\ -1 & 2 & 4 & 7 \\ -3 & -6 & 26 & 2 \end{bmatrix}$

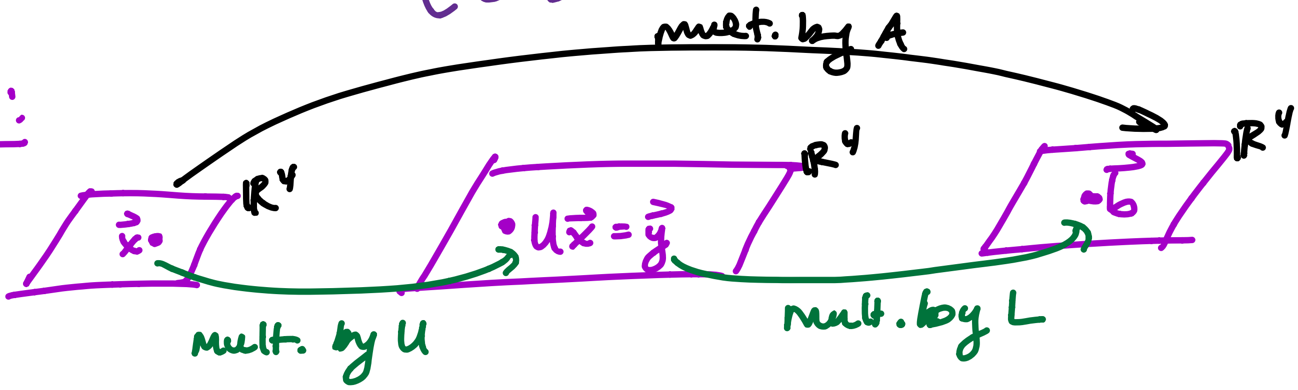
$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 4 & -2 & 1 \end{bmatrix}$ $\quad U = \begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$A = LU$ (check this for yourself)

Use this factorization to solve $A\vec{x} = \vec{b}$, for

$$\vec{b} = \begin{bmatrix} 12 \\ 0 \\ -6 \\ 5 \end{bmatrix}$$

Soln:



i.e. $A\vec{x} = \vec{b} \iff L(U\vec{x}) = \vec{b}$

$\iff L\vec{y} = \vec{b}$ } Solve 2 eqns.

① $L\vec{y} = \vec{b}$

② $U\vec{x} = \vec{y}$

2.5

$$[L\ \ \vec{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 12 \\ 3 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -6 \\ -3 & 4 & -2 & 1 & 5 \end{bmatrix}$$

augmented matrix for ① $L\vec{y}=\vec{b}$

$$\overset{RREF}{\equiv} \begin{bmatrix} 1 & 0 & 0 & 0 & 12 \\ 0 & 1 & 0 & 0 & -36 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 197 \end{bmatrix} = [I\ \ \vec{y}] \Rightarrow \vec{y} = \begin{bmatrix} 12 \\ -36 \\ 6 \\ 197 \end{bmatrix}$$

how many steps? 6 mult.
6 add.

$$[U\ \ \vec{y}] = \begin{bmatrix} 1 & -2 & -2 & -3 & 12 \\ 0 & -3 & 6 & 0 & -36 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 197 \end{bmatrix}$$

augmented matrix for $U\vec{x}=\vec{y}$

$$\overset{RREF}{\equiv} \begin{bmatrix} 1 & 0 & 0 & 0 & -1719 \\ 0 & 1 & 0 & 0 & -770 \\ 0 & 0 & 1 & 0 & -391 \\ 0 & 0 & 0 & 1 & 197 \end{bmatrix} = [I\ \ \vec{x}]$$

how many steps? 6 mult.
6 add.
4 div.

$$\Rightarrow \vec{x} = \begin{bmatrix} -1719 \\ -770 \\ -391 \\ 197 \end{bmatrix}$$

- Finding $\vec{x}$ took 28 operations (FLOPs)
- Row reduction of $[A\ \vec{b}]$ to $[I\ \vec{x}]$ takes 62 FLOPs

$\Rightarrow$ Using $A=LU$ is more efficient!

2.5

LU Factorization Algorithm

1. Reduce A to row echelon form, REF(A)=U, using only row replacement operations. *Note:* this is not always possible.
2. Place entries in L such that the same sequence of row operations reduces L to I.

$E_p \cdots E_1 A = U$

these are lower Δ $\Rightarrow$ their product is also lower Δ.

$$\Rightarrow A = (E_p \cdots E_1)^{-1} U = LU$$

$$\Rightarrow L = (E_p \cdots E_1)^{-1}$$

i.e. $A \xmapsto{E_p \cdots E_1} U$

$L \xmapsto{E_p \cdots E_1} I$

Example: Find an LU factorization of

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

(3) (2) (-1)

$$[A \mid I] = \left[\begin{array}{ccccc|cccc} 2 & 4 & -1 & 5 & -2 & 1 & 0 & 0 & 0 \\ -4 & -5 & 3 & -8 & 1 & 0 & 1 & 0 & 0 \\ 2 & -5 & -4 & 1 & 8 & 0 & 0 & 1 & 0 \\ -6 & 0 & 7 & -3 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

(-4) (3)

$$= \left[\begin{array}{ccccc|cccc} 2 & 4 & -1 & 5 & -2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 2 & -3 & 2 & 1 & 0 & 0 \\ 0 & -9 & -3 & -4 & 10 & -1 & 0 & 1 & 0 \\ 0 & 12 & 4 & 12 & -5 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccccc|cccc} 2 & 4 & -1 & 5 & -2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 2 & -3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 5 & 3 & 1 & 0 \\ \underbrace{0 \quad 0 \quad 0 \quad 4 \quad 7}_{U} & & & & & \underbrace{-5 \quad -4 \quad 0 \quad 1}_{L^{-1}} & & & \end{array}\right]$$

Find L:

(5) (-5) (-2)

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 5 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ -5 & -4 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

(4) (-3)

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & -5 & 0 & 1 & 0 \\ 0 & -4 & 0 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{matrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{matrix}\;\middle|\;\underbrace{\begin{matrix} 1&0&0&0\\ -2&1&0&0\\ 1&-3&1&0\\ -3&4&0&1 \end{matrix}}_{L}\right]$$

<u>check</u>

$$\underbrace{\begin{bmatrix} 1&0&0&0\\ -2&1&0&0\\ 1&-3&1&0\\ -3&4&0&1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 2&4&-1&5&-2\\ 0&3&1&2&-3\\ 0&0&0&2&1\\ 0&0&0&4&7 \end{bmatrix}}_{u}$$

$$= \begin{bmatrix} 2&4&-1&5&-2\\ -4&-5&3&-8&1\\ 2&-5&-4&1&8\\ -6&0&7&-3&1 \end{bmatrix} = A$$

✓