3.1

Section 3.1 Introduction to Determinants

ELOs:

- Be able to find the determinant of a matrix recursively.
- Be able to find the determinant of a matrix using cofactor expansion.
- Be able to explain why the determinant of a triangular matrix is the product of the diagonal entries.

Introduction: This entire chapter is devoted to the study of determinants. First we'll compute determinants to give a number associated w/ any square matrix. Then we'll see what that number can tell us!!!

must be a square matrix

Definition 1 *For $n \geq 2$, the* <u>*determinant*</u> *of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} det(A_{1j})$ with alternating signs and where the entires $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. That is,*

$$\begin{aligned} det(A) &= a_{11} det(A_{11}) - a_{12} det(A_{12}) + \cdots + (-1)^{1+n} a_{1n} det(A_{1n}) \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} det(A_{1j}), \end{aligned}$$

where A_{ij} is the submatrix of A obtained by deleting the i^{th} row and j^{th} column.

The quantity $(-1)^{i+j} det(A_{ij}) = C_{ij}$ is called the **(i, j)-cofactor** *of A.*

Notation: $|A| = det(A)$.

Example: Compute determinant of A.
(expand on 1st row)

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & 1 \\ 5 & -3 & 7 \end{bmatrix}$$

Theorem 1 Cofactor Expansion *The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column.*

Cofactor expansion across the i^{th} **row** *is given by:*

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

Cofactor expansion across the j^{th} **column** *is given by:*

$$det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Checkerboard Pattern of Alternating Signs:

$$n\left\{\overbrace{\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n+1} & \cdots & & & (-1)^{n+n} \end{bmatrix}}^{n}\right.$$

- cofactor = ±1 * determinant of smaller matrix.
- don't be scared of all the words

Example: Find $\det(A)$ given

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 3 & 0 & 5 & -4 \\ 9 & -2 & 0 & 1 \\ 10 & 1 & 0 & 2 \end{bmatrix}.$$

Expand on any row or column.

> **Theorem 2** *If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.*

Why? Let A look like

$$A = \begin{bmatrix} a_{11} & * & * & * \cdots & * \\ 0 & a_{22} & * & \cdots & * \\ 0 & 0 & a_{33} & * \cdots & * \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

$$\Rightarrow \det(A) = a_{11} \begin{vmatrix} a_{22} & * & \cdots & * \\ 0 & a_{33} & * \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11}(a_{22}) \begin{vmatrix} a_{33} & * & \cdots & * \\ 0 & a_{44} & * \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix}$$

$$= \cdots = a_{11}(a_{22})(a_{33})\cdots(a_{n-2}) \begin{vmatrix} a_{n-1,n-1} & * \\ 0 & a_{nn} \end{vmatrix}$$

$$= a_{11} a_{22} a_{33} \cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$$

Ex Find $\det(A)$ for $A = \begin{bmatrix} 3 & 4 & 9 & -7 \\ 0 & -1 & 10 & 2 \\ 0 & 0 & 5 & -6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Ex Find $\det(A)$ for $A = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 4 & 0 \end{bmatrix}$.

Section 3.2 Properties of Determinants

ELOs:

- Be able to identify and use properties of determinants, including those related to row operations, transposes and products.
- Understand how the value of the determinant "determines" whether or not a matrix is invertible.

let's explore determinants of row-equivalent matrices.

Theorem 3 ***Determinant Properties*** *Let A and C be $n \times n$ matrices.*

(a) If a multiple of one row of A is added to another row to produce a matrix B, then

$$det(B) = \underline{\hspace{6cm}}$$

(use space below to explore this)

(b) If two rows of A are interchanged to produce B, then

$$det(B) = \underline{\hspace{6cm}}$$

(c) If one row of A is multiplied by k to produce B, then

$$det(B) = \underline{\hspace{6cm}}$$

(d) The determinant of the transpose of A is,

$$det(A^T) = \underline{\hspace{6cm}}$$

(e) The determinant of the product AC is,

$$det(AC) = \underline{\hspace{6cm}}$$

Example: let's do examples to motivate our answers for the theorem.

(b) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ let $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$\det(A) =$

Then $\det(B) =$

(c) $C = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$ $\det(C) =$

Example: (d) $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $\det(A^T) =$

(a) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $(-k)$ $D = \begin{bmatrix} a & b \\ c-ak & d-bk \end{bmatrix}$

$\det(D) =$

Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges. That is, REF(A)=U where U is an upper triangular matrix.

If there are r interchanges, then

$$detA = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Assume $E_p \cdots E_2 E_1 A = B$ where $B = RREF(A)$.

case 1: A^{-1} exists

$B = I$

$\Rightarrow \det(B) = \det(I) = 1 \neq 0$

$\Rightarrow \det(A) \neq 0$

case 2: A^{-1} DNE

$B \neq I$ (but B is square)

$\Rightarrow B$ has a row of zeroes

$\Rightarrow \det(B) = 0$

$\Rightarrow \det(E_p \cdots E_2 E_1 A) = 0$

$\Leftrightarrow \det(E_p) \cdots \det(E_1)\det(A) = 0$

not zero

$\Rightarrow \det(A) = 0$.

Theorem 4 *A square matrix A is invertible if and only if $\det(A) \neq 0$.*

This is a really big deal! This gives us quick way to check for existence of A^{-1}.

Yay!

Section 3.3 Cramer's Rule, Volume and Linear Transformations

ELOs:

- Be able to use Cramer's Rule to solve a linear system.
- Understand the geometric interpretation of the determinant (scales area in $\mathbb{R}^2$ and volume in $\mathbb{R}^3$).

Introduction: We can solve matrix equations using the theory of determinants. (not great to do this by hand for $n>3$)

Theorem 7 ***Cramer's Rule*** *Let A be an invertible $n \times n$ matrix. For any $\mathbf{b} \in \mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by*

$$x_i = \frac{det(A_i(\mathbf{b}))}{det(A)}, \quad i = 1, 2, \ldots, n.$$

$A_i(\mathbf{b})$ is defined as the matrix where the ith column of A is replaced by $\mathbf{b}$. That is,

$$A_i(\mathbf{b}) = \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{b} & \ldots & \mathbf{a}_n \end{bmatrix}.$$

ith column.

Proof:

Let $A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_i \cdots \vec{a}_n]$ and $I = [\vec{e}_1 \ \vec{e}_2 \cdots \vec{e}_i \cdots \vec{e}_n]$

Assume $A\vec{x} = \vec{b}$. Then

Example: Use Cramer's rule to solve the system

$$3x_1 - 5x_2 = 47$$
$$-2x_1 + x_2 = -15$$

Application to Engineering: Laplace Transform

- tool for solving differential eqns (DEs)
- converts system of DEs into linear system.

Example: Use Cramer's Rule to solve system. t is an unspecified parameter. (Find t-values where this system has solutions.)

$$2tx_1 + 5x_2 = 3$$
$$10x_1 + tx_2 = 3$$

Cramer's Rule gives us formula for A^{-1}.

Theorem 8 An Inverse Formula *Let A be an invertible $n \times n$ matrix. Then,*

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

where $adj(A)$ denotes the adjugate (or classical adjoint), the $n \times n$ matrix of cofactors $C^T = [C_{ji}]$.

Why does this formula work?

Let A be invertible matrix.

$\Rightarrow AA^{-1} = A[\vec{x}_1\ \vec{x}_2 \cdots \vec{x}_n] = [A\vec{x}_1\ A\vec{x}_2 \cdots A\vec{x}_n] = I$

call columns of A^{-1} $\vec{x}_i$

but $I = [\vec{e}_1\ \vec{e}_2 \cdots \vec{e}_n]$

$\Rightarrow A\vec{x}_1 = \vec{e}_1,\ A\vec{x}_2 = \vec{e}_2, \ldots, A\vec{x}_n = \vec{e}_n$

$\Rightarrow$ the j^{th} column of A^{-1} (namely $\vec{x}_j$) is a vector that satisfies $A\vec{x}_j = \vec{e}_j$

$\Rightarrow$ by Cramer's Rule,

$$\{(i,j)\text{-entry of } A^{-1}\} = \{i^{th} \text{ entry of } \vec{x}_j\}$$

$$= \frac{\det(A_i(\vec{e}_j))}{\det A}$$

What do we mean by $\det(A_i(\vec{e}_j))$?

$$\det(A_i(\vec{e}_j)) = (-1)^{i+j}\det A_{ji} = C_{ji}$$

(j,i) cofactor of A

ex: $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 5 \\ 5 & 0 & -2 \end{bmatrix} \Rightarrow A_1(\vec{e}_2) = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 1 & 5 \\ 0 & 0 & -2 \end{bmatrix}$

$\Rightarrow \det(A_1(\vec{e}_2)) = 0\cdot C_{11} + 1\cdot C_{21} + 0\cdot C_{31} = C_{21}$

So we build up A^{-1} like this,

$$A^{-1} = \begin{bmatrix} \frac{\det(A_1(\vec{e}_1))}{\det A} & \frac{\det(A_1(\vec{e}_2))}{\det A} & \cdots & \frac{\det(A_1(\vec{e}_n))}{\det(A)} \\ \vdots & & & \\ \frac{\det(A_n(\vec{e}_1))}{\det A} & - & - & \frac{\det(A_n(\vec{e}_n))}{\det A} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \underbrace{\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}} = \frac{C^T}{\det A}$$

transpose of cofactor matrix C

this is defined to be the adjugate (or classical adjoint) of A

i.e. $\mathrm{adj}(A)$

optional

Exercise: Show that in the 2×2 case the adjugate formula for A^{-1} gives

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$A^{-1} = \frac{C^T}{det(A)}$$

$$C_{11} = (-1)^{1+1}(a_{22}) = a_{22}$$
$$C_{12} = (-1)^{1+2}(a_{21}) = -a_{21}$$
$$C_{21} = (-1)^{2+1}(a_{12}) = -a_{12}$$
$$C_{22} = (-1)^{2+2}(a_{11}) = a_{11}$$

$$C^T = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Example: Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$. Find A^{-1} if it exists.

You finish.

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} =$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} =$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix} =$$

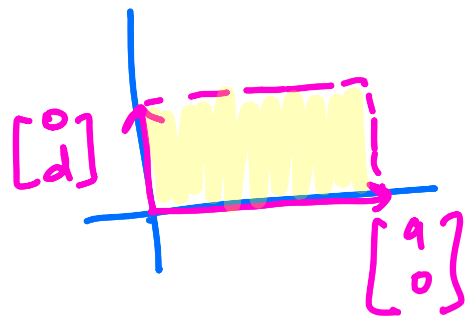
Cool fact/hint:
rather than finding det(A), you can use the fact that $adj(A) \cdot A = I\, det(A)$

Determinants as Area / Volume:

Let's first consider a 2x2 diagonal matrix

$$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \Rightarrow \det(A) = ad$$

notice geometrical interpretation



Area of the rectangle given by $\begin{bmatrix} 0 \\ b \end{bmatrix}$ and $\begin{bmatrix} a \\ 0 \end{bmatrix}$ as legs is $|ad|$.

$\Rightarrow |\det(A)| =$ area of rectangle given by columns of A.

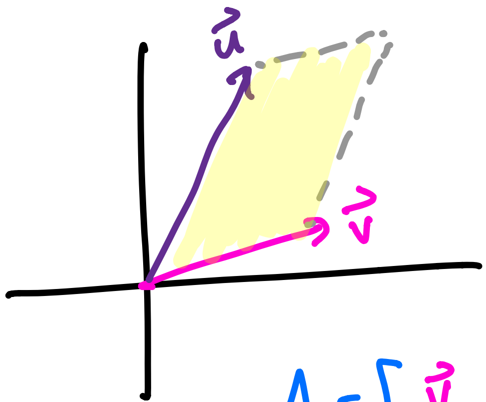
we know ① row swaps do not change $|\det(A)|$

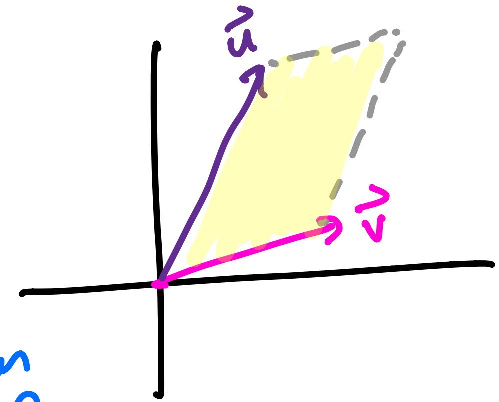
② adding multiple of one row to another does not change $|\det(A)|$.

③ $|\det(A)| = |\det(A^T)|$

$\Rightarrow$ column swaps and adding multiple of one column to another column of A also doesn't change $|\det(A)|$.

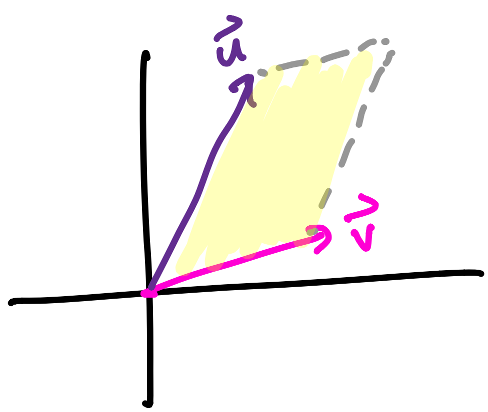
Qn: what do column swaps do to area of parallelogram?

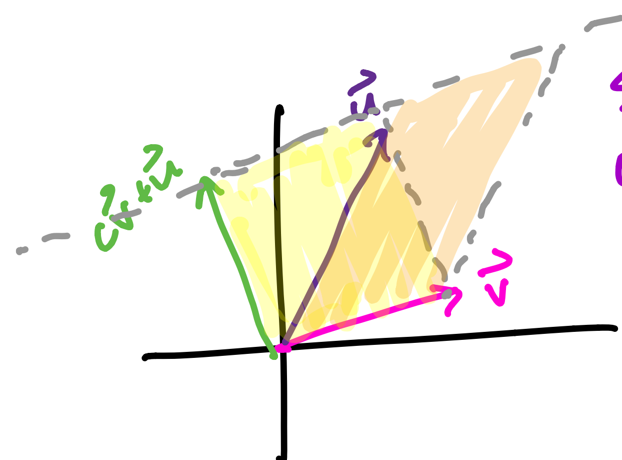




$A = [\vec{v} \;\; \vec{u}] \xrightarrow{\text{column swap}} B = [\vec{u} \;\; \vec{v}]$

i.e. no change in area.





same area just squished over

$A = [\vec{v} \;\; \vec{u}] \xrightarrow{c\vec{v}+\vec{u} \to \vec{u}} B = [\vec{v} \;\; c\vec{v}+\vec{u}]$

$\Rightarrow$ column swaps and add multiples of columns don't change area. $\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$ and area is $|det(A)|$.

Theorem 9 Area or Volume

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|det(A)|$.

If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|det(A)|$.

Theorem 10 Expansion Factors

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation determined by the 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{area\ of\ T(S)\} = |det(A)| \cdot \{area\ of\ S\}$$

Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation determined by the 3×3 matrix A. If S is a parallelepiped in $\mathbb{R}^3$, then

$$\{volume\ of\ T(S)\} = |det(A)| \cdot \{volume\ of\ S\}$$

i.e. det(A) gives us scaling factor.