4.1

Section 4.1 Vector Spaces and Subspaces

ELOs:

- Know the 10 axioms that define a vector space V.
- Determine if a set is a vector space using the axioms of vector spaces.
- Identify both examples and non-examples of subspaces.
- Show that the span of vectors in a vector space V is a subspace of V.

Introduction: Much of the theory we learned in Chapters 1 and 2 based on properties of $\mathbb{R}^n$ may be abstracted to generalized vector spaces.

Definition 1 *A vector space is a nonempty set V of objects, called* vectors, *which are defined by two operations, called* addition *and* multiplication by scalars, *which satisfy the 10 axioms listed below. The axioms must hold for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and scalars $c, d \in \mathbb{R}$.*

(1) $\mathbf{u} + \mathbf{v} \in V$

(2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(3) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(4) $\exists\ \mathbf{0} \in V$ *such that* $\mathbf{u} + \mathbf{0} = \mathbf{u}$

(5) $\forall\ \mathbf{u} \in V,\ \exists(-\mathbf{u}) \in V$ *such that* $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

(6) $c\mathbf{u} \in V$

(7) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(8) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(10) $1\mathbf{u} = \mathbf{u}$

By extension of the above axioms:

(i) $0\mathbf{u} = \mathbf{0}$

(ii) $c\mathbf{0} = \mathbf{0}$

(iii) $-\mathbf{u} = (-1)\mathbf{u}$

Examples of vector spaces:

Example 1: R^n where $n \geq 1$ is a positive integer. Vector addition and scalar multiplication are defined component-wise, and $\mathbf{0}$ is defined as the vector which has every entry equal to 0. The 10 axioms defining a vector space V reduce to properties of real number algebra.

Note: In Section 1.3, we showed that the above axioms hold for R^n.

Example 2: The space of $m \times n$ matrices with m, n fixed. Matrix addition and scalar multiplication are defined component-wise and the zero matrix 0 is defined as the matrix with every entry equal to 0. The 10 axioms defining a vector space V reduce to properties of real number algebra.

Note: In Section 2.1, we showed that the above axioms hold for $m \times n$ matrices with m, n fixed.

Example 3: $\mathbb{S} =$ **{doubly infinite sequences of numbers}**. An element of $\mathbb{S}$ can be written as

$$\{y_k\} = (\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots) \quad \{w_k\} = (\dots, w_{-1}, w_0, w_1, \dots)$$

(a) Define addition.

Let $\vec{u} = \{y_k\}$ and $\vec{v} = \{w_k\}$. Then

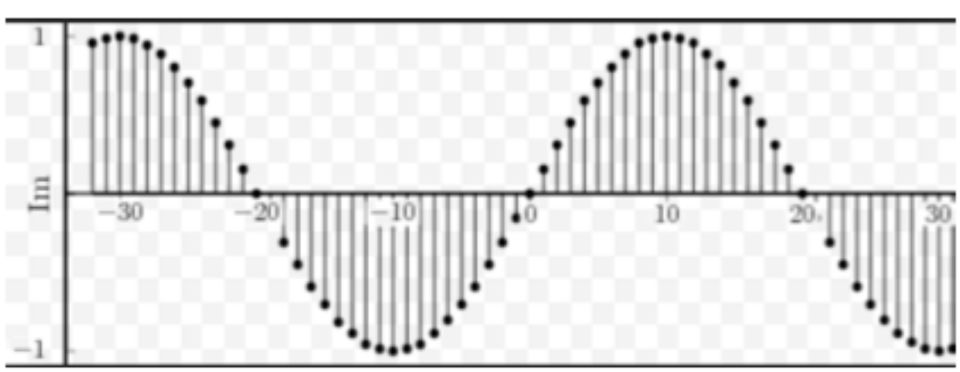
$$\vec{u} + \vec{v} = \{\dots, w_{-2}+y_{-2}, w_{-1}+y_{-1}, w_0+y_0, w_1+y_1, \dots\}$$

(b) Define scalar multiplication.

$$c\vec{u} = (\dots, cy_{-2}, cy_{-1}, cy_0, cy_1, cy_2, \dots)$$

(c) Checking the vector space axioms is equivalent to checking the axioms for $\mathbb{R}^n$ because vector addition and scalar multiplication are done entry by entry just with infinitely many entries in this case.

Note: We may consider $\mathbb{S}$ as the space of discrete-time signals. A signal may be visualized by its graph over its integer "domain."



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Example 4: $\mathbb{P}_n =$ **{polynomials of degree at most n where $n \geq 0$}**

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n, \quad a_0, \ldots, a_n \in \mathbb{R}$$

(a) Define addition.

(b) Define scalar multiplication.

(c) Check all of the axioms to show that V is a vector space.

Example 5: $V = \{$**all real-valued functions defined on a domain** $\mathbb{D}\}$ $= \{f : \mathbb{D} \to \mathbb{R}\}$ 4.1

(a) Define addition. $f, g \in V$ $t \in \mathbb{D}$.

$(f+g)(t) = f(t) + g(t)$

ex $f = \sin t$

$g = -5t^2 + \frac{1}{\sqrt{t}}$

$\mathbb{D}$ is some subset of $\mathbb{R}$ or all of $\mathbb{R}$

$\mathbb{D} = \{t \mid t > 0\}$

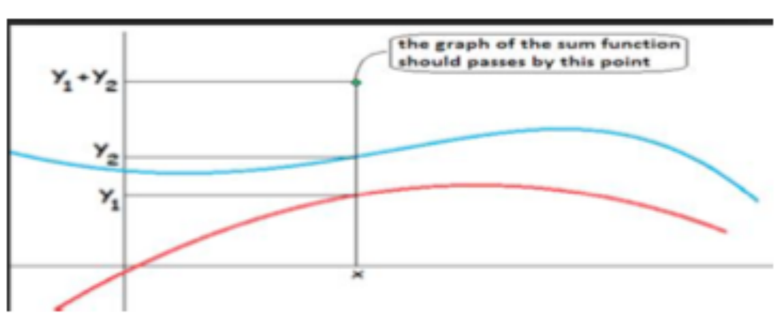
(b) Define scalar multiplication.

$c(f(t)) = cf(t)$

ex: $5f = 5\sin t$

(c) Check all of the axioms to show that V is a vector space.

As we did for the discrete-time signals, we may visualize functions as graphs over their domains..



a subspace is a smaller space that lives inside a bigger vector space

Definition 2 *A* <u>*subspace*</u> *of a vector space V is a subset H of V that has 3 properties:*

(a) The zero vector of V is in H.

(b) H is closed under addition.

(c) H is closed under scalar multiplication.

<u>Exercise</u>:

(a) Do all vector spaces have at least one subspace? If so, what is it?

(b) Is the set of all polynomials with real coefficients $\mathbb{P}$ a subspace of the vector space of all functions from $\mathbb{R} \to \mathbb{R}$? Why or why not?

(c) Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$? Why or why not?

(d) Is a plane in $\mathbb{R}^3$ not through the origin a subspace of $\mathbb{R}^3$? Why or why not?

Note:

Consider $H = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \middle| \, x, y \in \mathbb{R} \right\} \subset \mathbb{R}^3$

subset of

This set H IS a subspace of $\mathbb{R}^3$.

(a) $\vec{0} \in H$ ✓

(b) $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 0 \end{bmatrix} \in H$ ✓

(c) $c \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ 0 \end{bmatrix} \in H$ ✓

This subspace H looks and acts like $\mathbb{R}^2$ (we say it's isomorphic to $\mathbb{R}^2$).

4.1

Subspace spanned by a set

As in Chapter 1, a *linear combination* of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is given by

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p, \quad c_i \in \mathbb{R}$$

and $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is all possible linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

Exercise: Suppose $\mathbf{v}_1, \mathbf{v}_2 \in V$. Let $H = \text{span}\{\mathbf{v_1}, \mathbf{v_2}\}$. Show that H is a subspace. (of V).

(note! all vectors in H look like $a_1 v_1 + a_2 v_2$.)

- Check that the zero vector of V is in H.

- Check that H is closed under addition.

- Check that H is closed under scalar multiplication.

Theorem 1 *If V is a vector space, and $\mathbf{v_1}, \ldots, \mathbf{v_p} \in V$, then span$\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is a subspace of V.*

Note: we call span$\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ the subspace spanned by $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$.

Example: Let

$$H = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b, \in \mathbb{R} \right\} \subset \mathbb{R}^4.$$

Show that H is a subspace of $\mathbb{R}^4$.

notice:

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

ex Find all subspaces of $\mathbb{R}^3$.

Section 4.2 Null Spaces, Column Spaces and Linear Transformations 4.2

ELOs:

- Define the null space Nul A and column space Col A of $m \times n$ matrix A and linear transformation T.
- Compare and contrast Nul A and Col A.

Introduction: In Section 4.2, we'll generalize some of the theory we learned in Sections 1.4 (The Matrix Equation $A\mathbf{x} = \mathbf{b}$) and 1.5 (Solution Sets of Linear Systems), recognizing the solution set of a homogeneous matrix equation $A\mathbf{x} = 0$ and the set of all linear combinations of specified vectors as subspaces of $\mathbb{R}^k$.

The Null Space of a Matrix

Subspaces that occur naturally as solution sets of homogeneous systems.

Definition 1 *The null space of an $m \times n$ matrix A is*

$$Nul\ A = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

That is, Nul A is the set of all solutions to the homogeneous matrix equation, $A\mathbf{x} = \mathbf{0}$.

Example: Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. Is $\mathbf{u} \in$ Nul A?

(i.e. is it true or false that $A\vec{u} = \vec{0}$?)

Remember $\text{Nul}\,A = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$

Theorem 2 *The* null space *of an $m \times n$ matrix A is a* subspace of $\mathbb{R}^n$.

That is, the set of all solutions to the linear system $A\mathbf{x} = \mathbf{0}$ of m homogeneous equations in n unknowns is a subspace of $\mathbb{R}^n$.

Remember
Subspace defn: (H in V)
(a) $\vec{0} \in V \Rightarrow \vec{0} \in H$
(b) $\forall\, \vec{u}, \vec{v} \in H$, $\vec{u} + \vec{v} \in H$ also
(c) $\forall\, c \in \mathbb{R}$ and $\vec{u} \in H$, then $c\vec{u} \in H$ also

Question: Why is Nul A a subspace of $\mathbb{R}^n$?

Proof: Need to verify properties (a), (b) and (c) of subspace definition from Section 4.1.

(a) Show that $\mathbf{0} \in$ Nul A.

(b) Show that if $\mathbf{u}$, $\mathbf{v} \in$ Nul A, then $\mathbf{u} + \mathbf{v} \in$ Nul A (closure under addition).

(c) Show that if $\mathbf{u} \in$ Nul A and $c \in \mathbb{R}$, then $c\mathbf{u} \in$ Nul A (closure under scalar multiplication).

4.2

Solving the homogeneous matrix equation, $A\mathbf{x} = \mathbf{0}$, gives us an *explicit description* of Nul A. That is, we can write Nul A as the span of a collection of vectors.

Example: Find an *explicit description* or *spanning set* for Nul A where $A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$.

augmented matrix

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 7 & 18 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$$

pivot columns

Observations:

- Spanning set of Nul A is automatically linearly independent.

because the free variables are the weights/coefficients of the linear combo. of vectors that span nul A.

- If Nul $A \neq \{\mathbf{0}\}$, then the number of vectors in the spanning set of Nul A equals the number of free variables in $A\mathbf{x} = \mathbf{0}$

Definition 2 *The column space of an $m \times n$ matrix $A = [\mathbf{a}_1 \ \dots \ \mathbf{a}_n]$ is*

$$Col\ A = Span\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$$
$$= \{\mathbf{b} \in \mathbb{R}^m : A\mathbf{x} = \mathbf{b} \text{ where } \mathbf{x} \in \mathbb{R}^n\}$$

I like this one best.

That is, Col A is the set of all linear combinations of the columns of A

Question: Why is Col A a subspace of $\mathbb{R}^m$?

(a) $\vec{0} \in span\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$ ✓

(b) if $\vec{u}, \vec{v} \in span\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$, then $\vec{u} + \vec{v} \in span\{\vec{a}_1, \dots, \vec{a}_n\}$

(c) if $\vec{u} \in colA$, then $c\vec{u} \in span\{\vec{a}_1, \dots, \vec{a}_n\}$

Remember subspace defn: (H in V)

(a) $\vec{0} \in V \Rightarrow \vec{0} \in H$

(b) $\forall\ \vec{u}, \vec{v} \in H$, $\vec{u} + \vec{v} \in H$ also

(c) $\forall\ c \in \mathbb{R}$ and $\vec{u} \in H$, then $c\vec{u} \in H$ also

Theorem 3 *The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.*

Example: Find a matrix A such that $W = Col\ A$ where $W = \left\{ \begin{bmatrix} x_1 - 2x_2 \\ 3x_2 \\ x_1 + x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$

Observations:

- Col A is the *range* of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.
- The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^m$.

4.2

The Contrast Between Nul A and Col A

Question: How are Nul A and Col A related?

Solns:

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 0 & 0 & 1 \end{bmatrix}$. $\xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(a) Col $A \subset \mathbb{R}^k$ where $k =$ _____.

(b) Nul $A \subset \mathbb{R}^k$ where $k =$ _____.

(c) Find a nonzero vector in Col A, if one exists.

(d) Find a nonzero vector in Nul A, if one exists.

4.2

Compare and Contrast Nul A and Col A

$A: \mathbb{R}^n \to \mathbb{R}^m$

Nul (A)

(1) Nul A is a subspace of $\mathbb{R}^n$.

(2) Nul A is implicitly defined: that is, you are given only a condition, $A\vec{x}=\vec{0}$, that vectors $\mathbf{v}_i$ in Nul A must satisfy.

(3) Row operations on $[A \;\; \vec{0}]$ are required to find vectors in Nul A.

(4) No clear relation between Nul A and the entries in A.

(5) A vector $\mathbf{v}$ in Nul A has property that $A\vec{v}=\vec{0}$.

(6) Given a specific vector $\mathbf{v}$, we can determine if $\mathbf{v} \in$ Nul A by computing $A\vec{v}$.

(7) Nul $A = \{\mathbf{0}\}$ if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(8) Nul $A = \{\mathbf{0}\}$ if and only if $\mathbf{x} \mapsto A\mathbf{x}$ is 1-1.

Col A

(1) Col A is a subspace of $\mathbb{R}^m$.

(2) Col A is explicitly defined: that is, you are given how to build vectors in Col A.

(3) Vectors in Col A are displayed of the columns of A.

(4) Each column of A is in Col A.

(5) A vector in Col A has the property that $A\vec{x}=\vec{v}$ has at least one soln.

(6) Given a specific vector $\mathbf{v}$, we can determine if $\mathbf{v} \in$ Col A by doing row ops on $[A \;\; \vec{v}]$

(7) Col $A = \mathbb{R}^m$ if and only if $A\mathbf{x} = \mathbf{b}$ has a solution for every $\vec{b} \in \mathbb{R}^m$.

(8) Col $A = \mathbb{R}^m$ if and only if $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

4.2

Linear Transformations

Definition 3 *A* **linear transformation** $T : V \to W$ *is a function such that*

- $T(\mathbf{u}+\mathbf{v}) =$ $T(\vec{u}) + T(\vec{v})$
- $T(c\mathbf{u}) =$ $cT(\vec{u})$

for all $\mathbf{u}, \mathbf{v} \in V$ *and* $c \in \mathbb{R}$.

(a) The *kernel* *(or* **null space***) of* T *is* $\{\mathbf{u} \in V : T(\mathbf{u}) = \mathbf{0} \in W\} \subset V$.

(b) The *range* *of* T *is* $\{\mathbf{w} \in W : T(\mathbf{v}) = \mathbf{w}$ *for some* $\mathbf{v} \in V\} \subset W$.

Same ideas as NulA and colA but new vocab, applicable to lin. ops. on vector spaces.

Example: Let V be the vector space of real-valued functions f defined on an interval $[a, b]$ with the property that the functions are differentiable and their derivatives are continuous functions on $[a, b]$. Let W be the vector space of all continuous functions on the interval $[a, b]$. Let $D : V \longrightarrow W$ be the derivative transformation: $D(f) = f'$.

(a) What differentiation rules from Calculus show that D is a linear transformation?

(b) What subspace is the kernel of D?

(c) What is the range of D?

4.3

Section 4.3 Linearly Independent Sets; Bases

ELOs:

- Determine whether a set of vectors in a vector space V is linearly dependent or independent.
- Check whether or not a set of vectors is a basis of a vector space V.
- Understand the similarities and differences between a spanning set and a basis.

Introduction: In Section 4.3, we'll generalize some of the theory we learned in Sections 1.3 (Vector Equations and Span) and Section 1.7 (Linear Independence), to define a *basis*, a linearly independent, spanning set of a vector space.

Definition 1 *A set of vectors* $\{\mathbf{v}_1, \dots, \mathbf{v}_p\} \subset V$ *is said to be* linearly independent *if the vector equation*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. If a non-trivial solution exists, then we say that $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ *is* linearly dependent *and we can find a linear dependence relation among* $\mathbf{v}_1, \dots, \mathbf{v}_p$.

Recall: From Section 1.7, we know that

- a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ containing $\mathbf{0}$ is __________________________
- a set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is __________________________ if at least one vector is a scalar multiple of the other

Example: Let

$$\mathbf{p}_1(t) = 1,\ \mathbf{p}_2(t) = t,\ \mathbf{p}_3(t) = 4 - t.$$

Is $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ linearly independent in $\mathbb{P}$?

$\mathbb{P}$ = set of all real-coefficient polynomials

Exercise: Let

$$\mathbf{p}_1(t) = 1,\ \mathbf{p}_2(t) = t,\ \mathbf{p}_3(t) = 4 - t^2.$$

Is $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ linearly independent in $\mathbb{P}$?

Theorem 2 *An indexed set* $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ *of two or more vectors with* $\mathbf{v}_1 \neq \mathbf{0}$ *is linearly dependent if and only if some vector* $\mathbf{v}_j$ *with* $j > 1$ *is a linear combination of the preceding* $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

Note: How is this different than lin. ⫫ of cols of A? We can't necessarily write augmented matrix because this is more general.

Definition 2 *Let $H \subset V$ be a subspace. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\} \subset V$ is a* <u>*basis*</u> *for H if*

(i) *$\mathcal{B}$ is a linearly independent set.*

(ii) *$H = Span\{\mathbf{b_1}, \dots, \mathbf{b_p}\}$.*

<u>Example:</u> Let $\mathbf{e_1}, \dots, \mathbf{e_n}$ be the columns of the $n \times n$ identity matrix. That is, $I_n = [\mathbf{e}_1 \ \dots \ \mathbf{e}_n]$.
Recall: $\mathbf{e}_i$ is the $n \times 1$ vector with the i^{th} entry equal to 1 and all other entries 0.

(a) Are the columns of I_n linearly independent or linearly dependent?

(b) What is the span of the columns of I_n?

(c) Does $\{\mathbf{e_1}, \dots, \mathbf{e_n}\}$ form a basis for $\mathbb{R}^n$? Why or why not?

<u>Exercise:</u> Let $A = [\mathbf{a_1} \dots \mathbf{a_n}]$ be an $n \times n$ invertible matrix.

(a) Are the columns of A linearly independent or linearly dependent?

(b) What is the span of the columns of A?

(c) Does $\{\mathbf{a_1}, \dots, \mathbf{a_n}\}$ form a basis for $\mathbb{R}^n$? Why or why not?

note: indeed there are infinitely many bases for $\mathbb{R}^n$!!!

<u>Exercise:</u> Consider $S = \{1, \ t, \ t^2, \dots, \ t^n\}$ the *standard basis* for $\mathbb{P}_n$. To verify this set is a basis for $\mathbb{P}_n$, we need to check

(a) Is the set of vectors linearly independent?

(b) Does the set of vectors span $\mathbb{P}_n$?

4.3

Key Idea: Bases are the most efficient or minimal spanning sets of a vector space.

Example: Let

notice: $5\vec{v}_1 + 3\vec{v}_2 = \begin{bmatrix} 0 \\ 10 \\ -5 \end{bmatrix} + \begin{bmatrix} 6 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}, \text{ and } \; H = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.$$

Show that span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} =$ span$\{\mathbf{v}_1, \mathbf{v}_2\}$. Find a basis for H.

i.e. Prove

- Show span$\{\mathbf{v}_1, \mathbf{v}_2\} \subset$ span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

- Show span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subset$ span$\{\mathbf{v}_1, \mathbf{v}_2\}$.

- Find a basis for H.

basis of $H = \{\vec{v}_1, \vec{v}_2\}$

Theorem 3 *(**Spanning Set Theorem**) Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\} \subset V$ and $H =$ span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.*

- *If some $\mathbf{v}_k \in S$ is a linear combination of the remaining vectors in S, the set formed by removing $\mathbf{v}_k$ still spans H.*

- *If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.*

Bases for Nul A and Col A

Given $A = \begin{bmatrix} 1 & -1 & 0 & -1 & 5 \\ 2 & -3 & -1 & -4 & 8 \\ 2 & -2 & 0 & -2 & 2 \\ 1 & 2 & 3 & 5 & 1 \end{bmatrix}$ and $RREF(A) = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Example: Find a basis for Nul A.

For $A\vec{x} = \vec{0}$, we get $\left\{ \begin{array}{l} x_1 = -x_3 - x_4 \\ x_2 = -x_3 - 2x_4 \\ x_3 = x_3 \\ x_4 = x_4 \\ x_5 = 0 \end{array} \right\} \iff \vec{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow \text{Nul}A = \text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \Rightarrow$ $\boxed{\text{basis for Nul}A = \left\{ \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}}$

Example: Find a basis for Col A.

$\text{col}A = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 2 \\ 1 \end{bmatrix} \right\}$

$\text{col}(RREF(A)) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \Rightarrow$ $\boxed{\text{basis for col}(A) \text{ is } \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 2 \\ 1 \end{bmatrix} \right\}}$

Theorem 4 *The pivot columns of a matrix A form a basis for Col A.*

Yay!

Notice from last example:

$$A = \begin{array}{c} \begin{array}{ccccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & -1 & 5 \\ 2 & -3 & -1 & -4 & 8 \\ 2 & -2 & 0 & -2 & 2 \\ 1 & 2 & 3 & 5 & 1 \end{bmatrix} \end{array}$$

$$B = RREF(A) = \begin{array}{c} \begin{array}{ccccc} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\vec{b}_3 = \vec{b}_1 + \vec{b}_2$$

$$\vec{b}_4 = \vec{b}_1 + 2\vec{b}_2$$

and $\vec{a}_3 = \vec{a}_1 + \vec{a}_2$ check: $\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 3 \end{bmatrix}$ ✓

$\vec{a}_4 = \vec{a}_1 + 2\vec{a}_2$ check: $\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -6 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ -2 \\ 5 \end{bmatrix}$ ✓

i.e. same linear combination coefficients can be used to describe columns of A and columns of $RREF(A)$. :)

Section 4.4 Coordinate Systems

ELOs:

- Understand how vector spaces with bases $\mathcal{B}$ containing n vectors behave like $\mathbb{R}^n$.
- Explain in words and pictures different coordinate systems.
- Utilize the Unique Representation Theorem and describe a coordinate mapping.

wow!

Key Idea: n-dimensional vector spaces behave like $\mathbb{R}^n$. In choosing a basis, we are choosing a coordinate system to make a vector space look like $\mathbb{R}^n$. we say n-dim vector space is isomorphic to $\mathbb{R}^n$

Theorem 7 ***The Unique Representation Theorem***

Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space V. Then, for each $\mathbf{x} \in V$, there exist ***unique*** *$c_1, \ldots, c_n \in \mathbb{R}$ such that*

$$\mathbf{x} = c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n}.$$

Proof: $\mathcal{B}$ spans $V \Rightarrow \exists\, c_1, \ldots, c_n \in \mathbb{R}$ s.t. $\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$. ①

Assume $\exists\, d_1, \ldots, d_n$ s.t. $\vec{x}$ can also be written as $\vec{x} = d_1\vec{b}_1 + \cdots + d_n\vec{b}_n$. ②

$\Rightarrow$ ① − ② gives $\vec{x} - \vec{x} = (c_1 - d_1)\vec{b}_1 + \cdots + (c_n - d_n)\vec{b}_n$

$\Leftrightarrow \vec{0} = (c_1 - d_1)\vec{b}_1 + \cdots + (c_n - d_n)\vec{b}_n$

but this is homogeneous vector eqn, and $\vec{b}_1 \perp\!\!\!\perp \cdots \perp\!\!\!\perp \vec{b}_n$

$\Rightarrow c_1 - d_1 = 0, \ldots, c_n - d_n = 0$ must be true. $\Rightarrow c_i = d_i \;\forall\, i = 1, \ldots, n$.

i.e. $c_1, \ldots, c_n$ are unique weights.

Definition 1 *Suppose $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ is a basis for a vector space V and $\mathbf{x} \in V$. The coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of $\mathbf{x}$) are the weights $c_1, \ldots, c_n \in \mathbb{R}$ such that*

$$\mathbf{x} = c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n}.$$

The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is called the coordinate mapping (determined by $\mathcal{B}$) where

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in R^n$$

is the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$ or the $\mathcal{B}$-coordinate vector of $\mathbf{x}$.

Example: Let $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\mathbf{x} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{E}}$.

$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = b_1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + b_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

let $b_1 = 2$ and $b_2 = 3$

$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ✓

$[x]_{\mathcal{E}} = ?$

4.4

A Graphical Interpretation of Coordinates

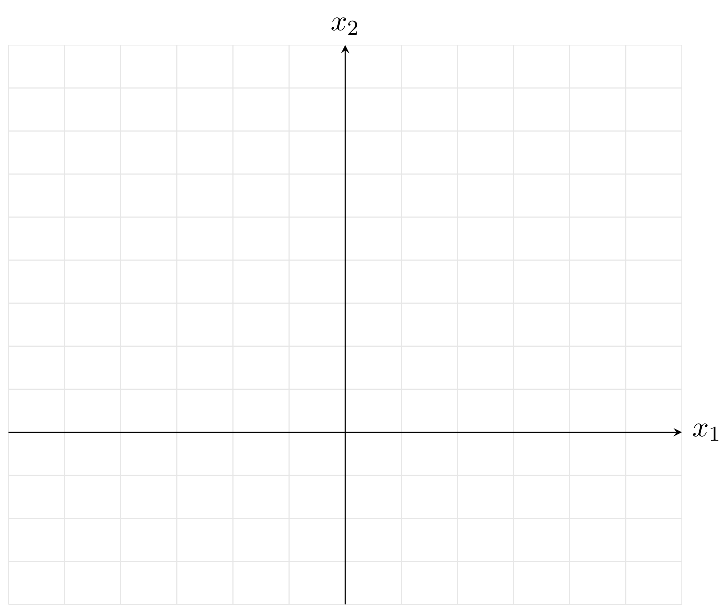
Key Idea: A coordinate system defines a one-to-one mapping of points in a set to $\mathbb{R}^n$.

Example: Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ be a non-standard basis of $\mathbb{R}^2$.

(a) Suppose $\mathbf{x} \in \mathbb{R}^2$ and $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find the standard coordinates for $\mathbf{x}$. That is, find $[\mathbf{x}]_{\mathcal{E}}$ where $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$.

(b) Find the $\mathcal{B}$-coordinates for the vector $\mathbf{b} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$.

(c) Interpret your work from parts (a) and (b) geometrically in terms of the coordinate system generated by $\mathcal{B}$.



Coordinates in $\mathbb{R}^n$

Definition 2 *Let* $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ *be a basis and* $\mathcal{P}_\mathcal{B} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \dots \mathbf{b}_n]$ *be the change-of-coordinates matrix from* $\mathcal{B}$ *to the standard basis in* $\mathbb{R}^n$*. Then, the vector equation*

$$\mathbf{x} = c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n}.$$

is equivalent to

$$\mathbf{x} = \mathcal{P}_\mathcal{B}[\mathbf{x}]_\mathcal{B}$$

Observation:

$$\vec{x} = P_B[\vec{x}]_B \iff \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [\vec{b}_1 \ \vec{b}_2 \cdots \vec{b}_n]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Also, $P_B^{-1}\vec{x} = [\vec{x}]_B$.

Must P_B^{-1} exist?

Example: Given $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\mathbf{x} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$.

(a) Find the change-of-coordinates matrix $\mathcal{P}_\mathcal{B}$ from $\mathcal{B}$ to the standard basis in $\mathbb{R}^2$, and the change-of-coordinates matrix $\mathcal{P}_\mathcal{B}^{-1}$ from the standard basis in $\mathbb{R}^2$ to $\mathcal{B}$.

(b) Use $\mathcal{P}_\mathcal{B}^{-1}$ to find $[\mathbf{x}]_\mathcal{B}$.

4.4

The Coordinate Mapping

Theorem 8 *Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space V. Then, the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a* ***one-to-one*** *linear transformation from V* ***onto*** *$\mathbb{R}^n$.*

Proof: Let $\vec{u}, \vec{v} \in V$. Then $\vec{u} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$ and $\vec{v} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n$ for some $a_i, c_i \in \mathbb{R}$, $\forall i = 1, \ldots, n$.

① $\Rightarrow \vec{u} + \vec{v} = (a_1 + c_1)\vec{b}_1 + \cdots + (a_n + c_n)\vec{b}_n$

$$\Leftrightarrow [\vec{u} + \vec{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 + c_1 \\ \vdots \\ a_n + c_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [\vec{u}]_{\mathcal{B}} + [\vec{v}]_{\mathcal{B}}.$$

and ② $p\vec{u} = pc_1\vec{b}_1 + \cdots + pc_n\vec{b}_n \Leftrightarrow [p\vec{u}]_{\mathcal{B}} = \begin{bmatrix} pc_1 \\ \vdots \\ pc_n \end{bmatrix} = p\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = p[\vec{u}]_{\mathcal{B}}$.

$\Rightarrow x \mapsto [x]_{\mathcal{B}}$ is a linear mapping.

We still need to prove that $x \mapsto [x]_{\mathcal{B}}$ is 1-1 and onto....

Example: Consider the standard basis for $\mathbb{P}_3 : \mathcal{B} = \{1, t, t^2, t^3\}$. $\vec{p}_1 = 1$, $\vec{p}_2 = t$, $\vec{p}_3 = t^2$, $\vec{p}_4 = t^3$

Polynomials in $\mathbb{P}_3$ behave like vectors in $\mathbb{R}^4$. Since

$$\mathbf{x} = c_0 + c_1t + c_2t^2 + c_3t^3 = \underline{\qquad}\mathbf{p}_1 + \underline{\qquad}\mathbf{p}_2 + \underline{\qquad}\mathbf{p}_3 + \underline{\qquad}\mathbf{p}_4,$$

we find

$$[\mathbf{x}]_{\mathcal{B}} = [c_0 + c_1t + c_2t^2 + c_3t^3]_{\mathcal{B}} =$$

and say that the vector space $\mathbb{P}_3$ is *isomorphic* to $\mathbb{R}^4$.

Note: We say that a vector space V is *isomorphic* to a vector space W if every vector space computation in V is accurately reproduced in W and vice versa.

4.4

Example: Use coordinate vectors to determine if $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a linearly independent set where

$$\mathbf{p}_1 = 1 - t, \qquad \mathbf{p}_2 = 2 - t + t^2, \qquad \mathbf{p}_3 = 2t + 3t^2$$

Exercise: Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\} = \left\{ \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$. Let $H = span\{\mathbf{b}_1, \mathbf{b}_2\}$. Find $[\mathbf{x}]_{\mathcal{B}}$ given $\mathbf{x} = \begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix}$.

$$\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2 \iff \begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$\iff \begin{cases} 9 = 3c_1 & ① \\ 13 = 3c_1 + c_2 & ② \\ 15 = c_1 + 3c_2 & ③ \end{cases} \iff \begin{array}{l} ①\ c_1 = 3 \\ ②\ 13 = 3(3) + c_2 \\ \quad c_2 = 4 \end{array}$$

check ③

$15 = 3 + 3(4)$ ✓

$$\Rightarrow \vec{x} = 3\vec{b}_1 + 4\vec{b}_2 \Rightarrow [x]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

4.5

Section 4.5 Dimension of a Vector Space

ELOs:

what does "dimension" really mean, for any vector space?

- Find the dimension of a vector space.
- Relate the dimension of Nul A and Col A to the number of pivots.

Key Idea: We showed in Section 4.4 that a vector space V with a basis $\mathcal{B}$ containing n vectors is *isomorphic* to $\mathbb{R}^n$. n represents the *dimension* of V and does not depend on the choice of basis.

Theorem 9 *If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be* linearly dependent.

Proof: (we'll sketch the proof here to give you the idea)

① Form set of p coordinate vectors s.t. all coordinate vectors live in $\mathbb{R}^n$ $\Rightarrow \exists$ nontrivial soln to homogeneous eqn.

② Then we know that set of p coordinate vectors in $\mathbb{R}^n$ is lin. dependent.

③ There's a linear xform from coordinate vectors to p vectors in V $\Rightarrow \exists$ nontrivial soln to homogeneous eqn of our original p vectors from V $\Rightarrow$ those p vectors are lin. dep.

Note: this proof idea basically uses isomorphism between V and $\mathbb{R}^n$.

Theorem 10 *If a vector space V has a basis on n vectors, then every basis of V must consist of exactly ______ vectors.*

This gives us an invariant of the vector space V. A different choice of basis yields the same number.

4.5

Definition 1 *If V is spanned by a finite set, then V is* finite-dimensional*, and the* dimension of V*, written* $\dim(V)$*, is the number of vectors in a basis for V.*

If V is NOT *spanned by a finite set, then V is said to be* infinite-dimensional.

Example:

(a) $\dim(\{\mathbf{0}\}) =$ 0 (by definition)

(b) $\dim(\mathbb{R}^n) =$

(c) $\dim(\mathbb{P}_n) =$

(d) $\dim(\mathbb{P}) =$

Example: Find a basis and the dimension of the subspace $W = \left\{ \begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$.

Answer:

Key Idea: The subspaces of $\mathbb{R}^n$ can be classified by dimension.

1) 0-dim. subspace: $\{\vec{0}\}$

2) 1-dim. subspace: any subspace spanned by a single vector (a line thru origin)

3) 2-dim subspace: any subspace spanned by two lin. ⫫ vectors (a plane thru origin)

4) 3-dim subspace: only $\mathbb{R}^3$ itself.

Subspaces of a Finite-Dimensional Space

> **Theorem 11** *Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded to a basis for H. Also, H is finite-dimensional and* dim $H \leq$ dim V.

EX ① let $H = \text{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(a) H is a subspace of $\mathbb{R}^m$ for $m = ?$

(b) what is $\dim(H)$?

② Let $G = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \right\}$. what is $\dim(G)$?

Knowing dimension of V makes it easier to check if something is a basis.

Theorem 12 ***The Basis Theorem***
Let V be a p-dimensional vector space where $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

Example: Show that $\{\underbrace{t}_{\vec{v}_1}, \underbrace{1-t}_{\vec{v}_2}, \underbrace{1+t-t^2}_{\vec{v}_3}\}$ is a basis for $\mathbb{P}_2$.

(we know $\{1, t, t^2\}$ is standard basis for P_2)

$$\star \quad c_1 t + c_2(1-t) + c_3(1+t-t^2) = 0$$

Key Idea: If V is a p-dimensional vector space where $p \geq 1$, then to determine whether a given a set of p vectors is a basis for V, we need only verify either the set of vectors is linearly independent or the set spans V.

4.5

The Dimensions of Nul A and Col A

1) $\dim(\text{Nul } A)$ = # of free variables in $A\vec{x}=\vec{0}$.

2) $\dim(\text{col } A)$ = # pivot columns in A.

Example: Given $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix}$.

(a) Find dim(Nul A).

(b) Find dim(Col A).

Key Idea: The dimension of Nul A is the number of free variables in the equation $A\vec{x}=\vec{0}$, and the dimension of Col A is the number of basic variables or pivot columns in A.

4.6

Section 4.6 Rank

ELOs:

- Identify the rank and nullity of a matrix.
- Use the rank-nullity theorem to determine properties of matrices, including number of pivot positions and dimensions of the null space, row space and column space.
- Represent geometrically the relationship between Row A and Nul A, and Col A and Nul A^T.
- Extend the invertible matrix theorem.

Key Idea: An exploration of the "hidden" relationships between the rows and columns of a matrix through the lens of vector space concepts.

Let A be an $m \times n$ matrix.

- each row of A has n entries & can be identified as vector in $\mathbb{R}^n$.
- set of all linear combinations of row vectors is called Row space of A (which is subspace of $\mathbb{R}^n$)

The Row Space

Example: Given $A = \begin{bmatrix} 1 & -1 & 0 & -1 & 5 \\ 2 & -3 & -1 & -4 & 8 \\ 2 & -2 & 0 & -2 & 2 \\ 1 & 2 & 3 & 5 & 1 \end{bmatrix}$. Then, $A^T = \begin{bmatrix} 1 & 2 & 2 & 1 \\ -1 & -3 & -2 & 2 \\ 0 & -1 & 0 & 3 \\ -1 & -4 & -2 & 5 \\ 5 & 8 & 2 & 1 \end{bmatrix}$.

Find Row A. *Hint:* Consider how Row A is related to Col A^T.

Definition 1 *Let A be an $m \times n$ matrix where each row can be identified with a vector in $\mathbb{R}^n$. Let $\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ be the rows of A. The **row space** of A is*

$$Row\ A = span\{\mathbf{r}_1, \ldots, \mathbf{r}_m\} \subset \mathbb{R}^n.$$

Note: *Row A=* Col A^T.

4.6

Theorem 13 *If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.*

This is distinctly different from column space! It's a bit surprising.

Bases for Row A, Col A and Nul A

Example: Given $A = \begin{bmatrix} 1 & -1 & 0 & -1 & 5 \\ 2 & -3 & -1 & -4 & 8 \\ 2 & -2 & 0 & -2 & 2 \\ 1 & 2 & 3 & 5 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = RREF(A).$

(a) Find a basis for Col A. Find dim(Col A).

(b) Find a basis for Nul A. Find dim(Nul A).

(c) Find a basis for Row A. Find dim(Row A).

(d) dim(Col A)+dim(Nul A)=_____. dim(Row A)+dim(Nul A)=_____.

WARNING: Row ops do not change dependence relations between columns but they do change dependence relations between rows. ⇒ If 1st 3 rows of RREF(A) are lin. ⊥, that's not necessarily true of first 3 rows of A.

Theorem 14 ***The Rank-Nullity Theorem*** defined as
Let A be an $m \times n$ matrix.

- *dim(Col A) = dim(Row A) := rank A = number of pivots*
-
$$rank\ A + dim(Nul\ A) = \underline{\ n\ }$$
$$\{number\ of\ basic\ variables\} + \{number\ of\ free\ variables\} = \_\_\_\_\_$$
$$\{number\ of\ pivot\ columns\} + \{number\ of\ non\text{-}pivot\ columns\} = \_\_\_\_\_$$
-
$$rank\ A^T + dim(Nul\ A^T) = \_\_\_\_\_$$

Observation:

Example continued: $A^T = \begin{bmatrix} 1 & 2 & 2 & 1 \\ -1 & -3 & -2 & 2 \\ 0 & -1 & 0 & 3 \\ -1 & -4 & -2 & 5 \\ 5 & 8 & 2 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 9/2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = RREF(A^T).$

(a) Find a basis for Nul A^T. Find dim(Nul A^T).

$x_1 = -\frac{9}{2}x_4$
$x_2 = 3x_4$
$x_3 = -5/4\ x_4$
x_4 free

$\Rightarrow \vec{x} = x_4 \begin{bmatrix} -9/2 \\ 3 \\ -5/4 \\ 1 \end{bmatrix}$

$\Rightarrow$ basis for Nul(A^T) $= \left\{ \begin{bmatrix} -9/2 \\ 3 \\ -5/4 \\ 1 \end{bmatrix} \right\}$

dim(Nul A^T) = 1

(b) Find rank A^T.

rank$(A^T) = 4 -$ dim(Nul A^T) = 3

Geometric Interpretation:

COOL FUN FACT
Row A is orthogonal to Nul A

(We'll see more about this in Chapter 6)

4.6

Application: A scientist solves a homogeneous system of 50 equations in 54 variables and finds that exactly 4 of the unknowns are free variables. Can the scientist be certain that any associated nonhomogeneous system (with the same coefficients) has a solution?

The Invertible Matrix Theorem (continued) continued from chp 2

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ __________ matrix.

(c) A has n pivot postions.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the __________ solution.

(e) The columns of A form a linearly __________ set.

(f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is __________.

(g) The equation $A\mathbf{x} = \mathbf{b}$ has __________ solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A __________ $\mathbb{R}^n$.

(i) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ __________ $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix C such that $CA =$ __________.

(k) There is an $n \times n$ matrix D such that $AD =$ __________

(l) A^T is an __________ matrix.

(m) The columns of A form a basis of $\mathbb{R}^n$.

(n) Col A = $\mathbb{R}^n$

(o) dim(Col A) = n

(p) rank A = n

(q) Nul A = $\{\vec{0}\}$

(r) dim(Nul A) = 0

<u>Ex</u> Think about these qns + answer.

1) A 5×9 matrix has $\dim(\text{Nul } A) = 2$.

What is $\text{rank } A$?

2) Could a 5×8 matrix have $\dim(\text{Nul } A) = 2$?

why or why not?

4.7

Section 4.7 Change of Basis

ELOs:

- Find a coordinate system for an n-dimensional vector space V given a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$.
- Be able to change coordinate systems given a change of basis.

Key Idea: Find the connection between different coordinate systems of a vector space V.

i.e. how can we go back + forth between 2 different bases of V?

Warm-Up: Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$.

(a) Given $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Find $\mathbf{x}$.

(b) Given $\mathbf{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{B}}$.

Example: Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \qquad \text{and} \qquad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2.$$

Suppose $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{C}}$.

use regular algebra

$$[\vec{x}]_{\mathcal{B}} = 3\vec{b}_1 + 1\vec{b}_2 = 3(4\vec{c}_1 + \vec{c}_2) + (-6\vec{c}_1 + \vec{c}_2) = 6\vec{c}_1 + 4\vec{c}_2$$

$$\Rightarrow [\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

OR

use linear algebra

$$[\vec{x}]_{\mathcal{C}} = [3\vec{b}_1 + \vec{b}_2]_{\mathcal{C}} = 3[\vec{b}_1]_{\mathcal{C}} + 1[\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \Rightarrow [\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

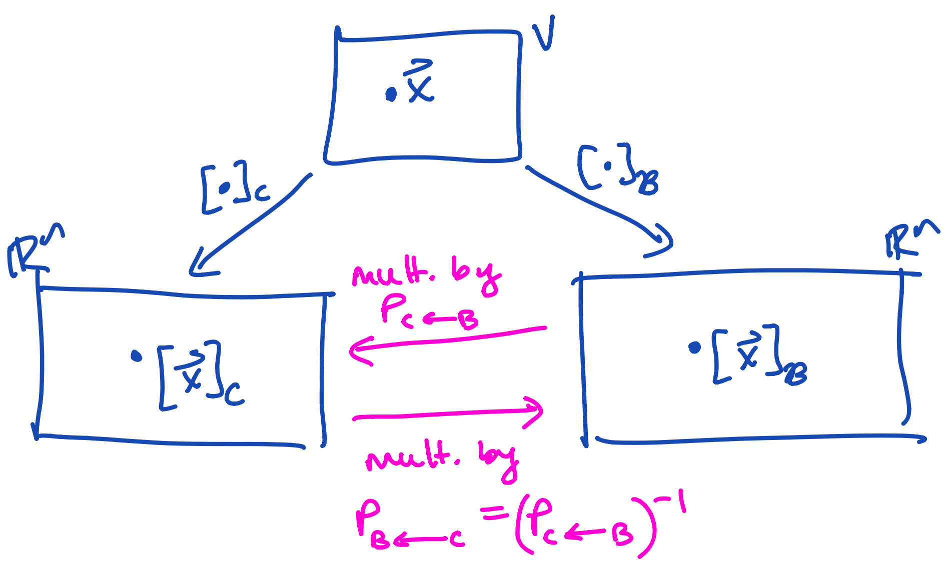
Theorem 15 ***Change of Basis*** *Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases of a vector space V. Then, there is a unique $n \times n$ matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ such that*

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

where the columns $P_{\mathcal{C} \leftarrow \mathcal{B}}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = [[\mathbf{b}_1]_{\mathcal{C}} \; [\mathbf{b}_2]_{\mathcal{C}} \ldots [\mathbf{b}_n]_{\mathcal{C}}]$$

$P_{\mathcal{C} \leftarrow \mathcal{B}}$ is called the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and is invertible.



Note: This is generalization of eqn we had in §4.4.

4.7

Change of Basis in $\mathbb{R}^n$ let $V = \mathbb{R}^n$, $\mathcal{E} = \{\vec{e}_1, \ldots, \vec{e}_n\}$ is standard basis. $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a different basis. Then $[\vec{x}]_{\mathcal{E}} = \vec{x}$.

$\Rightarrow$ formula in Thm 15 becomes $[\vec{x}]_{\mathcal{E}} = \boxed{\vec{x} = P_{\mathcal{E} \leftarrow \mathcal{B}} [\vec{x}]_{\mathcal{B}}}$

Example: Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, and consider the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1,\ \mathbf{c}_2\}$.

(a) Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

$P_{C \leftarrow B} = \left[[\vec{b}_1]_C \ [\vec{b}_2]_C \right]$

(b) Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

$P_{B \leftarrow C} = (P_{C \leftarrow B})^{-1}$