Section 6.1 Inner Product, Length, Orthogonality

note: if you had calculus 3 already an inner product is a dot product.

ELOs:

- Compute inner products (dot products) in $\mathbb{R}^n$ and describe their properties.
- Find the length of a vector in $\mathbb{R}^n$ and the distance between two vectors.
- Use the inner product to check if two vectors are orthogonal.

Goal: Introduce geometric concepts of length, distance and orthogonality in vector spaces.

Motivation:

concepts of length, distance, perpendicular are all understood for $\mathbb{R}^2$ & $\mathbb{R}^3$, but we need to extend these ideas for $\mathbb{R}^n$, $n>3$.

Warm-up: Let $\mathbf{u}$ be a vector in $\mathbb{R}^n$. We may consider vectors in $\mathbb{R}^n$ as $n \times 1$ matrices and define the transpose $\mathbf{u}^\mathbf{T}$ as a $1 \times n$ matrix. Then the matrix product $\mathbf{u}^\mathbf{T}\mathbf{u}$ is a 1×1 matrix, which we write as a scalar without brackets.

ex if $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\vec{u}\cdot\vec{u} = \vec{u}^T\vec{u} = [1 \quad 2 \quad 3]\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= 1^2 + 2^2 + 3^2 = 14$

ex if $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 5 \\ 1 \end{bmatrix}$ then $\vec{u}\cdot\vec{u} = [1 \quad 0 \quad -2 \quad 5 \quad 1]\begin{bmatrix} 1 \\ 0 \\ -2 \\ 5 \\ 1 \end{bmatrix} = 1+0+4+25+1$

$= 31$

Definition 1 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in* $\mathbb{R}^n$. *The scalar,* $\mathbf{u}^T\mathbf{v} = \mathbf{u}\cdot\mathbf{v}$, *is called the inner product (or dot product) of* $\mathbf{u}$ *and* $\mathbf{v}$

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Properties of Inner Products

Theorem 1 *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$.

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ commutativity

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ distributivity

(c) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$ commutativity & associativity w/ scalar multiplication

(d) $\mathbf{u} \cdot \mathbf{u} \geq 0$ *and* $\mathbf{u} \cdot \mathbf{u} = 0 \iff \mathbf{u} = \mathbf{0}$

Repeated application of (b) and (c) yields

$$(c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p) \cdot \mathbf{w} = c_1(\mathbf{u}_1 \cdot \mathbf{w}) + \cdots + c_p(\mathbf{u}_p \cdot \mathbf{w})$$

note! $\boxed{\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}}$

Example:

Compute $(\vec{u}+\vec{v}) \cdot \vec{w}$ if $\vec{u} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 2 \\ 9 \end{bmatrix}$.

Definition 2 *The* <u>*length (or norm)*</u> *of* $\mathbf{v}$ *is the nonnegative scalar* $\| \mathbf{v} \|$ *defined by*

$$\boxed{\| \mathbf{v} \| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}}$$

$$\boxed{\| \mathbf{v} \|^2 = \mathbf{v} \cdot \mathbf{v}} = \vec{v}^T \vec{v}$$

note: this extends what happens in $\mathbb{R}^2$ & $\mathbb{R}^3$.

A unit vector, **u**, *is a vector of length 1. To create a unit vector or to "normalize" a vector, divide the vector by its length. That is,*

$$\mathbf{u} = \frac{\mathbf{v}}{\| \mathbf{v} \|}$$

where **u** *is in the same direction as* **v**.

Example: For $\vec{v} = \begin{bmatrix} 4 \\ -2 \\ 3 \\ 0 \end{bmatrix}$, find a unit vector $\vec{u}$.

Observe: For any scalar c, $\| c\mathbf{v} \| = |c| \, \| \mathbf{v} \|$

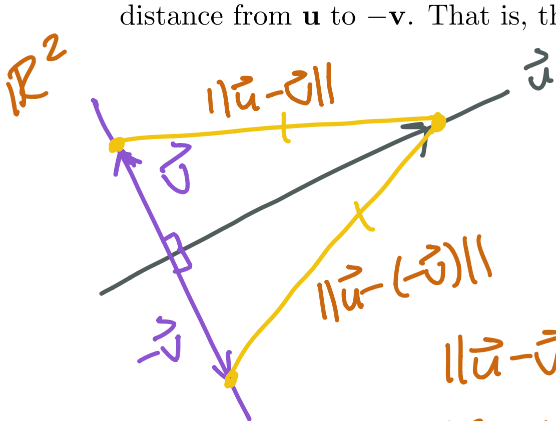
Distance in $\mathbb{R}^n$

> **Definition 3** *The <u>distance between $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$</u> is*
>
> $$dist(\mathbf{u},\mathbf{v}) = \| \mathbf{u} - \mathbf{v} \| = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+\cdots+(u_n-v_n)^2}$$

<u>Example:</u> Find distance between $\vec{u} = \begin{bmatrix} 5 \\ 1 \\ 0 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ -4 \\ 1 \\ 9 \end{bmatrix}$.

Orthogonal Vectors

Two lines are geometrically perpendicular if and only if the distance from $\mathbf{u}$ to $\mathbf{v}$ is the same as the distance from $\mathbf{u}$ to $-\mathbf{v}$. That is, the squares of the distances are the same.



remember $\|\vec{u}-\vec{v}\|$ = distance from $\vec{u}$ to $\vec{v}$.

For $\vec{u} \perp \vec{v}$ to be true, we need

$$\|\vec{u}-\vec{v}\| = \|\vec{u}-(-\vec{v})\| \iff \|\vec{u}-\vec{v}\|^2 = \|\vec{u}-(-\vec{v})\|^2$$

$$\iff (u_1-v_1)^2+(u_2-v_2)^2 = (u_1+v_1)^2+(u_2+v_2)^2$$

$$\iff (u_1^2+u_2^2)+(v_1^2+v_2^2)-2u_1v_1-2u_2v_2 = (u_1^2+u_2^2)+(v_1^2+v_2^2)+2u_1v_1+2u_2v_2$$

$$\iff \vec{u}\cdot\vec{u}+\vec{v}\cdot\vec{v}-2\vec{u}\cdot\vec{v} = \vec{u}\cdot\vec{u}+\vec{v}\cdot\vec{v}+2\vec{u}\cdot\vec{v}$$

$$\iff \|\vec{u}\|^2+\|\vec{v}\|^2-2\vec{u}\cdot\vec{v} = \|\vec{u}\|^2+\|\vec{v}\|^2+2\vec{u}\cdot\vec{v}$$

$$\iff -2\vec{u}\cdot\vec{v} = 2\vec{u}\cdot\vec{v} \iff \underbrace{\vec{u}\cdot\vec{v}=0}_{\text{iff condition for } \perp \text{ in } \mathbb{R}^2}$$

Definition 4 *Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are orthogonal (to each other) if $\mathbf{u} \cdot \mathbf{v} = 0$.*

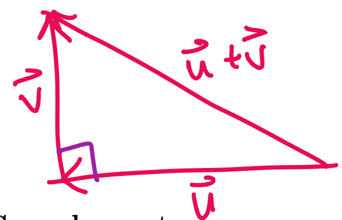
Observe: The zero vector is orthogonal to every vector in $\mathbb{R}^n$.

This expands idea from $\mathbb{R}^2$ to $\mathbb{R}^n$ $\forall n = 2, 3, \ldots$

Pythagorean Theorem

Theorem 2 *Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\| \mathbf{u}+\mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2$.*

in $\mathbb{R}^2$



Orthogonal Complements

Example: In $\mathbb{R}^3$, the z-axis, i.e. any vector of the form $\begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$ where $c =$ constant, is orthogonal to the xy-plane, i.e. the subspace in $\mathbb{R}^3$ spanned by $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

So we'd say $W^\perp = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ when $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Observation: The orthogonal complement of W is the collection of all vectors in $\mathbb{R}^n$ orthogonal to W :

$$W^\perp = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$ called "W perp"

Facts about $W^\perp$:

- $W^\perp$ is subspace of $\mathbb{R}^n$.
- $\vec{x} \in W^\perp$ iff $\vec{x}$ orthogonal to every vector in a spanning set of W.

Theorem 3 *Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:*

$$(Row\ A)^\perp = Nul\ A \text{ and } (Col\ A)^\perp = Nul\ A^T$$

A $m \times n$

note: $Nul\ A \subset \mathbb{R}^n$ | $Col\ A \subset \mathbb{R}^m$

$Row\ A \subset \mathbb{R}^n$ | $Nul\ A^T \subset \mathbb{R}^m$

Section 6.2 Orthogonal Sets

ELOs:

- Define and give examples of orthogonal sets and orthogonal bases.
- Find an orthogonal projection in $\mathbb{R}^n$.

Definition 1 *A set of vectors* $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ *in* $\mathbb{R}^n$ *is an* *orthogonal set* *if each pair of distinct vectors from the set is orthogonal. That is,* $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ *when* $i \neq j$.

Example: Show that $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set.

Exercise: Construct an orthogonal set in $\mathbb{R}^3$ that contains three vectors.

Theorem 4 *If* $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ *is an orthogonal set of nonzero vectors in* $\mathbb{R}^n$*, then* S *is linearly independent and hence is a basis for the subspace spanned by* S.

Pf: Assume $\exists$ $c_1, \ldots, c_p$ s.t.
$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p = \vec{0}$, i.e. we're exploring if S is linearly independent set.

Then $(c_1\vec{u}_1 + \cdots + c_p\vec{u}_p) \cdot \vec{u}_1 = \vec{0} \cdot \vec{u}_1$ would be true.

$\Leftrightarrow$ $c_1(\vec{u}_1 \cdot \vec{u}_1) + c_2(\vec{u}_2 \cdot \vec{u}_1) + \cdots + c_p(\vec{u}_p \cdot \vec{u}_1) = 0$

$\Leftrightarrow$ $c_1(\vec{u}_1 \cdot \vec{u}_1) = 0$ since $\vec{u}_2 \cdot \vec{u}_1, \ldots, \vec{u}_p \cdot \vec{u}_1 = 0$ since S is orthogonal set.

But $\vec{u}_1 \neq \vec{0} \Rightarrow \vec{u}_1 \cdot \vec{u}_1 \neq 0 \Rightarrow c_1 = 0$ must be true.

By same argument for $\vec{u}_2, \ldots, \vec{u}_p$, we get $c_2 = \cdots = c_p = 0$.

$\Leftrightarrow$ S is lin. indep set. #

Definition 2 *An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is an orthogonal set.*

Note: We can readily compute the weights/coefficients in a linear combination of orthogonal basis vectors.

Let $\vec{w} = c_1\vec{u}_1 + \cdots + c_n\vec{u}_n$ where $S = \{\vec{u}_1, \dots, \vec{u}_n\}$ is orthogonal basis of $\mathbb{R}^n$.

Then $\vec{w}\cdot\vec{u}_1 = (c_1\vec{u}_1 + \cdots + c_n\vec{u}_n)\cdot\vec{u}_1$

$\iff \vec{w}\cdot\vec{u}_1 = c_1(\vec{u}_1\cdot\vec{u}_1)$

$\iff \boxed{c_1 = \dfrac{\vec{w}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}}$ likewise, $\boxed{c_j = \dfrac{\vec{w}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j}}$

note: If you had calc 3, do you notice this is the projection coefficient?

Theorem 5 *Let $\{\mathbf{u}_1, \dots \mathbf{u}_p\}$ be an orthogonal basis for subspace $W \subset \mathbb{R}^n$. For each $\mathbf{y} \in W$, the weights in the linear combination*

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p$$

are given by

$$c_j = \frac{\mathbf{y}\cdot\mathbf{u}_j}{\mathbf{u}_j\cdot\mathbf{u}_j} \qquad (j = 1, \dots, p)$$

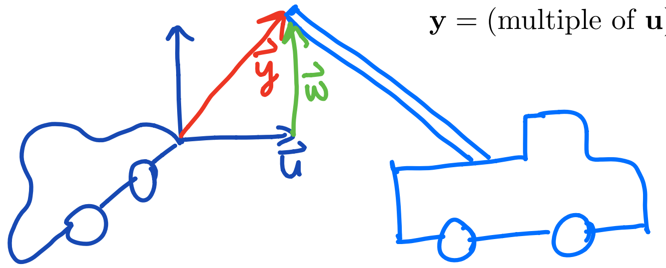
i.e. $c_j = \dfrac{\vec{y}\cdot\vec{u}_j}{\|\vec{u}_j\|^2}$

Example: $S = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^2$. Find $\begin{bmatrix} a \\ b \end{bmatrix}$ s.t. $a\begin{bmatrix} 1 \\ -1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

note: without an orthog. basis, we have to solve a system of eqns. With an orthog. basis, we can use formula from Thm 5, i.e. projections.

Orthogonal Projections

Problem: Suppose we have a preferred vector $\mathbf{u}$. How can we decompose any vector $\mathbf{y} \in \mathbb{R}^n$ into the sum of two vectors - one a multiple of $\mathbf{u}$, and the other orthogonal to $\mathbf{u}$?



$$\mathbf{y} = (\text{multiple of } \mathbf{u}) + (\text{multiple of a vector } \perp \mathbf{u})$$

let $\vec{w}$ be a vector that's $\perp$ to $\vec{u}$ (as shown).

we want $\vec{y} = \alpha \vec{u} + \beta \vec{w}$ where $\alpha, \beta \in \mathbb{R}$ (constants).

$\iff \beta \vec{w} = \vec{y} - \alpha \vec{u}$ and $\vec{w}$ is $\perp$ to $\vec{u}$

$\implies \underbrace{0 = \beta \vec{w} \cdot \vec{u}}_{\text{since } \vec{w} \perp \vec{u}} = (\vec{y} - \alpha \vec{u}) \cdot \vec{u}$

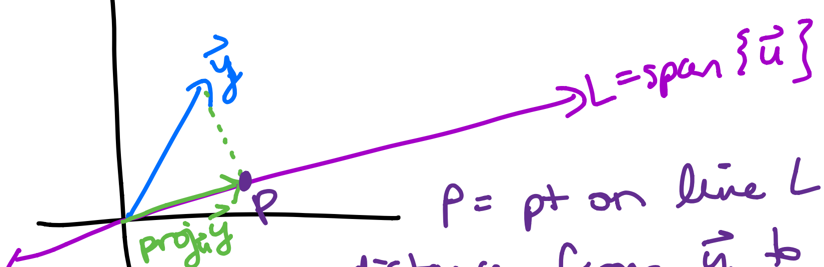
$\iff 0 = \vec{y} \cdot \vec{u} - \alpha \vec{u} \cdot \vec{u}$

$\iff \alpha = \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \implies \boxed{\alpha \vec{u} = \dfrac{\vec{y} \cdot \vec{u}}{\|\vec{u}\|^2} \vec{u}}$ this is orthog projection of $\vec{y}$ onto $\vec{u}$

$$\boxed{\operatorname{proj}_{\vec{u}} \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}}$$

Example: Let $\mathbf{y} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$. Then write $\mathbf{y}$ as the sum of two orthogonal vectors, one in span$\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

Geometrically:



$P =$ pt on line L that's closest to $\vec{y}$.

distance from $\vec{y}$ to L is $\|\vec{y} - \operatorname{proj}_L \vec{y}\|$.

Orthonormal Sets

Definition 3 *A set of vectors* $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ *in* $\mathbb{R}^n$ *is an orthonormal set if it is an orthogonal set of unit vectors. If* $W = span\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$*, then* $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ *is an orthonormal basis of* W *since the set of* p *vectors is automatically linearly independent by Theorem 4.*

Example: Show that $V = \left\{ \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$ is an orthonormal set.

① check each vector is unit vector. and ② check each pair of vectors are orthogonal.

Theorem 6 *An* $m \times n$ *matrix* U *has orthonormal columns if and only if* $U^T U = I$.

notice U^T kind of acts like an inverse even though U^{-1} doesn't exist if $m \neq n$

Theorem 7 *Let* U *be an* $m \times n$ *matrix with orthonormal columns, and let* $\mathbf{x}$ *and* $\mathbf{y}$ *be in* $\mathbb{R}^n$. *Then*

(a) $\| U\mathbf{x} \| = \| \mathbf{x} \|$

(b) $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ pf $(U\vec{x}) \cdot (U\vec{y}) = (U\vec{x})^T U\vec{y} = \vec{x}^T U^T U \vec{y} = \vec{x}^T \vec{y} = \vec{x} \cdot \vec{y}$.

(c) $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ *if and only if* $\mathbf{x} \cdot \mathbf{y} = 0$ because $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$.

Example: Does U have orthonormal columns? (Hint: compute $U^T U$.)

$$U = \begin{bmatrix} -2/3 & 1/\sqrt{5} \\ 1/3 & 2/\sqrt{5} \\ 2/3 & 0 \end{bmatrix}$$

Definition 4 *An orthogonal matrix is a square invertible matrix* U *such the* $U^{-1} = U^T$. *An orthogonal matrix has orthonormal columns and orthonormal rows.*

So if U is square matrix and its columns + rows are orthonormal, then $U^T U = I \Rightarrow U^T = U^{-1}$.