1. Definitions and Concepts

A) (10 points) Complete the following definitions **by using complete sentences!**

A matrix A is called symmetric if

A number λ is called an eigenvalue of A if

A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is called an orthogonal set if

The function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if

If A is a matrix then $NulA$ is

B) (2 points) Explain why you know that the matrix

$$B = \begin{bmatrix} 1 & -5 & -7 & 16 \\ -5 & 6 & 1 & -1 \\ -7 & 1 & 0 & 5 \\ 16 & -1 & 5 & 2 \end{bmatrix}$$

is diagonalizable.

C) (4 points) Suppose that A is a 10×5 matrix with three pivots. Then what are:

Rank A

dim Col A

dim Nul A

dim Row A

D) (4 points) Write down matrices A and B in reduced echelon form with the following properties:

i) The column space of A is 2 dimensional

ii) The null space of B is a 2 dimensional plane in $\mathbb{R}^5$.

2) Computations and Interpretations

A) (6 points) Row reduce the following matrix to reduced echelon form

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}.$$

Then determine if the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has a solution.

B) (6 points) Compute a least squares solution $\hat{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}.$$

C) (8 points) Given that A and B are row equivalent, **write down bases for Col A, Row A, and Nul A.** Please be careful and make sure your answer is a **basis**, not some other description of the space.

$$A = \begin{pmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

3. Diagonalization and Similarity

A) (10 points) True of False (no explanation needed)

i) Invertible matrices are always diagonalizable.

ii) Symmetric matrices are the only matrices that can be orthogonally diagonalized.

iii) If a real matrix has a complex eigenvalue then it is not diagonalizable.

iv) If the characteristic polynomial of A is $\lambda^2(\lambda - 2)$ then A is not diagonalizable.

v) The matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is diagonalizable.

B) Consider the following symmetric matrix:

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

The eigenvalues of A are -2 and 7.

i) (4 points) To save computation, you are given that $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ are all eigenvectors of A. Use this information to write down bases for the eigenspaces coming from $\lambda = -2$ and $\lambda = 7$.

ii) (2 points) Using the information given and what you found from i) write down the characteristic polynomial of A in factored form.

iii) (4 points) Write down any matrix P, and a diagonal matrix D such that $A = PDP^{-1}$. Is it possible to choose P so that $P^{-1} = P^T$? If yes, explain how you could find such a P (but don't do the computation)

4. Let $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

A) (4 points) Write down the corresponding quadratic form $\mathbf{x}^T A\mathbf{x}$

B) (4 points) Explain why the matrix from part A is diagonalizable. Say what this means we could do to the quadratic form to make it simpler.

C) (4 points) What is the determinant of the matrix A?

D) (4 points) Consider the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\mathbf{x} \mapsto A\mathbf{x}$.
Is T 1-1? Why or why not?
Is T onto? Why or why not?

E) (4 points) Is the vector $\mathbf{e}_3$ an eigenvector of A? Why or why not?

5.

A) (6 points) Write down a matrix A that rotates $\mathbb{R}^2$ through an angle of 90 degrees counter-clockwise about the origin. Then write a matrix B that reflects $\mathbb{R}^2$ across the line $y = -x$.

B) (4 points) Is the set of all polynomials $\{2x^2 + b \mid b \in \mathbb{R}\}$ a subspace of $\mathbb{P}_2$? Why or why not?

C) (4 points) Suppose that V is an abstract vector space with basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and T is a linear transformation that satisfies

$$T(\mathbf{b}_1) = \mathbf{b}_2 + 2\mathbf{b}_3$$

$$T(\mathbf{b}_2) = -\mathbf{b}_1 + \mathbf{b}_3$$

$$T(\mathbf{b}_2 + \mathbf{b}_3) = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$$

write down the matrix for T with respect to the basis $\mathcal{B}$.

D) (6 points) Suppose A is a real 4×4 matrix. You know the following:

- Nul A is 2 dimensional
- $A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$
- $\det(A - 4I) = 0$.

What are the eigenvalues of A?

Is A diagonalizable? Why or why not?

If possible, write down the characteristic polynomial of A

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Date: April 27, 2017

Final Exam

(200 points)

Show all of your work. You may not use a calculator.

1. **Examples**

Give examples of the following.

(a) **(5 points)** A matrix A whose nullspace is a 3-dimensional subspace of $\mathbb{R}^5$.

(b) **(5 points)** A matrix A which defines a negative-definite quadratic form on $\mathbb{R}^4$.

(c) **(5 points)** A matrix A such that $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ is a solution of $A\mathbf{x} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$.

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(d) **(5 points)** A matrix A such that the volume of the parallelepiped in $\mathbb{R}^3$ deterined by the columns of A is 12.

(e) **(5 points)** A matrix A such that $\text{rank}A = 2$ and $\det A = 0$.

(f) **(5 points)** A matrix A with a two-dimensional eigenspace.

(g) **(5 points)** A vector space V whose objects are functions with $\dim V = 5$.

(h) **(5 points)** An orthonormal basis for $\mathbb{R}^2$ which is **not** the standard basis $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

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2. **Computations**

(a) **(10 points)** Solve the linear system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= -5 \\ x_1 + x_2 - x_3 &= -1 \end{aligned}$$

Write your solution in parametric vector form.

(b) **(10 points)** Find a least squares solution $\hat{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$ for $A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$.

(c) **(10 points)** The matrices A and B below are row equivalent. Use this information to write down bases for ColA, RowA, and NulA.

$$A = \begin{pmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & 6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -2 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(d) **(10 points)** Consider the subspace $W = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 9 \\ 1 \\ 5 \end{pmatrix} \right\}$ of $\mathbb{R}^3$. Compute an orthogonal basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ of W.

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5. **Singular Value Decomposition**

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & 1 \end{pmatrix}$. An orthogonal diagonalization of $A^T A$ is given by

$$\begin{pmatrix} 2 & 4 & 0 \\ 4 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 10 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \end{pmatrix}.$$

(a) **(13 points)** Use the information above to compute the singular value decomposition $A = U\Sigma V^T$. (*Hint: You can immediately write down V and Σ using information given above.*)

(b) **(7 points)** What are the singular values of $A^T = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ -1 & 1 \end{pmatrix}$? What does this tell you about the rank of A^T? (*Hint: You don't have to do any calculations to do this problem.*)

6. **Linear Transformations**

(a) **(8 points)** Write down a matrix A that rotates $\mathbb{R}^2$ ninety degrees counterclockwise about the origin. Write down a matrix B that reflects $\mathbb{R}^2$ across the line $y = -x$.

$A =$ $\qquad\qquad\qquad$ $B =$

(b) Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation defined by $T(p(t)) = \begin{pmatrix} p(-1) \\ p(2) \end{pmatrix}$.

i. **(5 points)** Find the matrix M for T relative to the standard basis $\mathcal{E} = \{1, t, t^2\}$ of $\mathbb{P}_2$.

ii. **(5 points)** Compute a basis for Nul$M$.

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iii. **(5 points)** Use your answer in part ii. to write down a basis for ker$T$. (*Hint: You can check your answer to this problem by applying the transformation T to your basis.*)

7. **Quadratic Forms**

Let $A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$.

(a) **(4 points)** Write down the corresponding quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

(b) **(4 points)** Explain why the matrix A is diagonalizable.

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(c) **(4 points)** What is the determinant of A?

(d) **(4 points)** Two eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = 2$. What are the minimum and maximum values of $Q(\mathbf{x})$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$?

(e) **(4 points)** Consider the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\mathbf{x} \mapsto A\mathbf{x}$. Is T one-to-one? Why or why not? Is T onto? Why or why not?

8. **Invertible Matrices**

(10 points) Which of the following matrices are invertible? Circle all the apply. You do not need to justify your choice.

(a) $A = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix}$. (Note that the columns of A are orthonormal.)

(b) A matrix with a trivial nullspace.

(c) A matrix A with $\det A = 12$.

(d) A 4×4 matrix with rank 2.

(e) $A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(f) A 3×2 matrix that represents a one-to-one linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$.

(g) The matrix $(A - 2I)$, where 2 is an eigenvalue of the matrix A.

(h) A matrix with a 2-dimensional nullspace.

(i) An $n \times n$ matrix with linearly independent columns .

(j) A 3×3 matrix A such that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^3$.

(c) Consider the subspace $W = \text{span}\left\{\begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}9\\1\\5\end{pmatrix}\right\}$ of $\mathbb{R}^3$.

- **(3 points)** Compute an orthogonal basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ of W.

- **(3 points)** Let $\mathbf{y} = \begin{pmatrix}-9\\2\\4\end{pmatrix}$. Find the closest point in W to $\mathbf{y}$.

5. (12 points) **Fitting data with a line**
Suppose you have some data with three points

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

(a) Use the following steps to find the line that best approximates these points.

- **(4 points)** Write down the three equations in two variables m and c that would have to be satisfied for each of these points to go through the line $x_2 = mx_1 + c$.

- **(4 points)** Write down the matrix equation that corresponds to the linear system from part (a).

- **(4 points)** Find the equation for the line that is the closest possible line to these three points; that is, find the least squares solution.