1. Definitions and Concepts

A) (10 points) Complete the following definitions **by using complete sentences!**

A matrix A is called symmetric if

$A^T = A$

A number λ is called an eigenvalue of A if

A $n \times n$

$\exists \vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \lambda \vec{x}$

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is called an orthogonal set if

$\vec{v_i}^T \vec{v_j} = 0 \quad \forall\, i,j = 1, \ldots, n \ \& \ i \neq j$

The function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if

For $\vec{u}, \vec{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$

① $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

and ② $T(c\vec{u}) = cT(\vec{u})$

If A is a matrix then $NulA$ is (A $m \times n$)

vector space containing all vectors $\vec{x} \in \mathbb{R}^n$

s.t. $A\vec{x} = \vec{0}$.

B) (2 points) Explain why you know that the matrix

$$B = \begin{bmatrix} 1 & -5 & -7 & 16 \\ -5 & 6 & 1 & -1 \\ -7 & 1 & 0 & 5 \\ 16 & -1 & 5 & 2 \end{bmatrix}$$

is diagonalizable.

$B = B^T$, i.e. it's symmetric

C) (4 points) Suppose that A is a 10×5 matrix with three pivots. Then what are:

5 columns, 3 pivot columns

Rank A = 3

dim Col A = 3

dim Nul A = 2

dim Row A = 3

D) (4 points) Write down matrices A and B in reduced echelon form with the following properties:

REF

i) The column space of A is 2 dimensional

ex $A = \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 4 & 6 \end{bmatrix}$

ii) The null space of B is a 2 dimensional plane in $\mathbb{R}^5$.

ex $B = \begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix}$

$\Rightarrow$ Nul $B = \text{span}\left\{ \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$x_1 = -2x_4 + x_5$

$x_2 = -x_5$

$x_3 = -3x_4$

x_4, x_5 free

$\Rightarrow \vec{x} = \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_5$

2) Computations and Interpretations

A) (6 points) Row reduce the following matrix to reduced echelon form

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}.$$

Then determine if the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has a solution.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 4 & 5 & 6 & 7 & 2 \\ 6 & 7 & 8 & 9 & 3 \end{bmatrix} \equiv \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & -3 & -6 & -9 & -2 \\ 0 & -5 & -10 & -15 & -3 \end{bmatrix} \equiv \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & 2 & 3 & 2/3 \\ 0 & 1 & 2 & 3 & 3/5 \end{bmatrix}$$

(-6) (-4) $\quad (-\tfrac{1}{3}) \quad (-\tfrac{1}{5}) \quad (-1)$

$$\equiv \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & 2 & 3 & 2/3 \\ 0 & 0 & 0 & 0 & -1/15 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -1 & -2 & -1/3 \\ 0 & 1 & 2 & 3 & 2/3 \\ 0 & 0 & 0 & 0 & -1/15 \end{bmatrix}$$

(-2)

$$\Rightarrow \boxed{\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

bottom row of augmented matrix $\Rightarrow$ $\boxed{\text{N.S. for } A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$

B) (6 points) Compute a least squares solution $\hat{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}.$$

if unique solution exists, then it is $\hat{x} = (A^TA)^{-1}A^T\vec{b}$

$$A^TA = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1+1+4 & -2-2+10 \\ -2-2+10 & 4+4+9+25 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{6(42)-6(6)} \begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix} = \frac{1}{6}\left(\frac{1}{36}\right) \begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 7 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\hat{x} = \frac{1}{36} \begin{bmatrix} 7 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 9 & -9 & -3 & 9 \\ -3 & 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$$

$$\hat{x} = \frac{1}{36} \begin{bmatrix} 48 \\ -12 \end{bmatrix} = \boxed{\begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}}$$

C) (8 points) Given that A and B are row equivalent, **write down bases for Col A, Row A, and Nul A.** Please be careful and make sure your answer is a **basis**, not some other description of the space.

$$A = \begin{pmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \begin{matrix} \\ \frac{1}{2} \\ \cdot\frac{1}{5} \\ \\ \end{matrix}$$

$$\text{RREF}(A) = \begin{bmatrix} 1 & -3 & 0 & 5 & 0 \\ 0 & 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

basis for $\text{col}(A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix} \right\}$

basis for $\text{Nul}(A) = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3/2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

basis for $\text{row}(A) = \left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Null Space computations:

$x_1 = 3x_2 - 5x_4$

$x_3 = \frac{3}{2} x_4$

$x_5 = 0$

x_2, x_4 free

$$\Rightarrow \vec{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -5 \\ 3/2 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_4$$

3. Diagonalization and Similarity

A) (10 points) True of False (no explanation needed)

i) Invertible matrices are always diagonalizable.

False (a matrix can be invertible but not have eigenspace w/ dimension n)

ii) Symmetric matrices are the only matrices that can be orthogonally diagonalized.

True (this is an iff statement)

iii) If a real matrix has a complex eigenvalue then it is not diagonalizable.

False (It depends on how many eigenvectors the matrix has.)

iv) If the characteristic polynomial of A is $\lambda^2(\lambda - 2)$ then A is not diagonalizable.

False (A may or may not be diagonalizable. It depends on how many eigenvectors $\lambda = 0$ has.)

v) The matrix $\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A}$ is diagonalizable. True

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$\Rightarrow \exists$ 2 distinct eigenvalues

$\Rightarrow \exists$ 2 eigenvectors

$\Rightarrow \exists$ a basis for $\mathbb{R}^2$ from eigenvectors

$\Rightarrow A$ is diagonalizable.

B) Consider the following symmetric matrix:

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

The eigenvalues of A are -2 and 7.

i) (4 points) To save computation, you are given that $\overset{\lambda=-2}{\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}}, \overset{\lambda=7}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \overset{\lambda=7}{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}, \overset{\lambda=7}{\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}$ are all eigenvectors of A. Use this information to write down bases for the eigenspaces coming from $\lambda = -2$ and $\lambda = 7$.

this vector lin. combo of 2 previous vectors

$$\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} \Rightarrow \lambda=-2$$

$$\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -14 \\ 0 \end{bmatrix} \Rightarrow \lambda=7$$

$$\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 7 \end{bmatrix} \Rightarrow \lambda=7$$

$$\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ -14 \\ 7 \end{bmatrix} \Rightarrow \lambda=7$$

$\Rightarrow$ basis for $E_{\lambda=-2} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\}$ basis for $E_{\lambda=7} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\}$

ii) (2 points) Using the information given and what you found from i) write down the characteristic polynomial of A in factored form.

$\det(A-\lambda I) = k(\lambda+2)(\lambda-7)^2$ for some constant $k \in \mathbb{R}$

iii) (4 points) Write down any matrix P, and a diagonal matrix D such that $A = PDP^{-1}$. Is it possible to choose P so that $P^{-1} = P^T$? If yes, explain how you could find such a P (but don't do the computation)

$$P = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -2 \\ -2 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Yes, we can choose P s.t. $P^{-1} = P^T$, but we'd have to do Gram-Schmidt process on vectors for $E_{\lambda=7}$ in order to get orthogonal basis for eigenspace.

4. Let $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

A) (4 points) Write down the corresponding quadratic form $\mathbf{x}^T A\mathbf{x}$

$$Q(\vec{x}) = \vec{x}^T A \vec{x} = 3x_1^2 + 2x_2^2 + 0x_3^2 + 4x_1x_2 + 0x_1x_3 + 2x_2x_3$$

$$= \boxed{3x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2x_3}$$

B) (4 points) Explain why the matrix from part A is diagonalizable. Say what this means we could do to the quadratic form to make it simpler.

- it's symmetric $\Rightarrow$ it's orthogonally diagonalizable.
- This means we can make a transformation $\vec{x} = P\vec{y}$ where P is orthogonal s.t. $Q(\vec{y}) = \vec{y}^T D \vec{y}$ w/ D = diagonal matrix.

$\Rightarrow Q(\vec{y})$ has no cross terms.

C) (4 points) What is the determinant of the matrix A?

$$\det(A) = \begin{vmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1 \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= -1(3-0) = -3$$

D) (4 points) Consider the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\mathbf{x} \mapsto A\mathbf{x}$.
Is T 1-1? Why or why not?
Is T onto? Why or why not?

$\det(A) \neq 0 \Rightarrow A^{-1}$ exists

$\Rightarrow T$ (represented by A) is both onto and 1-1.

E) (4 points) Is the vector $\mathbf{e}_3$ an eigenvector of A? Why or why not?

$$A\vec{e}_3 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \neq \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

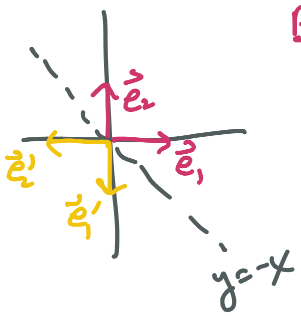
$\Rightarrow$ no $\vec{e}_3$ is not an eigenvector of A.

5.

$\Rightarrow A$ is 2×2

A) (6 points) Write down a matrix A that rotates $\mathbb{R}^2$ through an angle of 90 degrees counter-clockwise about the origin. Then write a matrix B that reflects $\mathbb{R}^2$ across the line $y = -x$.

2×2 generic rotation matrix $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\quad \theta = 90° \Rightarrow \boxed{A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$



$B\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$B\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\Rightarrow \boxed{B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}$

B) (4 points) Is the set of all polynomials $H = \{2x^2 + b \mid b \in \mathbb{R}\}$ a subspace of $\mathbb{P}_2$? Why or why not?

NO

$\mathbb{P}_2 = \{ax^2 + cx + d \mid a, c, d \in \mathbb{R}\}$

any of these arguments will work

(a) is $0 \in H$?

let $w(x) = 2x^2 + b$ then $w \in H$. If we let $b = 0$, then we still have $2x^2 \Rightarrow 0 \notin H$

$\Rightarrow H$ is NOT a subspace of $\mathbb{P}_2$.

(b) $w_1 = 2x^2 + b_1$ $\quad w_2 = 2x^2 + b_2$, $\quad w_1, w_2 \in H$

$\Rightarrow w_1 + w_2 = 2x^2 + b_1 + 2x^2 + b_2 = 4x^2 + (b_1 + b_2) \notin H$

$\Rightarrow H$ is NOT a subspace of $\mathbb{P}_2$.

(c) $w = 2x^2 + b$, $\quad c \in \mathbb{R}$

$\Rightarrow cw = 2cx^2 + bc \notin H \Rightarrow H$ is NOT a subspace of $\mathbb{P}_2$.

(unless we restrict $c = 1$, which negates $c \in \mathbb{R}$ statement)

C) (4 points) Suppose that V is an abstract vector space with basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and T is a linear transformation that satisfies

$$T(\mathbf{b}_1) = \mathbf{b}_2 + 2\mathbf{b}_3$$

$$T(\mathbf{b}_2) = -\mathbf{b}_1 + \mathbf{b}_3$$

$$T(\mathbf{b}_2 + \mathbf{b}_3) = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$$

T linear $\Rightarrow T(\vec{b}_2) + T(\vec{b}_3) = \vec{b}_1 + \vec{b}_2 + \vec{b}_3$

write down the matrix for T with respect to the basis $\mathcal{B}$. $\Leftrightarrow -\vec{b}_1 + \vec{b}_3 + T(\vec{b}_3) = \vec{b}_1 + \vec{b}_2 + \vec{b}_3$

I'll call it A

$T(\vec{b}_3) = 2\vec{b}_1 + \vec{b}_2$

$$\boxed{A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}}$$

D) (6 points) Suppose A is a real 4×4 matrix. You know the following:

- Nul A is 2 dimensional $\Rightarrow \text{rank } A = 2 \Rightarrow A^{-1}$ DNE $\Rightarrow \lambda_1 = 0$
- $A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ $\Rightarrow \lambda_2 = 1$ w/ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$
- $\det(A - 4I) = 0.$ $\Rightarrow \lambda_3 = 4$

note: $\text{Nul} A = \text{Nul}(A - 0I)$
$\Rightarrow \dim(\text{Nul} A) = 2$ means
that $\dim(E_{\lambda=0}) = 2$
$\Rightarrow \exists$ 2 eigenvectors for $\lambda_1 = 0$

What are the eigenvalues of A?

$\boxed{\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 4}$

Is A diagonalizable? Why or why not?

we have λ_1 w/ 2-dim eigenspace
λ_2 w/ 1-dim "
λ_3 " " "

$\Rightarrow$ eigenspace is 4d & A is 4×4

$\Rightarrow$ $\boxed{A \text{ IS diagonalizable}}$

If possible, write down the characteristic polynomial of A

$\det(A - \lambda I) = \boxed{k\lambda^2(\lambda - 1)(\lambda - 4) \text{ for some } k = \text{constant} \in \mathbb{R}.}$

MATH 2270-001 Spring 2017
Instructor: Anna Romanova
Date: April 27, 2017

Name: ______________________

Final Exam

(200 points)

Show all of your work. You may not use a calculator.

1. **Examples**

Give examples of the following.

(a) **(5 points)** A matrix A whose nullspace is a 3-dimensional subspace of $\mathbb{R}^5$.

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank$(A) = 2$
$\Rightarrow$ dim (Nul A) = 3
and Nul $A \subset \mathbb{R}^5$

(b) **(5 points)** A matrix A which defines a negative-definite quadratic form on $\mathbb{R}^4$.

means all eigenvalues are negative

$$A = \begin{bmatrix} -1 & & & 0 \\ & -2 & & \\ & & -3 & \\ 0 & & & -4 \end{bmatrix}$$

(c) **(5 points)** A matrix A such that $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ is a solution of $A\mathbf{x} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$.

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

we need $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ to be in column space of A

i.e. $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ is linear combo of cols of A.

$\Rightarrow$ pick two cols of A to be $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

we can choose A 2

$$\Rightarrow \begin{bmatrix} 0 & -1 & a \\ 5 & 0 & b \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

$\Leftrightarrow \begin{cases} -1 - 3a = -1 \Leftrightarrow a = 0 \\ 10 - 3b = 5 \Leftrightarrow -3b = -5 \Leftrightarrow b = 5/3 \end{cases}$

$\Rightarrow$ let $A = \begin{bmatrix} 0 & -1 & 0 \\ 5 & 0 & 5/3 \end{bmatrix}$

10

(d) **(5 points)** A matrix A such that the volume of the parallelepiped in $\mathbb{R}^3$ determined by the columns of A is 12.

i.e. $12 = |\det A|$

let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ then $\det A = -12$ & $|\det A| = 12$

(e) **(5 points)** A matrix A such that rank$A = 2$ and det$A = 0$.

$\Rightarrow A$ is $n \times n$, s.t. $n \geq 2$

$\det A = 0$
$\Rightarrow A$ is square + A^{-1} DNE

$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 5 & 7 \\ 3 & 1 & 4 \end{bmatrix}$

this column is linear combo of 1st 2 columns $\Rightarrow A^{-1}$ DNE

(f) **(5 points)** A matrix A with a two-dimensional eigenspace.

easy answer $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

somewhat more creative answer $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$

(g) **(5 points)** A vector space V whose objects are functions with dim$V = 5$.

$V = \text{span}\{1, x, x^2, x^3, x^4\} = \mathbb{P}_4$

or $V = \text{span}\{|x|, \sin x, e^x, \ln x, x^2\}$

(h) **(5 points)** An orthonormal basis for $\mathbb{R}^2$ which is **not** the standard basis $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

orthog basis $= \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\} \Rightarrow$ orthonormal basis $= \left\{ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \right\}$

11

2. **Computations**

(a) **(10 points)** Solve the linear system

$$x_1 + 2x_2 - 3x_3 = -5$$
$$x_1 + x_2 - x_3 = -1$$

Write your solution in parametric vector form.

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$(-1)\left[\begin{array}{ccc|c} 1 & 2 & -3 & -5 \\ 1 & 1 & -1 & -1 \end{array}\right] \equiv \left[\begin{array}{ccc|c} 1 & 2 & -3 & -5 \\ 0 & -1 & 2 & 4 \end{array}\right] \equiv \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & -4 \end{array}\right]$$

$$\iff \boxed{\begin{aligned} x_1 &= 3 - x_3 \\ x_2 &= -4 + 2x_3 \\ x_3 &\text{ free} \end{aligned} \iff \vec{x} = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} x_3}$$

(b) **(10 points)** Find a least squares solution $\hat{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$ for $A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$. $\iff$ solve $A^T A\vec{x} = A^T\vec{b}$

$$A^T A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix} \Rightarrow (A^T A)^{-1} = \frac{1}{20-9}\begin{bmatrix} 2 & -3 \\ -3 & 10 \end{bmatrix}$$

$$\Rightarrow \vec{x} = (A^T A)^{-1} A^T \vec{b} = \frac{1}{11}\begin{bmatrix} 2 & -3 \\ -3 & 10 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 2 & 3 & -3 \\ -3 & 1 & 10 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 11 \\ -11 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

12

(c) **(10 points)** The matrices A and B below are row equivalent. Use this information to write down bases for ColA, RowA, and NulA.

$$A = \begin{pmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & 6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -2 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(this is same problem as C on page 4)

(d) **(10 points)** Consider the subspace $W = \text{span}\left\{ \underset{\vec{\omega}_1}{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}, \underset{\vec{\omega}_2}{\begin{pmatrix} 9 \\ 1 \\ 5 \end{pmatrix}} \right\}$ of $\mathbb{R}^3$. Compute an orthogonal basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ of W.

$$\vec{v}_1 = \vec{\omega}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \vec{\omega}_2 - \frac{\vec{\omega}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix} 9 \\ 1 \\ 5 \end{bmatrix} - \frac{18+1+5}{4+1+1}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \right\}}$$

5. **Singular Value Decomposition**

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & 1 \end{pmatrix}$. An orthogonal diagonalization of $A^T A$ is given by

$$A^TA = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \overset{P}{\begin{pmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{pmatrix}} \overset{D}{\begin{pmatrix} 10 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \overset{P^T}{\begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \end{pmatrix}}.$$

(a) **(13 points)** Use the information above to compute the singular value decomposition $A = U\Sigma V^T$. (*Hint: You can immediately write down V and Σ using information given above.*)

$\lambda_1 = 10,\ \lambda_2 = 2,\ \lambda_3 = 0$ for A^TA

$\Rightarrow \sigma_1 = \sqrt{10},\ \sigma_2 = \sqrt{2},\ \sigma_3 = 0$ (singular values of A)

$$\Rightarrow \boxed{\Sigma = \begin{bmatrix} \sqrt{10} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}}$$

and $$\boxed{V = \begin{bmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}}$$

remember

U orthogonal (& square) 2×2

$U^TU = I$

and V is 3×3 orthogonal

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 5/\sqrt{5} \\ 5/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} \sqrt{5} \\ \sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

(b) **(7 points)** What are the singular values of $A^T = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ -1 & 1 \end{pmatrix}$? What does this tell you about the rank of A^T? (*Hint: You don't have to do any calculations to do this problem.*)

$A = U\Sigma V^T \Rightarrow A^T = (U\Sigma V^T)^T = V\Sigma^T U^T$

and $\Sigma = \begin{bmatrix} \sqrt{10} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \Rightarrow \Sigma^T = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \boxed{\sigma_1 = \sqrt{10},\ \sigma_2 = \sqrt{2}}$ for A^T SVD.

$\text{rank}(A^T) = \#$ nonzero singular values of A^T $\Rightarrow \boxed{\text{rank}(A^T) = 2}$

14

6. **Linear Transformations**

(a) **(8 points)** Write down a matrix A that rotates $\mathbb{R}^2$ ninety degrees counterclockwise about the origin. Write down a matrix B that reflects $\mathbb{R}^2$ across the line $y = -x$.

this is same problem as #5 on page 8

$A =$ $B =$

(b) Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation defined by $T(p(t)) = \begin{pmatrix} p(-1) \\ p(2) \end{pmatrix}$.

i. **(5 points)** Find the matrix M for T relative to the standard basis $\mathcal{E} = \{1, t, t^2\}$ of $\mathbb{P}_2$.

$$M = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$T(p(t)) = \begin{bmatrix} p(-1) \\ p(2) \end{bmatrix}$$

$$\Rightarrow T(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T(t) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$T(t^2) = \begin{bmatrix} (-1)^2 \\ (2)^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

ii. **(5 points)** Compute a basis for Nul$M$.

$$\begin{array}{c} (-1) \\ \leftarrow \end{array} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \equiv \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \equiv \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\equiv \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow$$

Null space

$$\left.\begin{array}{l} x_1 = -2x_3 \\ x_2 = -x_3 \\ x_3 \text{ free} \end{array}\right\} \Rightarrow \vec{x} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} x_3$$

basis for

$$\Rightarrow \text{Nul}(M) = \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

iii. **(5 points)** Use your answer in part ii. to write down a basis for ker$T$. (*Hint: You can check your answer to this problem by applying the transformation T to your basis.*)

$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$ corresponds to $\boxed{\{-2-t+t^2\}}$

check $T(-2-t+t^2) = \begin{bmatrix} -2-(-1)+(-1)^2 \\ -2-2+2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

7. **Quadratic Forms**

Let $A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$.

(a) **(4 points)** Write down the corresponding quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

$\boxed{Q(\vec{x}) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 0x_1x_3 + 4x_2x_3}$

(b) **(4 points)** Explain why the matrix A is diagonalizable.

$\boxed{A \text{ is symmetric}}$

needed later: $\det(A-\lambda I) = \det\begin{bmatrix} 3-\lambda & 2 & 0 \\ 2 & 2-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{bmatrix}$

$= (3-\lambda)[(2-\lambda)(1-\lambda)-4] - 2[2(1-\lambda)-0]$

$= -(\lambda^3 - 6\lambda^2 + 3\lambda + 10)$

$= -(\lambda-5)(\lambda-2)(\lambda+1)$

$\Rightarrow$ eigenvalues are $\lambda_1 = 5,\ \lambda_2 = 2,\ \lambda_3 = -1$

16

(c) **(4 points)** What is the determinant of A?

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\det A = 3 \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} = 3(2-4) - 2(2-0)$$
$$= 3(-2) - 4 = -6 - 4 = -10$$

(d) **(4 points)** Two eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = 2$. What are the minimum and maximum values of $Q(\mathbf{x})$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$?

(see work above, on last page)

$\lambda_1 = 5$, $\lambda_2 = 2$, $\lambda_3 = -1 \implies$

$$\boxed{\begin{array}{l} \min Q(\vec{x}) \text{ s.t. } \|\vec{x}\|^2 = 1 \quad \text{is} \quad -1 \\ \max Q(\vec{x}) \text{ s.t. } \|\vec{x}\|^2 = 1 \quad \text{is} \quad 5 \end{array}}$$

(e) **(4 points)** Consider the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\mathbf{x} \mapsto A\mathbf{x}$. Is T one-to-one? Why or why not? Is T onto? Why or why not?

Since $\det A \neq 0$, then A^{-1} exists

$\implies$ $\boxed{T \text{ is both 1-1 and onto.}}$

17

8. **Invertible Matrices**

(10 points) Which of the following matrices are invertible? Circle all the apply. You do not need to justify your choice.

(a) $A = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix}$. (Note that the columns of A are orthonormal.)

if columns are orthonormal, then columns of A are lin. indep $\Rightarrow A^{-1}$ exists

(b) A matrix with a trivial nullspace.

$\Rightarrow \text{rank}(A) = n$ for A $n \times n$

(c) A matrix A with $\det A = 12$.

$\det A \neq 0 \Rightarrow A^{-1}$ exists

(d) A 4×4 matrix with rank 2.

no, columns of A are not lin. indep.

(e) $A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

$\Rightarrow \det A = 1(3)(5)(1) = 15 \neq 0$

(f) A 3×2 matrix that represents a one-to-one linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$.

no, A^{-1} can only exist if A is square

(g) The matrix $(A - 2I)$, where 2 is an eigenvalue of the matrix A.

$\lambda = 2 \Rightarrow (A-2I)\vec{x} = \vec{0}$ has nonzero solution $\Rightarrow (A-2I)^{-1}$ DNE

(h) A matrix with a 2-dimensional nullspace.

no, the null space of invertible matrix has to be 0-dimensional

(i) An $n \times n$ matrix with linearly independent columns .

(j) A 3×3 matrix A such that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^3$.

no, $A\vec{x} = \vec{b}$ would have to have exactly one solution $\forall \vec{b} \in \mathbb{R}^3$.

18

(c) Consider the subspace $W = \text{span}\left\{ \overset{\vec{w}_1}{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}, \overset{\vec{w}_2}{\begin{pmatrix} 9 \\ 1 \\ 5 \end{pmatrix}} \right\}$ of $\mathbb{R}^3$.

- **(3 points)** Compute an orthogonal basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ of W.

$$\vec{u}_1 = \vec{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \vec{w}_2 - \frac{\vec{w}_2 \cdot \vec{u}_1}{\|\vec{u}_1\|^2}\vec{u}_1 = \begin{bmatrix} 9 \\ 1 \\ 5 \end{bmatrix} - \frac{18+1+5}{4+1+1}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

$$\boxed{\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \right\}}$$

- **(3 points)** Let $\mathbf{y} = \begin{pmatrix} -9 \\ 2 \\ 4 \end{pmatrix}$. Find the closest point in W to $\mathbf{y}$.

$\hat{y}$ is closest to $\vec{y}$ s.t. $\hat{y} = \text{proj}_W \vec{y}$

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\|\vec{u}_1\|}\vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\|\vec{u}_2\|^2}\vec{u}_2 = \frac{-18+2+4}{4+1+1}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \frac{-9-6+4}{1+9+1}\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -2 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix}}$$

5. (12 points) **Fitting data with a line**
Suppose you have some data with three points

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \overset{①}{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}, \overset{②}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \overset{③}{\begin{pmatrix} 0 \\ 4 \end{pmatrix}}.$$

(a) Use the following steps to find the line that best approximates these points.

- **(4 points)** Write down the three equations in two variables m and c that would have to be satisfied for each of these points to go through the line $x_2 = mx_1 + c$.

$$\left.\begin{aligned} ①\ \ 0 &= m(2)+c \\ ②\ \ 1 &= m(1)+c \\ ③\ \ 4 &= m(0)+c \end{aligned}\right\} \iff \boxed{\left\{\begin{aligned} 2m+c&=0 \\ m+c&=1 \\ c&=4 \end{aligned}\right.}$$

- **(4 points)** Write down the matrix equation that corresponds to the linear system from part (a).

$$\boxed{\begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}}$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

- **(4 points)** Find the equation for the line that is the closest possible line to these three points; that is, find the least squares solution.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

least squares solns: $A^T A \hat{x} = A^T \vec{b}$
$\hat{x}$ satisfy

$$A^T A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}$$

$$\Rightarrow (A^T A)^{-1} = \frac{1}{15-9} \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix}$$

$$\Rightarrow \hat{x} = (A^T A)^{-1} A^T \vec{b} = \frac{1}{6} \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

$$\hat{x} = \frac{1}{6} \begin{bmatrix} 3 & 0 & -3 \\ -1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -12 \\ 22 \end{bmatrix} = \begin{bmatrix} -2 \\ 11/3 \end{bmatrix} = \begin{bmatrix} m \\ c \end{bmatrix}$$

$\Rightarrow$ best fit line is $x_2 = m x_1 + c$

i.e. $\boxed{x_2 = -2 x_1 + 11/3}$