Math2270

Overview of "Factorizations"

Section 2.5: **LU Factorization**

A is an mxn matrix.

Then, $A=LU$.

U is an mxn matrix that is created by finding an echelon form of A.

L is a unit upper triangular nxn matrix, such that the diagonal entries are all one.

One way to find the LU factorization is to augment A with the identity matrix and the do row operations. You transform $[A\,I]$ to $[U\,L^{-1}]$ and then you can figure out L from there.

Do all matrices have LU factorization? No.

Is the LU factorization unique if it exists? No. We can choose a different REF of A to be U.

Section 5.3: **Diagonalization of A**

A is nxn matrix.

A is diagonalizable means that A can be written as $A=P\,D\,P^{-1}$ for some nxn invertible P matrix and D is an nxn diagonal matrix.

Fact: A is diagonalizable iff there exists an eigenvector basis of $\mathbb{R}^n$. (In other words, if A has exactly n eigenvectors, then A is diagonalizable.)

To find P and D , find the eigenvalues of A and those eigenvalues become the diagonal entries of D . And the eigenvectors, in order corresponding to the order of the eigenvalues in D , are the column vectors of P . Then, you can compute P^{-1} .

Are all square matrices diagonalizable? No. Only those square matrices that have a full set of eigenvectors.

If a matrix is diagonalizable, are the P and D matrices given by $A=P\,D\,P^{-1}$ unique? No. We can change the order of the eigenvalues in D, and thus that changes P.

Section 6.4: **QR Factorization**

A is an mxn matrix with linearly independent columns.

Then $A=QR$ such that

Q is an mxn matrix with columns that are an orthonormal basis for column space of A and

R is an nxn upper triangular matrix with positive diagonal entries.

Remember that since Q is filled with orthonormal columns, then $Q^T Q=I$ is true.

To find Q , use Gram-Schmidt process to create an orthonormal basis for column space of A . Those basis vectors are the columns of Q . Then, you can compute $R=Q^T A$ to find R .

Does $A=QR$ for all matrices A ? No, only those A with linearly independent columns.

If $A=QR$, is this factorization unique? No. We can rearrange the columns of Q and end up with different R.

Section 7.1: **Orthogonal Diagonalization of A**

A is nxn matrix.

A is orthogonally diagonalizable means that A can be written as $A=PDP^T$ for some nxn invertible orthogonal P matrix (i.e. $P^{-1}=P^T$) and D is an nxn diagonal matrix.

Fact: A is orthogonally diagonalizable iff A is symmetric, i.e. $A=A^T$.

To find P and D , find the eigenvalues of A and those eigenvalues become the diagonal entries of D . Then, take the eigenvectors of A and ensure that they are orthonormal (you might have to do Gram-Schmidt process to guarantee this is true, and then normalize each of those vectors). The orthonormal basis vectors you just created for the eigenspace of A form the columns of P , again in the same order as their corresponding eigenvalues are listed in D .

Are all square matrices orthogonallydiagonalizable? No. Only those square matrices that are symmetric.

If a matrix is orthogonally diagonalizable, are the P and D matrices given by $A=PDP^T$ unique? No. We can change the order of the eigenvalues in D, and thus that changes P.

Section 7.4: **SVD**

A is an mxn matrix.

rank(A) = r, where $r \le n$

$A=U\Sigma V^T$

U is an mxm orthogonal matrix

V is an nxn orthogonal matrix

D is an rxr diagonal matrix of singular values of $A^T A$, such that $\sigma_1 \ge \sigma_2 \ge ... \ge \sigma_r > 0$ and

$$D=\begin{bmatrix} \sigma_1 & 0 & . & . & . & 0 \\ 0 & \sigma_2 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & \sigma_r \end{bmatrix}$$

(Remember that the singular values are just the square roots of the corresponding eigenvalues of $A^T A$ since we know all eigenvalues are non-negative and the singular values are specifically inserted in D in order from biggest to smallest.)

To find the SVD:

The columns of V are the orthonormal basis vectors for the eigenspace of $A^T A$, in order corresponding to σ_1 , σ_2 , ... , σ_r .

$\Sigma=\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ is size mxn matrix.

And, $U=[A\vec{v}_1 \;\; A\vec{v}_2 \;\; ... \;\; A\vec{v}_m]$ where $\vec{v}_i$ is the ith eigenvector, i.e. the ith column of V.

Does $A=U\Sigma V^T$ for all matrices A ? YES!!!

Is the SVD factorization unique? YES!!!