Key Definitions: Sections 1.1-1.9

- A linear combination of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is

- $span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is

- For $A_{m\times n}$ and $\mathbf{x} \in \mathbb{R}^n$, $A\mathbf{x}$ is

- The homogeneous matrix equation is

- The nonhomogeneous matrix equation is

- For a matrix to be in reduced row echelon form (RREF), the following four conditions must be met:

  (a)

  (b)

  (c)

  (d)

- The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent if

- The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent if

- A linear transformation is a function $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ that preserves

  (i)

  (ii)

- The standard matrix for the linear transformation, $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is

- A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is onto $\mathbb{R}^m$ if

- A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is one-to-one if

Major Theorems: Sections 1.1-1.9

Theorem 1 ***Uniqueness of Reduced Row Echelon Form:***
Each matrix is row equivalent to one and only one reduced row echelon form matrix.

Theorem 2 ***Existence and Uniqueness*** *A linear system is consistent if and only if the augmented column does not have a pivot position. A solution is unique if an only if there are no free variables.*

Theorem 3 ***Equivalent Descriptions***
If A is an $m \times n$ matrix with columns $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ and if $\boldsymbol{b} \in \mathbb{R}^m$,

$$\textit{the matrix equation} \quad A\boldsymbol{x} = \boldsymbol{b}$$

has the same solution set as the

$$\textit{the vector equation} \quad x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \ldots + x_n\boldsymbol{a}_n = \boldsymbol{b}$$

which has the same solution set as the linear system of m equations in n unknowns/variables whose

$$\textit{augmented matrix is} \quad [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n \quad \mathbf{b}].$$

Theorem 4 ***Logically Equivalent Statements***
Let A be an $m \times n$ matrix. Then, the following statements are logically equivalent (i.e the statements are all true or all false).

(a) For each $\mathbf{b} \in \mathbb{R}^m$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

Theorem 5 ***Linearity of Matrix Multiplication***
If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then

(a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

(b) $A(c\mathbf{u}) = c(A\mathbf{u})$.

Theorem 6 ***Parametric Vector Form***
Suppose the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent for a given $\mathbf{b}$ and let $\mathbf{p}$ be a solution. Then, the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

$$\mathbf{w} = \mathbf{p} + \mathbf{v}_h,$$

where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Theorem 7 *A set* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v_n}\}$ *of two or more vectors is linearly dependent if and only if at least one of the vectors in* S *is a linear combination of the others.*

Theorem 8 *If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \in \mathbb{R}^m$ *is linearly dependent if* $n > m$*.*

Theorem 9 *If a set* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \in \mathbb{R}^m$ *contains the zero vector, then the set is linearly dependent.*

Theorem 10 *Let* $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *be a linear transformation. Then there exists a* ***unique*** *matrix* A *such that*

$$T(\mathbf{x}) = A\mathbf{x} \qquad \forall\ \mathbf{x} \in \mathbb{R}^n$$

Moreover, A *is the* $m \times n$ *matrix whose* j*th column is the vector* $T(\mathbf{e_j})$*, where* $\mathbf{e_j}$ *is the* j*th column of the identity matrix in* R^n*:*

$$A = \begin{bmatrix} T(\mathbf{e_1}) & T(\mathbf{e_2}) & \ldots & T(\mathbf{e_n}) \end{bmatrix}$$

called the ***standard matrix for the linear transformation*** T*.*

Theorem 11 *Let* $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *be a linear transformation. Then* T *is* *1-to-1* *if and only if the equation* $T(\mathbf{x}) = \mathbf{0}$ *has only the trivial solution,* $\mathbf{x} = \mathbf{0}$*.*

Theorem 12 *Let* $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *be a linear transformation, and let* A *be the standard matrix for* T*. Then:*

(a) T *maps* $\mathbb{R}^n$ *onto* $\mathbb{R}^m$ *if and only if the columns of* A *span* $\mathbb{R}^m$*.*

(b) T *is* *1-to-1* *if and only if the columns of* A *are linearly independent.*

Chapter 1 Supplementary Exercises: 1, 3, 5, 7, 11, 12, 17, 19, 20, 21

Supplemental Practice Problems:

1) Linearly (In)Dependent Sets

   (a) Give an example of two vectors in $\mathbb{R}^2$ that are linearly dependent.
   (b) Give an example of two vectors in $\mathbb{R}^2$ that are linearly independent.
   (c) Give an example of three vectors in $\mathbb{R}^2$ that are linearly dependent.
   (d) Give an example of three vectors in $\mathbb{R}^2$ that are linearly independent.
   (e) Give an example of three vectors in $\mathbb{R}^3$ that are linearly dependent.
   (f) Give an example of three vectors in $\mathbb{R}^3$ that are linearly independent.

2) Find all solutions, if any, to the following systems of equations.

   (a)
   $$\begin{cases} x_1 - 3x_2 = -3 \\ -x_1 + x_2 = -1 \\ 2x_1 - 5x_2 = -4 \end{cases}$$

   (b)
   $$\begin{cases} -2x_1 - x_2 + 3x_3 = 5 \\ 3x_1 + 2x_2 - 5x_3 = -2 \end{cases}$$

3) Find all solutions, if any, to the following matrix equations.

   (a) $\begin{bmatrix} 2 & -4 & 10 \\ 3 & 1 & 1 \\ -2 & 3 & -8 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 2 \\ -1 & 0 & -1 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

4) Consider the matrix equation $\begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$.

   (a) Show that the equation has a unique solution and find that solution.
   (b) Write the corresponding system of equations and graph the two corresponding lines in $\mathbb{R}^2$. Geometrically, how do you interpret your solution from (a)?
   (c) Write the corresponding linear combination problem. Verify that your solution from (a) gives the correct linear combination.

5) Consider the matrix equation $\begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

   (a) Show that the system has no solution.
   (b) Graph the lines of the corresponding system of equations. How does this graph relate to the fact that there is no solution?
   (c) Graph the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ along with the column vectors, $\mathbf{a}_1$ and $\mathbf{a}_2$, of the matrix. How can you interpret the fact that there is no solution in terms of linear combinations?

6) Consider the following vectors in $\mathbb{R}^3$.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$$

For each of the sets below, determine whether the set is linearly dependent or independent. If the set is linearly dependent, give a dependency relation between the vectors.

(a) $\{\mathbf{u}, \mathbf{v}\}$

(b) $\{\mathbf{u}, \mathbf{x}\}$

(c) $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$

(d) $\{\mathbf{u}, \mathbf{v}, \mathbf{y}\}$

7) Find all solutions, if any, to the following linear combination (or vector equation) problems.

(a) Determine if $\mathbf{w} = \begin{bmatrix} 5 \\ 6 \\ -12 \end{bmatrix}$ is a linear combination of $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$.

(b) Determine if $\mathbf{w} = \begin{bmatrix} -1 \\ 13 \end{bmatrix}$ is a linear combination of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

8) Homogeneneous, $A\mathbf{x} = \mathbf{0}$ and Nonhomogeneous Systems, $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \neq \mathbf{0}$

(a) What condition(s) on the row echelon form of the matrix A guarantee(s) that the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions?

(b) What condition(s) on the row echelon form of the matrix A guarantee(s) that the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ always has at least one solution no matter the entries of $\mathbf{b}$?

(c) What condition(s) of the numbers of rows and columns of A always give infinitely many solutions to the homogeneous problem?

(d) What condition(s) on the numbers of rows and columns of A guarantee that there will be lots of vectors $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ is inconsistent?

9) Consider the homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$ with the matrix A and its reduced row echelon form given below:

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 4 & 1 \\ -2 & -4 & 0 & -2 & -2 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find and express the solution, if any, to this system in linear combination form.

(b) Are the columns of A linearly independent or dependent?

(c) For what $\mathbf{b} \neq \mathbf{0} \in \mathbb{R}^3$, does a solution exist? Find a solution to such a nonhomogeneous matrix equation.

10) Suppose $\{\mathbf{v}_1, \mathbf{v_2}, \mathbf{v_3}\}$ is a linearly dependent set in $\mathbb{R}^n$. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Explain why $\{T(\mathbf{v}_1), T(\mathbf{v_2}), T(\mathbf{v_3})\}$ must be linearly dependent in $\mathbb{R}^m$.