# 8.1 Matrices & Systems of Equations

**Defn** $\boxed{\text{Matrix}}$

$$[a_{ij}] = A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

this is an $m \times n$ matrix

$m$ ↑ # of rows

$n$ ↑ # of columns

It is a structured array of numbers.

$a_{ij}$ is called an <u>entry</u> of the matrix

<u>order</u> of a matrix is $m \times n$ (read "m by n") always given first w/ # of rows

<u>square matrix</u> ⇒ when $m = n$

<u>row matrix</u> ⇒ a $1 \times n$ matrix (i.e. only has one row)

<u>column matrix</u> ⇒ an $m \times 1$ matrix (i.e. only has one column)

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Will use matrices to solve linear systems of eqns.

<u>ex</u>

$$\begin{aligned} 3x - 2y + z &= 5 \\ x + y + 2z &= 1 \\ -x \quad\quad - z &= 0 \end{aligned}$$

<u>augmented matrix</u>

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 5 \\ 1 & 1 & 2 & 1 \\ -1 & 0 & -1 & 0 \end{array}\right]$$

$$\Longleftrightarrow \quad A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

<u>Coefficient matrix</u>

variable matrix (column) matrix

constant matrix

$$AX = B$$

8.1 (cont)

Ex 1 What size (or order) are these matrices?

(a) $\begin{bmatrix} -2 & 5 & 1 \\ 7 & 6 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 3 & 2 & 3 & 1 \\ 4 & 4 & 7 & 4 & 4 \\ 9 & 8 & 6 & 5 & -1 \end{bmatrix}$

row-echelon form ⇒

1. all zero rows at bottom
2. has leading 1 in every nonzero row
3. all entries below leading 1 are zero.

reduced row-echelon form ⇒

row-echelon form **and** all entries above leading 1 are also zero

Ex 2 Indicate if these matrices are in (a) row-echelon form, (b) reduced row-echelon form, or (c) neither.

(a) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 5 \\ 1 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 5 & 6 & 3 \\ 0 & 1 & 2 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 7 & 9 \\ 0 & 1 & 3 & 5 \end{bmatrix}$

8.1 (cont)

Ex3 Use an augmented matrix to solve these systems of linear eqns.

(a) $2x - y + 3z = 24$
$2y - z = 14$
$7x - 5y = 6$

8.1 (cont)

Ex 3 (cont)

(overdetermined system)

(b) $x + 2y = 0$
$x + y = 6$
$3x - 2y = 8$

(c) $3x - 2y + z = 15$
$-x + y + 2z = -10$
$5x - 4y - 3z = 35$

# 8.2 Operations w/ Matrices

**Matrix Addition** $\Rightarrow A+B$

if $A=[a_{ij}]$ and $B=[b_{ij}]$ <u>and</u> A and B are the same size, then $A+B=[a_{ij}+b_{ij}]$

(i.e. the sum is the same size matrix where each entry is the sum of the A & B corresponding entries).

**Scalar multiplication** (<u>scalar</u> = real # or constant)

$cA=[ca_{ij}]$ i.e. every entry of A gets multiplied by c.

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<u>ex</u> $A=\begin{bmatrix} 2 & 5 & 1 \\ 3 & 0 & 2 \end{bmatrix}$ $B=\begin{bmatrix} 4 & 7 & -1 \\ -2 & -3 & 0 \end{bmatrix}$

$$\Rightarrow A+B=\begin{bmatrix} 2+4 & 5+7 & 1+-1 \\ 3+-2 & 0+-3 & 2+0 \end{bmatrix}=\begin{bmatrix} 6 & 12 & 0 \\ 1 & -3 & 2 \end{bmatrix}$$

<u>ex</u> $$3B=3\begin{bmatrix} 4 & 7 & -1 \\ -2 & -3 & 0 \end{bmatrix}=\begin{bmatrix} 3(4) & 3(7) & 3(-1) \\ 3(-2) & 3(-3) & 3(0) \end{bmatrix}=\begin{bmatrix} 12 & 21 & -3 \\ -6 & -9 & 0 \end{bmatrix}$$

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**Properties of Matrix Addition & Scalar multiplication**

1. $A+B=B+A$
2. $A+(B+C)=(A+B)+C$
3. $(cd)A=c(dA)$
4. $1A=A$
5. $c(A+B)=cA+cB$
6. $(c+d)A=cA+dA$

## 8.2 (cont)

**Ex 1** $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix}$

Find (a) $A+B$ (b) $A-B$ (c) $3A-2B$

## Matrix Multiplication

$A = [a_{ij}]$ $m \times n$ matrix

$B = [b_{ij}]$ $n \times p$ matrix

$\Rightarrow AB = [c_{ij}]$ such that

$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$

$AB$ is size $m \times p$

☆ # cols of A **must** match # rows of B

## Identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

- must be square matrix
- 1's on diagonal and zeros everywhere else

8.2 (cont)

Ex 2 Find $AB$, if possible.

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$$

Matrix Multiplication Properties

① $A(BC) = (AB)C$

② $A(B+C) = AB + AC$

③ $(A+B)C = AC + BC$

④ $c(AB) = (cA)B = A(cB)$

? is $AB = BA$ ?

Ex3 Find $AB$, if possible.

$$A = \begin{bmatrix} 3 \\ -1 \\ 5 \\ 7 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & -6 \end{bmatrix} + \begin{bmatrix} 7 & -1 \end{bmatrix} + \begin{bmatrix} -8 & 9 \end{bmatrix}$$

8.2 (cont)

**Ex 4** Find $AB$, $BA$ and $A^2$, if possible.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$

**Ex 5** Rewrite

$$\begin{aligned} x - y + 4z &= 17 \\ x + 3y \quad &= -11 \\ 2y + 5z &= 0 \end{aligned}$$

as $AX = B$.

# 8.3 Inverse of a Square Matrix

(This is like an inverse function.) An inverse matrix for A will be denoted as $A^{-1}$ ($-1$ is **not** an exponent here).

**Defn** If A is $n \times n$ matrix, and $AA^{-1} = A^{-1}A = I$, then $A^{-1}$ is inverse of A.

It's a multiplicative inverse. Notice A must be square to even consider it having an inverse.

**Ex 1** Show B is inverse of A.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix}$$

8.3 (cont)

**EX 2** Find $A^{-1}$ (if it exists) ~~for~~

Process for finding $A^{-1}$

1. augment A w/ I
2. perform rows ops until left side looks like I.
3. The right side is $A^{-1}$

(a) $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

8.3 (cont)

Ex 3 Find $A^{-1}$ given

a)
$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$$

System of Eqns w/ unique solution

If $AX=B$, then
$X = A^{-1}B$, if $A^{-1}$ exists.

Pf $AX = B$
$\Leftrightarrow A^{-1}AX = A^{-1}B$
$\Leftrightarrow IX = A^{-1}B$
$\Leftrightarrow X = A^{-1}B$

(b) use $A^{-1}$ from (a) to solve system
$$\begin{aligned} x + 2y + 2z &= 0 \\ 3x + 7y + 9z &= 1 \\ -1x - 4y - 7z &= 2 \end{aligned}$$