# 8.4 The Determinant of a Square Matrix

**determinant**

| 2x2 matrix | 3x3 matrix |
|---|---|
| $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow$ <br> $\det(A) = \lvert A \rvert$ <br> $= \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ | $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow \det(A) = \lvert A \rvert =$ <br> $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$ <br> $= a(ei - hf) - b(di - gf) + c(dh - ge)$ |

Alternate way to do $\lvert A \rvert$ for 3x3 matrix $\Rightarrow$ (☆ this only works for 3×3, not a larger square matrix!)

subtract these products

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

add these products

**History**

Notice ~~for~~

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

, it has solution $x = \dfrac{ce - bf}{ae - bd}$ and

$$y = \frac{af - cd}{ae - bd}$$

and if $A = \begin{bmatrix} a & b \\ d & e \end{bmatrix}$, then $\lvert A \rvert = ae - bd \Rightarrow$

denominator of soln for linear system is determinant of coefficient matrix.

## 8.4 (cont)

**Vocab**

**minor** $\Rightarrow$ if $A$ is square matrix, $M_{ij}$, the minor of the entry $a_{ij}$ is the determinant of the matrix left after we delete row $i$ and column $j$ from $A$.

**cofactor** $\Rightarrow$ cofactor $C_{ij}$ of entry $a_{ij}$ (of square matrix $A$) is $(-1)^{i+j} M_{ij}$.

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**Ex 1** Find all $M_{ij}$ + $C_{ij}$ for $A = \begin{bmatrix} 0 & 1 & 5 \\ 2 & -1 & 3 \\ 4 & 0 & 1 \end{bmatrix}$

8.4 (cont)

EX 2 Find $|A|$ for these matrices.

(a) $A = \begin{bmatrix} 3 & 5 \\ 2 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & -1 & 3 \\ 4 & 0 & 3 \\ 2 & 0 & -3 \end{bmatrix} = A$

8.4 (cont)

Ex 2 (cont)

(c) $A = \begin{bmatrix} -1 & 3 & 0 & 1 \\ 2 & 0 & 4 & -3 \\ -2 & 1 & 7 & 0 \\ 3 & 2 & 0 & 5 \end{bmatrix}$

8.4 (cont)

Ex 3 Solve for x.

$$\begin{vmatrix} x+4 & -2 \\ 7 & x-5 \end{vmatrix} = 0$$

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# 8.5 Applications of Matrices & Determinants

Remember in the last section, we noticed for

$ax + by = c$ the solution is
$dx + ey = f$

$$x = \frac{ce - bf}{ae - bd} \quad \& \quad y = \frac{af - cd}{ae - bd}$$

we can rewrite that as:

**Cramer's Rule**

$$\boxed{x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}}$$

Yet another way to solve a system of 2 linear eqns

We can expand it to bigger systems:

For a system of n *linear* eqns w/ n variables, and coefficient matrix A. If $|A| \neq 0$, then solution is

$$x_1 = \frac{|A_1|}{|A|}, \; x_2 = \frac{|A_2|}{|A|}, \; \ldots, \; x_n = \frac{|A_n|}{|A|}$$

where $|A_i|$ = determinant of matrix A w/ column i replaced w/ the constant column from the system of eqns.

Ex 1 Use Cramer's Rule to solve

$$-x + 2y - 3z = 3$$
$$-2x \quad - z = 1$$
$$3x - 4y + 4z = 0$$

8.5 (cont)

Ex1 (space)

Area of Triangle

w/ vertices $(x_1, y_1)$ $(x_2, y_2)$ $(x_3, y_3)$ is

$$A = \pm \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

(choose $\pm$ in order to make area positive)

Test for Collinearity

if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \Rightarrow (x_1, y_1), (x_2, y_2) + (x_3, y_3)$ are collinear.

8.5 (cont)

**Two-Pt Form of Eqn of a line**

thru $(x_1, y_1)$ & $(x_2, y_2)$

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Ex 2 Find an eqn of a line through $(1,5)$ & $(0,-2)$

Ex 3 Find the area of the triangle w/ vertices $(4,5)$, $(0,0)$ and $(5,-2)$.

8.5 (cont)

Ex 4 Are these pts collinear?
(6,1) (-3,-5) and (10,2)

Encoding /Decoding

| A | B | C | D | E | .... |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | |

Ex 5 MATH IS FUN

Will "encode" this message by separating the message into 1 X 3 matrices, and replacing the letters w/ the corresponding #s from the alphabet. (0 = blank)

[M A T] [H _ I] [S _ F] [U N _]

$\Rightarrow$ [13 1 20] [8 0 9] [19 0 6] [21 14 0]

and multiply each of these matrices, in order, by

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$$

to encode.

8.5 (cont)

$$[13 \quad 1 \quad 20]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} = [-4 \quad -7 \quad -105]$$

$$[8 \quad 0 \quad 9]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} = [-1 \quad -20 \quad -47]$$

$$[19 \quad 0 \quad 6]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} =$$

$$[21 \quad 14 \quad 0]\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} =$$

$\Rightarrow$ Sender sends this message

$[-4 \quad -7 \quad -105] \quad [-1 \quad -20 \quad -47] \quad [\qquad] \quad [\qquad]$

Then, to decode, the receiver multiplies each encoded $1 \times 3$ matrix by $A^{-1}$.

(I calculated $A^{-1}$ to be $\begin{bmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix}$.)

8.5 (cont)

Decode message:

$$\begin{bmatrix} -4 & -7 & -105 \end{bmatrix} \begin{bmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -1 & -20 & -47 \end{bmatrix} \begin{bmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix} =$$

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