# 9.4 Mathematical Induction

This is a strategy or form of mathematical proof.... built upon pattern recognition. :)
Induction is a formal way to prove that our pattern recognition hypothesis is correct.

**Mathematical Induction**

Let $P_n$ be a statement involving $n$, $n \in \mathbb{N}$.
If ① $P_1$ is true and
② $P_k \Rightarrow P_{k+1}$ $\forall\, k \in \mathbb{N}$, then
$P_n$ is true for all $n \in \mathbb{N}$.

Ex 1 Let's prove that
$$3+6+9+\dots+3n = \frac{3}{2}n(n+1) = P_n.$$

pf ① If $n=1$, then $\frac{3}{2}(1)(1+1) = \frac{3}{2}(2) = 3 = P_1$
and we know also that $3=3$. ✓

② Assume $3+6+9+\dots+3k = \frac{3}{2}k(k+1) = P_k$

Look at $3+6+9+\dots+3(k+1)$.

(☆) $3+6+9+\dots+3(k+1) = 3+6+9+\dots+3k+3(k+1)$

and we know $3+6+9+\dots+3k = \frac{3}{2}k(k+1)$.

$\Rightarrow$ (☆) becomes $\frac{3}{2}k(k+1)+3(k+1)$

$$= (k+1)\left(\frac{3}{2}k+3\right)$$
$$= (k+1)\frac{3}{2}(k+2)$$
$$= \frac{3}{2}(k+1)(k+2) = P_{k+1}$$

$\Leftrightarrow 3+6+9+\dots+3(k+1) = P_{k+1}$

9.4 (cont)

This means the truth of $P_k$ implies $P_{k+1}$ is also true. So, we're done with the proof.

Ex2 Use induction to prove

$$1^3+2^3+3^3+4^3+\ldots+n^3 = \frac{n^2(n+1)^2}{4}. \quad \left(P_n = \frac{n^2(n+1)^2}{4}\right)$$

What is $P_{k+1}$ here?

9.4 (cont)

Ex 3  Prove  $2n^2 > (n+1)^2$ whenever $n \geq 3$

9.4 (cont)

Ex 4 Prove a factor of $(2^{2n+1}+1)$ is 3.

9.4 (cont)

Ex 5 Find a formula for

$$S_n = \frac{1}{4} + \frac{1}{12} + \frac{1}{24} + \frac{1}{40} + \cdots + \frac{1}{2n(n+1)}.$$

## 9.4 (cont)

**Ex6** Find the sum of these series.

(a) $\sum_{n=1}^{20} (n^3 - 2n)$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

(b) $\sum_{i=1}^{10} (4 - \frac{1}{2}i + i^2)$

# 9.5 Binomial Theorem

binomial $\Rightarrow$ polynomial w/ 2 terms

notice pattern:

$(x+y)^0 = 1$

$(x+y)^1 = x+y$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Pascal's Triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

coefficients of binomial expansion

**Binomial Thm**

$$(x+y)^n = x^n + nx^{n-1}y + \dots + {}_nC_r x^{n-r}y^r + \dots + nxy^{n-1} + y^n$$

and $\underbrace{\binom{n}{r} \neq {}_nC_r = \frac{n!}{(n-r)!\,r!}}_{\text{called binomial coefficients}}$ (read "n choose r")

Pascal's Triangle entries tell us how many ways we can do boolean problems.

ex How many ways can I roll 3 heads and 2 tails w/ 5 coins?

HHHTT, HHTHT, HTHHT, THHHT, HHTTH, HTHTH,
THHTH, HTTHH, THTHH, TTHHH   10 ways

note ${}_5C_3 = {}_5C_2 = \frac{5!}{2!\,3!} = \frac{5 \cdot 4}{2} = 10$ (see it in Pascal's $\Delta$)

## 9.5 (cont)

**Ex1** Evaluate (numerically).

(a) $\binom{8}{7}$

(b) ${}_{12}C_5$

**Ex2** Evaluate (using Pascal's $\Delta$).

(a) ${}_6C_2$

(b) $\binom{5}{4}$

**Ex3** Use Binomial Thm to expand.

$(x-2)^5$

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9.5 (cont)

**Ex 4** Expand $(2\sqrt{t} - 1)^3$ and simplify.

**Ex 5** Find the $5^{th}$ term in the expansion of $(5a+6b)^5$

**Ex 6** Find the coefficient of the $x^2y^8$ term in the expansion of $(4x-y)^{10}$.