2.5 Zeros of Polynomial Functions

Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero (root) in $\mathbb{C}$.

$\Rightarrow$ f has precisely n linear factors, i.e.

$$f(x) = a_n(x-c_1)(x-c_2)\cdots(x-c_n)$$

$$c_1, c_2, \ldots, c_n \in \mathbb{C}.$$

- Complex zeros ALWAYS occur in conjugate pairs!!

so if $(a+bi)$ is a zero of $f(x)$ so is $(a-bi)$

- Every polynomial can be factored into linear and quadratic factors (w/ $\mathbb{R}$ coefficients)

Ex 1 Find all zeros of $f(x)$ given

$$f(x) = 2(x-3)(x+1)(x-4i)(x+4i)$$

2.5 (cont)

Ex 2 Given $x = 3+i$ is a zero of $f(x) = 2x^3 - 11x^2 + 14x + 10$, find all other zeros.

How Do I Find Zeros of a Polynomial?

Rational Zero Test

If $f(x)$ has integer coefficients, every rational zero of $f(x)$ has form $\frac{p}{q}$ where p = factor of constant term and q = factor of leading coefficient

Descartes Rule of Signs

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ $a_i \in \mathbb{R}$

($a_0 \neq 0$)

1. # of positive $\mathbb{R}$ roots = # of variations in signs of $f(x)$ or less than that by an even #
2. # of negative $\mathbb{R}$ roots = # of variations in signs of $f(-x)$ or less than that by even #

2.5 (cont)

Upper / Lower Bounds For Zeros

If $f(x)$ polynomial w/ $\mathbb{R}$ coefficients, and leading coefficient is positive, then

when we divide $f(x)$ by $x-c$ (using synthetic div.)

① if $c>0$ & each # in last row is ≥ 0, then c is an upper bound for $\mathbb{R}$ zeros of f.

② if $c<0$ & #s in last row are alternately + and − (zero entries count as + or −), then c is a lower bound for $\mathbb{R}$ zeros of f.

Ex 3 Find all the zeros of $f(x)=x^3+9x^2+27x+35$ and write $f(x)$ in factored form.

① Rational Root List ⇒ $\pm 1, \pm 5, \pm 7, \pm 35$

② Descarte's Rule of Signs ⇒ 0 +ve roots

$f(-x)=-x^3+9x^2-27x+35$ ⇒ 3 or 1 −ve roots

1 2 3

Try $x=-5$

$$\begin{array}{r|rrrr} -5 & 1 & 9 & 27 & 35 \\ & & -5 & -20 & -35 \\ \hline & 1 & 4 & 7 & 0 \end{array}$$

factored form

$\Rightarrow f(x) = \boxed{(x+5)(x^2+4x+7)} = 0$ ⟹ roots: $x=-5$, $-2 \pm i\sqrt{3}$

$x=-5$ $\quad x^2+4x+7=0$

$$x = \frac{-4 \pm \sqrt{16-4(1)(7)}}{2}$$

$$x = \frac{-4 \pm \sqrt{-12}}{2} = \frac{-4 \pm 2i\sqrt{3}}{2} = -2 \pm i\sqrt{3}$$

2.5 (cont)

Ex 4 Find all zeros of

$$f(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$$

Ex 5 Find all zeros of $f(x) = 3x^3 - 2x^2 + 15x - 10$

2.6 Rational Functions

Defn a rational fn is a ratio (quotient) of two polynomials; $f(x) = \frac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials

Domain ⇒ all input values where $D(x) \neq 0$.

Vertical Asymptote (VA)	Horizontal Asymptote (HA)
• line $x=a$, where $D(a)=0$	• line $y=b$ where $\lim_{x \to \pm\infty} f(x) = b$
• graph can NEVER touch or cross vert. asymptotes	• can touch or cross HA
• find VA by looking at disallowed domain values	• find by looking at end behavior of graph
• a RESTRICTION	• a DESCRIPTION

Ex 1 Find VA and HA for $f(x)$.

(a) $f(x) = \frac{2x^2}{x+1}$

HA:

VA:

(b) $f(x) = \frac{3x+2}{x-5}$

HA:

VA:

(c) $f(x) = \frac{x+4}{x^2-25}$

HA:

VA:

HA

• If degree$(N(x))$ > degree$(D(x))$, no HA.

• If degree$(N(x))$ < degree$(D(x))$, HA is $y=0$.

• If degree$(N(x))$ = degree$(D(x))$, HA is y = ratio of leading coefficients.

2.6 (cont)

Ex2 Analyze and graph
$$f(x) = \frac{x^2-2x-8}{x^2-9}$$

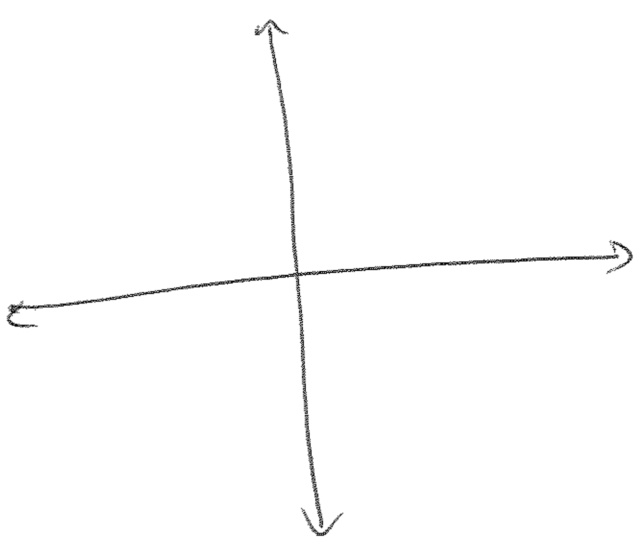
domain:

VA:

HA:

y-intercept(s):

x-intercept(s):



Analyze / Graph Rational Fns

1. Find domain, VA and HA.
2. Find x-int.
3. Find y-int.
4. Plot at least one pt on both sides of VA and x-int.
5. Fill in w/ smooth curves.

2.6 (cont)

Ex 3 Analyze and graph

$$g(x) = \frac{x^2+5x+8}{x+3}$$

Slant Asymptote

① This is only a possibility if degree $(N(x)) = 1 + \text{degree}(D(x))$

② To find, do long division.

③ you'll get something like

$f(x) = ax + b + \frac{c}{D(x)}$

then $y = ax + b$ is slant asymptote!

★ either have HA or slant asymptote, never both

2.7 Nonlinear Inequalities

For solving a linear inequality, it was basically the same as solving a linear eqn. We only had to remember to switch the inequality sign if we multiplied (or divided) by a -ve #.

For a nonlinear inequality, the strategy is slightly more complicated.

Ex 1 Solve $2(x-3)(x+1)(x-5) > 0$.

Strategy

① Get everything on one side, w/ 0 on other side.

② Factor completely.

③ Find all zeros of function.

④ Make # line for different intervals to test.

⑤ Choose soln intervals that satisfy inequality.

2.7 (cont)

Ex 2 Solve

(a) $x^2 + 2x > 3$

Vocab

Critical # ⇒ the variable value that makes the expression zero or undefined

(b) $\dfrac{3x}{x-1} \leq \dfrac{x}{x+4} + 3$

2.7 (cont)

Ex 3 A rectangular parking lot w/ a perimeter of 440 ft is to have an area of at least 8000 ft^2. Within what bounds must the length of the rectangle lie?