M1220 Final Quiz

① $D_x(x^{1+x}) = ?$

let $y = x^{1+x}$

$\ln y = (1+x)\ln x$

$\frac{1}{y}\, y' = \frac{(1+x)}{x} + \ln x$

$\boxed{(d)}$ $\quad y' = (x^{1+x})\left(\frac{1+x}{x} + \ln x\right)$

② $\int e^x \sin(e^x)\,dx = \int \sin u\, du = -\cos u + C$

$= -\cos(e^x) + C$

$\boxed{b}$ let $u = e^x$

$du = e^x\,dx$

③ $f(x) = x^5 + 2x^3 + 4x$ $\quad$ (1) $f^{-1}(7) = ? \iff f(?) = 7$

$f'(x) = 5x^4 + 6x^2 + 4 > 0$ $\quad$ (2) $(f^{-1})'(7) = \frac{1}{f'(?)}$

$\forall x \in \mathbb{R}$

$\Rightarrow f(x)$ monotonically increasing

$\Rightarrow f^{-1}(x)$ exists

$7 = x^5 + 2x^3 + 4x \iff x = 1$ or $? = 1 \Rightarrow f^{-1}(7) = 1$

$\boxed{d}$ $\quad f(1) = 7$

$(f^{-1})'(7) = \frac{1}{f'(1)} = \frac{1}{5+6+4} = \frac{1}{15}$

(4) $A = P\left(1+\frac{r}{n}\right)^{nt}$ $r = 0.09$ $n = 12$ $t = 10$ $P = 500$

$\boxed{c}$ $A = 500\left(1+\frac{0.09}{12}\right)^{12(10)} = 500(1.0075)^{120} = \$1225.68$

(5) $y = [\sin(x)+1]^{\cos x}$ at $x = \pi/2$

$y\left(\frac{\pi}{2}\right) = \left(\sin\frac{\pi}{2}+1\right)^{\cos\left(\frac{\pi}{2}\right)} = 2^0 = 1$

pt $(\pi/2, 1)$

$\ln y = (\cos x)\ln(\sin x+1)$

$\frac{1}{y}y' = -\sin x \ln(\sin x+1) + \frac{\cos^2 x}{\sin x+1}$

$y' = y\left(-\sin x \ln(\sin x+1) + \frac{\cos^2 x}{\sin x+1}\right)$

$y'(\pi/2, 1) = 1\left(-\sin(\pi/2)\ln\left(\sin\left(\frac{\pi}{2}\right)+1\right) + \frac{\cos^2(\pi/2)}{\sin(\pi/2)+1}\right)$

$= -1(\ln 2) + 0 = -\ln 2$

$m = -\ln 2$ pt $(\pi/2, 1)$ $\Rightarrow$ $y - 1 = -\ln 2\,(x - \pi/2)$

$y = (-\ln 2)x + 1 + \frac{\pi}{2}\ln 2$

$\boxed{a}$

(6) $\frac{dy}{dx} + 2xy - 2x = 0$ thru $(0, 3)$

$\frac{dy}{dx} + 2xy = 2x$ $e^{\int 2x\,dx} = e^{x^2}$

$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = 2xe^{x^2}$

$\boxed{e}$ $\frac{d}{dx}\left(e^{x^2}y\right) = 2xe^{x^2}$

$\int d(e^{x^2}y) = \int 2xe^{x^2}\,dx$ $u = x^2$ $du = 2x\,dx$

$e^{x^2}y = \int e^u\,du$

$e^{x^2}y = e^{x^2} + C$

$y = 1 + Ce^{-x^2}$

$3 = 1 + Ce^0$

$2 = C$

$\Rightarrow y = 1 + 2e^{-x^2}$

⑦ $\displaystyle\int \frac{x+9}{x^3+9x}\,dx = \int \frac{x+9}{x(x^2+9)}\,dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+9}\,dx$

$x+9 = Ax^2+9A+Bx^2+Cx$

$A+B=0$ $\quad C=1 \quad 9A=9$

$A=-B$ $\qquad\qquad\qquad A=1$

$\Rightarrow B=-1$

$\displaystyle = \int \frac{1}{x} + \frac{-x+1}{x^2+9}\,dx$

$\displaystyle = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+9}\,dx + \int \frac{1}{x^2+9}\,dx$

$=$ $\quad u=x^2+9$

$\frac{1}{2}\,du = x\,dx$

$\displaystyle = \ln|x| - \frac{1}{2}\int \frac{du}{u} + \frac{1}{9}\int \frac{1}{1+\left(\frac{x}{3}\right)^2}\,dx$

$\boxed{d}$ $\displaystyle = \ln|x| - \frac{1}{2}\ln(x^2+9) + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$

⑧ $\displaystyle\int \frac{\cos x(\sin x+\cos x)}{\sin x}\,dx = \int \cos x\,dx + \int \frac{\cos^2 x}{\sin x}\,dx$

$\displaystyle = \sin x + \int \frac{1-\sin^2 x}{\sin x}\,dx = \sin x - \int \sin x\,dx + \int \csc x\,dx$

$\boxed{b}$ $= \sin x + \cos x + \ln|\csc x - \cot x| + C$

⑨ $\displaystyle\int_0^4 \frac{x}{\sqrt{9+x^2}}\,dx = \frac{1}{2}\int_9^{25} u^{-1/2}\,du = \frac{1}{2}\left(\frac{u^{1/2}}{\frac{1}{2}}\right)\Big|_9^{25}$

$= \sqrt{u}\,\Big|_9^{25} = 5-3=2$

$\boxed{a}$ $u=9+x^2$

$\frac{1}{2}\,du = x\,dx$

$x=0,\ u=9$

$x=4,\ u=25$

⑩ $\displaystyle\int_3^7 \frac{2x}{\sqrt{x-3}}\,dx = \lim_{a\to 0}\int_a^4 \frac{2(u+3)}{\sqrt{u}}\,du = 2\lim_{a\to 0}\int_a^4 u^{1/2}+3u^{-1/2}\,du$

$\displaystyle = 2\left(\lim_{a\to 0}\frac{2}{3}u^{3/2} + 6u^{1/2}\right)\Big|_a^4$

$\boxed{d}$ $u=x-3 \Leftrightarrow x=u+3$

$du=dx$

$x=3,\ u=0$

$x=7,\ u=4$

$= 2\left(\frac{2}{3}(4^{3/2}-0) + 6(4^{1/2}-0)\right)$

$= \frac{4}{3}(8) + 12(2) = \frac{32}{3} + 24 = \frac{104}{3}$

(11) $\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx$

$u = \ln x \qquad v = \frac{1}{3}x^3$

$du = \frac{1}{x}dx \qquad dv = x^2 dx$

$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

$= \frac{1}{3}x^3\left(\ln x - \frac{1}{3}\right) + C$

$\boxed{d}$

(12) $\lim_{x\to 0} (1+\sin x)^{2/x} = \lim_{x\to 0} e^{\left(\frac{2}{x}\right)\ln(1+\sin x)}$

$= e^{\lim_{x\to 0} \frac{2\ln(1+\sin x)}{x}} \overset{L}{=} e^{\lim_{x\to 0} \frac{\frac{2\cos x}{1+\sin x}}{1}}$

$\boxed{b}$

$= e^2$

(13) $\lim_{x\to 0} \frac{\sin x - \tan x}{x^2 \sin x} = \lim_{x\to 0} \frac{\cancel{\sin x}\left(1 - \frac{1}{\cos x}\right)}{x^2 \cancel{\sin x}}$

$= \lim_{x\to 0} \frac{\left(1 - \frac{1}{\cos x}\right)}{x^2} = \lim_{x\to 0} \frac{\cos x - 1}{x^2 \cos x}$

$\boxed{c}$

$\overset{L}{=} \lim_{x\to 0} \frac{-\sin x}{2x\cos x - x^2 \sin x} \overset{L}{=} \lim_{x\to 0} \frac{-\cos x}{2\cos x - 2x\sin x - 2x\sin x - x^2\cos x}$

$= \frac{-1}{2-0-0-0} = \frac{-1}{2}$

(14) $\int_{1/2}^{2} \frac{dx}{x\sqrt[3]{\ln x}} = \int_{-\ln 2}^{\ln 2} u^{-1/3} du = \int_{-\ln 2}^{0} u^{-1/3} du + \int_{0}^{\ln 2} u^{-1/3} du$

$= \lim_{a\to 0^-} \int_{-\ln 2}^{a} u^{-1/3} du + \lim_{b\to 0^+} \int_{b}^{\ln 2} u^{-1/3} du$

$u = \ln x$

$du = \frac{1}{x}dx$

$= \lim_{a\to 0^-} \frac{3}{2}u^{2/3}\Big|_{-\ln 2}^{a} + \lim_{b\to 0^+} \frac{3}{2}u^{2/3}\Big|_{b}^{\ln 2}$

$x = 1/2, \; u = -\ln 2$

$x = 2, \; u = \ln 2$

$\boxed{a}$

$= \frac{3}{2}(0) - \frac{3}{2}(-\ln 2)^{2/3} + \frac{3}{2}(\ln 2)^{2/3} - \frac{3}{2}(0)$

$= -\frac{3}{2}(\ln 2)^{2/3} + \frac{3}{2}(\ln 2)^{2/3} = 0$

(15) $a_n = \frac{1}{\sqrt[3]{n}} + \frac{1}{\sqrt[n]{3}}$

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt[3]{n}} + \frac{1}{\sqrt[n]{3}}\right) = \lim_{n\to\infty}\left(\frac{1}{n^{1/3}} + \frac{1}{3^{1/n}}\right) = 0 + \frac{1}{3^0} = 1$$

$\boxed{b}$

(16) $\sum_{n=1}^{\infty} \frac{(-3)^n n^2}{(2n)!}$

ART $\lim_{n\to\infty} \frac{3^{n+1}(n+1)^2}{(2(n+1))!} \cdot \frac{(2n)!}{3^n n^2} = \lim_{n\to\infty} \frac{3(n+1)^2}{n^2(2n+2)(2n+1)}$

$\boxed{a}$

$\sim \frac{3n^2}{4n^4} = \frac{3}{4n^2} \to 0 < 1$

$\Rightarrow$ absolute convergence

(17) $\sum_{n=1}^{\infty} \frac{2n+7}{\sqrt{4n^4+5n+1}}$

LCT choose $b_n = \frac{1}{n}$, we know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, harmonic series.

$\boxed{c}$ $\lim_{n\to\infty} \frac{2n+7}{\sqrt{4n^4+5n+1}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{2n^2+7n}{\sqrt{4n^4+5n+1}} \sim \frac{2n^2}{\sqrt{4n^4}} = \frac{2n^2}{2n^2} = 1$

$\Rightarrow \sum a_n$ diverges

(18) $\sum_{n=1}^{\infty} \frac{3n^3+2n}{1+n^3}$ n^{th} term test $\lim_{n\to\infty} \frac{3n^3+2n}{1+n^3} = 3$

$\boxed{c}$ $\Rightarrow$ diverges

(19) $\sum_{n=0}^{\infty} \frac{(x-3)^n}{2^n+1}$ ART $\lim_{n\to\infty} \frac{|x-3|^{n+1}}{2^{n+1}+1} \cdot \frac{2^n+1}{|x-3|^n}$

$= |x-3| \lim_{n\to\infty} \frac{2^n+1}{2^{n+1}+1} = |x-3|\left(\frac{1}{2}\right) < 1$

$|x-3| < 2$

$-2 < x-3 < 2$

$1 < x < 5$

(19) (cont)

test end pts

① $x=1 \quad \sum_{n=0}^{\infty} \frac{(1-3)^n}{2^n+1} = \sum_{n=0}^{\infty} \frac{(-2)^n}{2^n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^n+1}$

$\boxed{c}$ $\lim_{n\to\infty} \frac{2^n}{2^n+1} = 1 \neq 0 \Rightarrow$ by n^{th} term test and AST diverges

② $x=5 \quad \sum_{n=0}^{\infty} \frac{2^n}{2^n+1}$ by n^{th} term test $\Rightarrow$ diverges

$\Rightarrow$ convergence set $(1,5)$

(20) $f(x) = \frac{2}{x-1}$ about $a=2$ order 4

$f(2) = \frac{2}{2-1} = 2$

$f^{(5)}(x) = \frac{-240}{(x-1)^6}$

$f'(x) = \frac{-2}{(x-1)^2}$ $\quad f'(2) = -2$

$f''(x) = \frac{4}{(x-1)^3}$ $\quad f''(2) = 4$

$f'''(2) = -12$

$f'''(x) = \frac{-12}{(x-1)^4}$

$f^{(4)}(2) = 48$

$f^{(4)}(x) = \frac{48}{(x-1)^5}$

$\boxed{a}$ $\Rightarrow f(x) = 2 + -2(x-2) + \frac{4}{2!}(x-2)^2 + \frac{-12}{3!}(x-2)^3 + \frac{48}{4!}(x-2)^4 + \cdots$

$f(x) \simeq 2 - 2(x-2) + 2(x-2)^2 - 2(x-2)^3 + 2(x-2)^4$

(21) $f(1.5) \simeq 2 - 2\left(\frac{-1}{2}\right) + 2\left(\frac{1}{4}\right) - 2\left(\frac{-1}{8}\right) + 2\left(\frac{1}{16}\right)$

$\boxed{c}$ $= 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{16+8+4+2+1}{8} = \frac{31}{8}$

error $R_4(x) = \left| \frac{f^{(5)}(c)}{5!}(x-2)^5 \right| \Rightarrow R_4(1.5) = \left| \frac{-240}{(c-1)^6\, 5!}(1.5-2)^5 \right|$

$c \in (1.5, 2)$ $\quad = \left| \frac{2\left(\frac{1}{2}\right)^5}{(c-1)^6} \right| = \left| \frac{1}{2^4(1.5-1)^6} \right| = \frac{1}{2^4\left(\frac{1}{2^6}\right)} = \frac{1}{\frac{1}{4}} = 4$

(22) $(-3, \sqrt{3})$

$x = -3 \qquad y = \sqrt{3}$

$r^2 = (-3)^2 + (\sqrt{3})^2$

$r^2 = 9 + 3 = 12$

$r = \pm 2\sqrt{3}$

$\tan\theta = \frac{\sqrt{3}}{-3} = \frac{1}{-\sqrt{3}}$

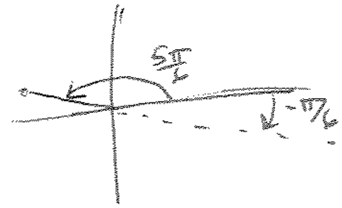
$= \frac{1/2}{-\sqrt{3}/2}$

$\Rightarrow \theta = \frac{5\pi}{6}$ or $-\frac{\pi}{6}$

$\boxed{b}$

$\left(2\sqrt{3}, \frac{5\pi}{6}\right)$

or $\left(-2\sqrt{3}, \frac{-\pi}{6}\right)$



(23) $\left(-1, \frac{5\pi}{4}\right)$
$\quad r \quad \theta$

$x = -1\cos\left(\frac{5\pi}{4}\right) = -1\left(\frac{-\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$

$y = -1\sin\left(\frac{5\pi}{4}\right) = -1\left(\frac{-\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\boxed{d}$

(24) $r^2 - 6r\cos\theta - 4r\sin\theta + 9 = 0$

$x^2 + y^2 - 6x - 4y + 9 = 0$

$(x^2 - 6x) + (y^2 - 4y) + 9 = 0$

$(x^2 - 6x + 9) + (y^2 - 4y + 4) = 4$

$(x-3)^2 + (y-2)^2 = 2^2$

circle w/ center $(3, 2)$ and $r = 2$

$\boxed{b}$

(25) $f(x) = \frac{3x^2}{4 - x^3} = \frac{3x^2}{4}\left(\frac{1}{1 - \left(\frac{x^3}{4}\right)}\right)$

$= \frac{3x^2}{4}\sum_{n=0}^{\infty}\left(\frac{x^3}{4}\right)^n = \sum_{n=0}^{\infty}\frac{3x^{3n+2}}{4^{n+1}}$

$\left|\frac{x^3}{4}\right| < 1$

$|x^3| < 4$

$|x| < \sqrt[3]{4}$

$\boxed{a}$