About this book

This book is intended to be an enjoyable supplement to the standard text and workbook material on geometry. Understanding what area is, what some of its basic properties are, and why they work are this book's primary goals. Beginning chapters are written at an advanced second-grade or third-grade level. Later materials may follow immediately or be used also in fourth, fifth and sixth grades.

Material is developed in a logical sequence and presupposes only knowledge of basic arithmetic of whole numbers and simple fractions. Some review of previous concepts is built into those that follow.

Each child will need a soft pencil with a good eraser, a straight-edge, and plenty of evenly lined graph paper. An adult guide should slowly read and re-read each page or concepts, allowing the children time to make the figures on a separate sheet of graph paper. The idea is that each child makes as many figures as possible—those shown in the book, those asked for by the book, and as many of their own conception as they can. The pace should be leisurely, and the children should be given plenty of time and hints to complete all the questions and exercises in each chapter. Also, the adult guide or, preferably, the children themselves should make up supplementary exercises like those in the particular chapter. Each child should be comfortable with the concepts in a chapter before going on to the next. There is little point in proceeding further until the individual child indicates readiness.

These materials were developed in, and have been used for several years by, the Open Classroom Program of the Salt Lake City School District.

Herbert Clemens
October, 1988

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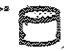
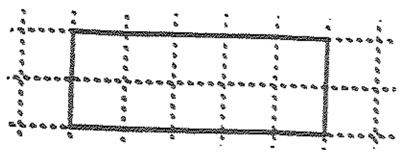
Cans of Paint

There once was a small person  with a small paint brush .
He used small cans of paint. 

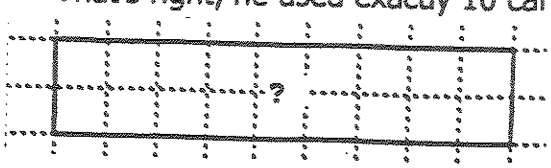
In fact, in each can of paint, there was exactly enough paint to paint exactly one square of this graph paper. 

His first job was to paint this:

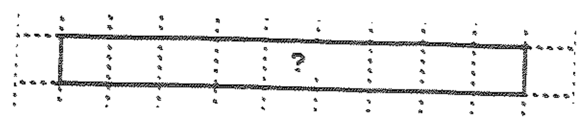


How many cans of paint did he use for that job? ________

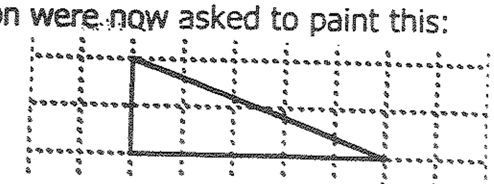
That's right, he used exactly 10 cans of paint. Now how about



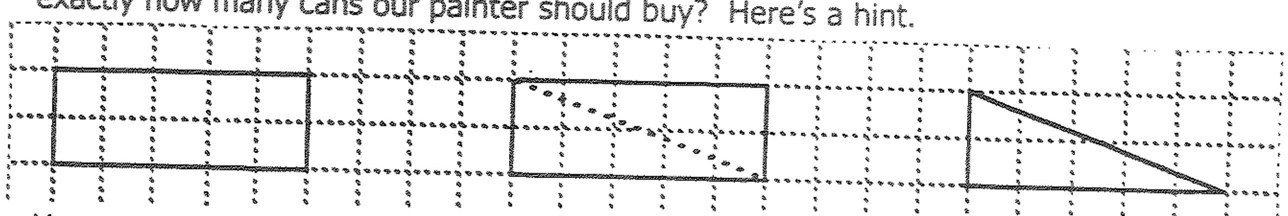
and



Suppose our little person were now asked to paint this:



How many cans of paint would he need? The correct answer comes out exactly even, that is, there won't be any parts of cans left over. Can you figure out exactly how many cans our painter should buy? Here's a hint.



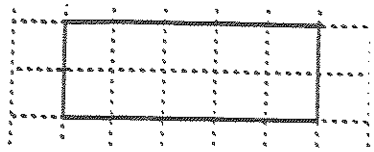
How many cans of paint does it take to paint the rectangle on the left?

What happens when you cut the rectangle like we are doing in the figure in the center?

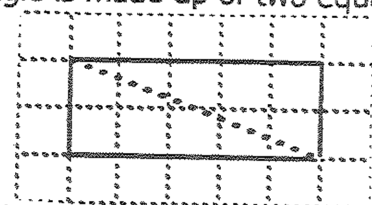
Can you say something about the two figures we get when we cut the rectangles along the dotted line?

So how many cans of paint does it take to paint the figure on the right?

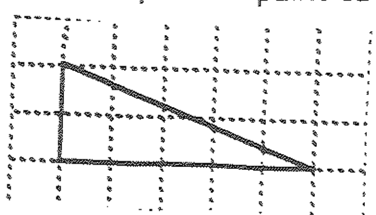
If you said the little person will need exactly 5 cans of paint to paint the triangle on the right, you are correct! It takes 10 cans to paint the rectangle.



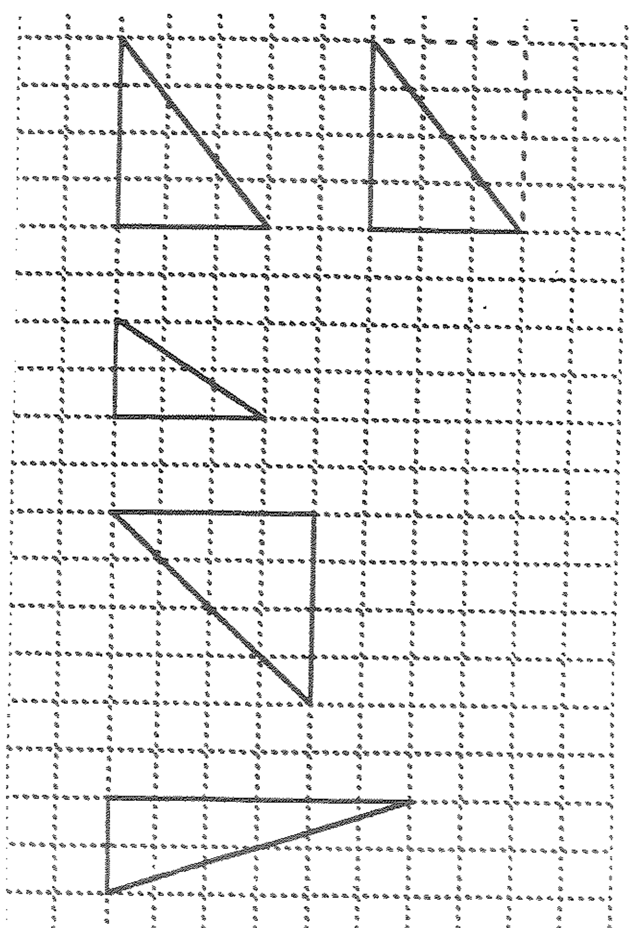
And the rectangle is made up of two equal triangles.



So it takes $5 = \frac{1}{2} x 10$ cans of paint to paint each of the two triangles.

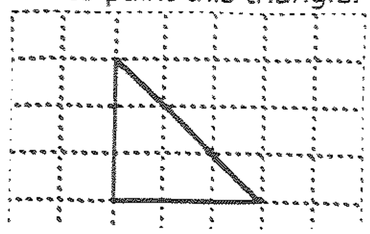


Here are some other triangles:

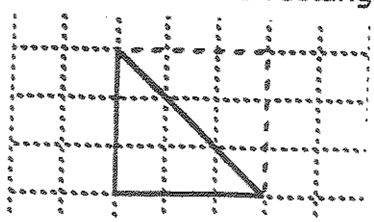


How many cans of paint will it take to paint the rectangle? How many cans to paint the triangle?

Sometimes the number of cans the little person has to use won't come out exactly even. Suppose he has to paint this triangle.



Let's figure out how much paint he needs, just like we did for the other triangles. First, we fill out to make a rectangle.

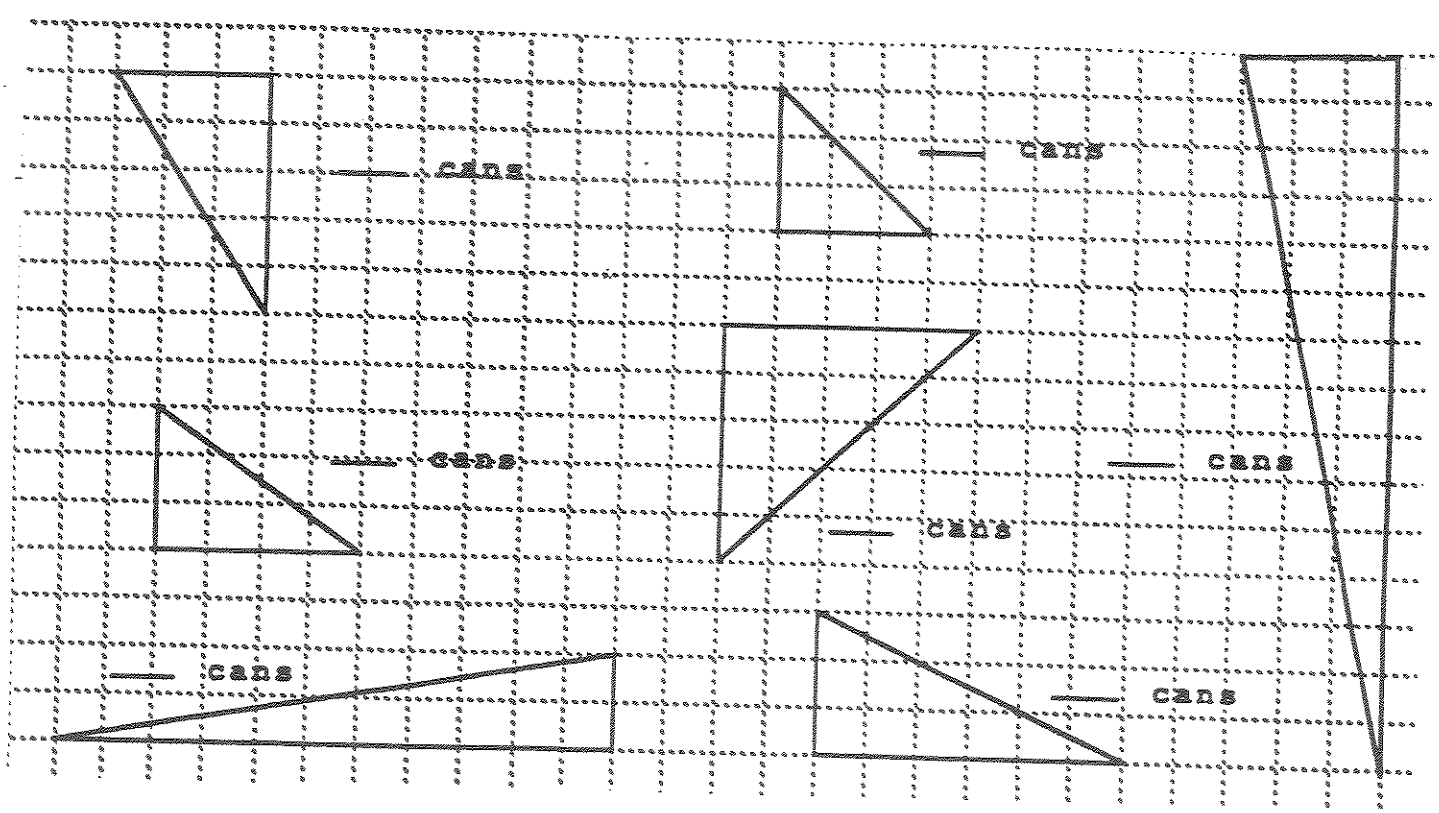


And count the squares. It will take 9 cans of paint to paint the rectangle. So, it will take half that much paint to paint the triangle. How much is half of 9?

Half of 8 is 4 .
Half of 1 is ½.
So, half of 9 is 4 ½

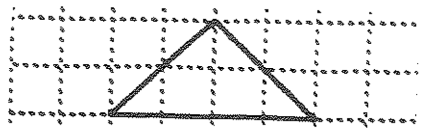
$\frac{1}{2} \times 8 = 4$
$\frac{1}{2} \times 1 = \frac{1}{2}$
$\frac{1}{2} \times 9 = 4\frac{1}{2}$

Here are some other triangles to paint.



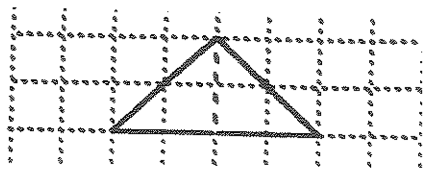
Painting Other Kinds of Triangles

The little person with the little paint cans got very good at his work and began to take on more difficult jobs, such as the following triangle.

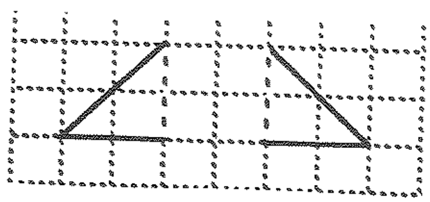


Can you figure out how many cans of paint he needs for this job?

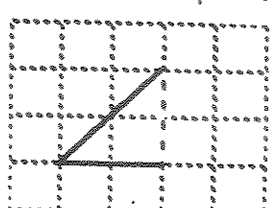
Here's a hint:



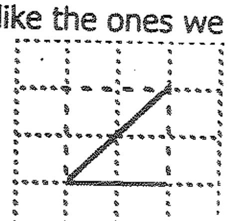
Does this help you figure out how many cans of paint our painter will need? The dotted line divides our triangle into two smaller ones.



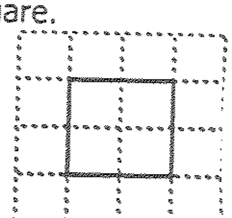
How many cans of paint does the little person need for one of these two small triangles?



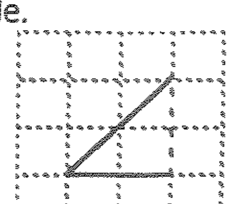
Well, this question is just like the ones we were doing before.



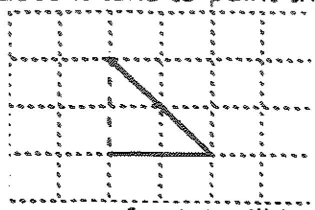
It takes 4 cans of paint to paint the square.



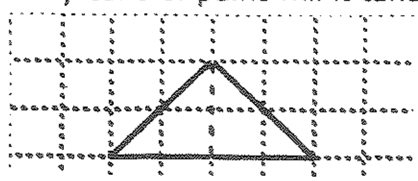
So, it takes 2 cans to paint the triangle.



How many cans of paint does it take to paint the following triangle?

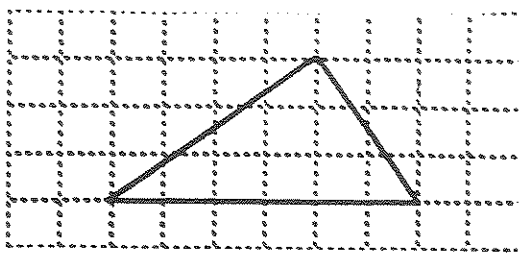


So, all together, how many cans of paint will it take to paint this?

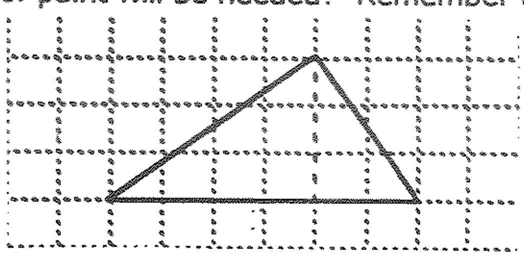


If you figured out that it will take 4 cans of paint altogether, you're right!

Here's another triangle to be painted.

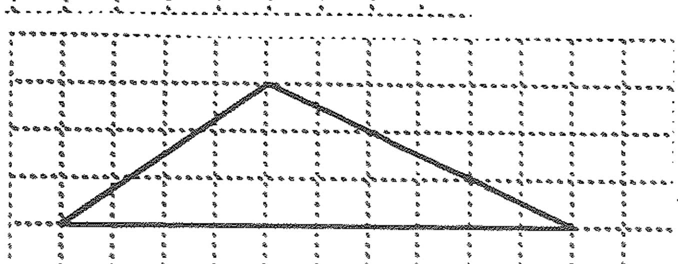


How many cans of paint will be needed? Remember the hint.



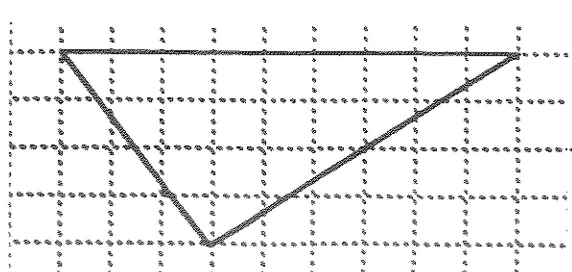
_____ cans

Another one



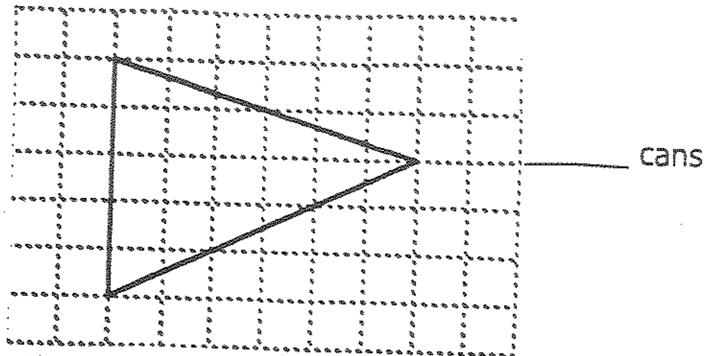
_____ cans

One standing on its head.



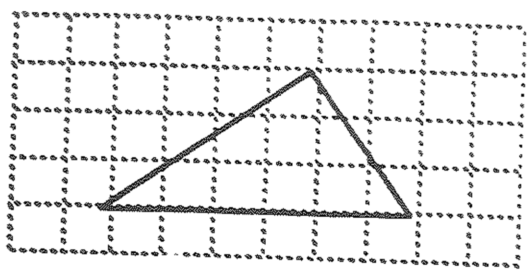
_____ cans

One standing on its side.

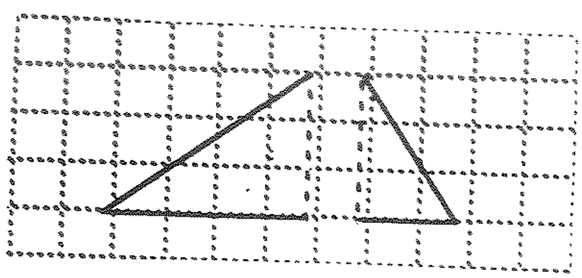


There is a lot of math in each of these triangles our little man has been painting. That is because there is more than one way to figure out how much paint is needed for each triangle.

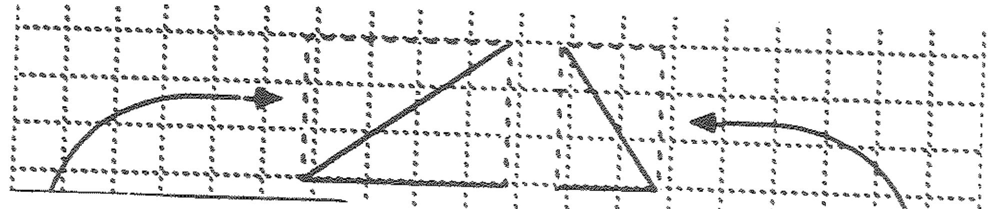
Let's do



To figure out how many cans of paint are needed, we split this triangle into two pieces.

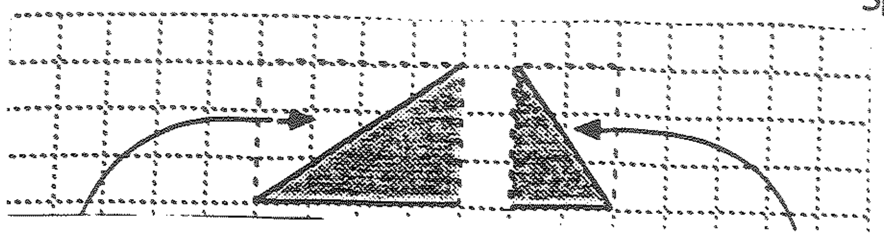


Each of the pieces is half of a rectangle.



It takes 3 x 4 = 12 cans of paint to paint this rectangle.

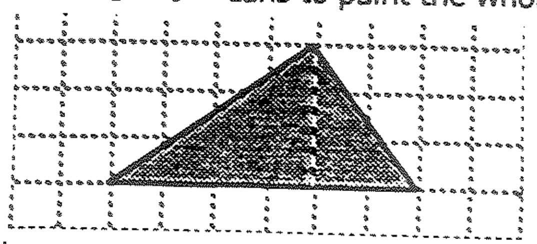
It takes 3 x 2 = 6 cans of paint to paint this rectangle.



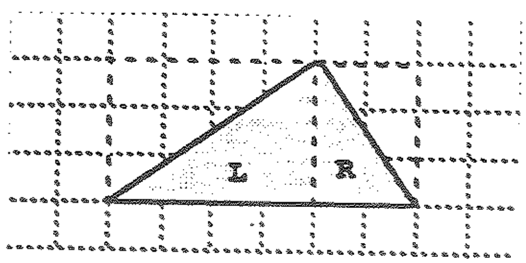
So, it takes ½ x 12 = 6 cans of paint to paint this triangle.

It takes ½ x 6 = 3 cans of paint to paint this triangle.

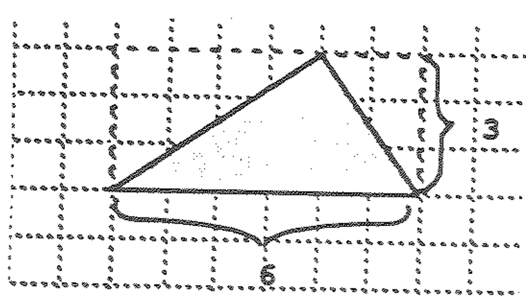
Altogether, it takes 6 + 3 = 9 cans to paint the whole triangle.



But, there is a quicker way to figure all this out.

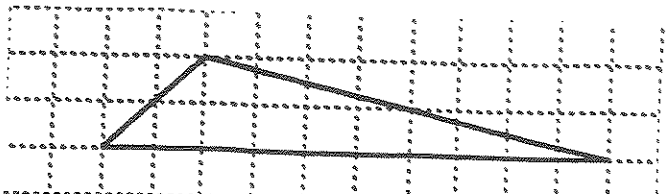


The left-hand piece of the shaded triangle is half of the left-hand rectangle, and the right-hand piece of the triangle is half of the right-hand rectangle. So all together, it takes half as much paint to paint the whole shaded triangle as it would take to paint the whole big rectangle.

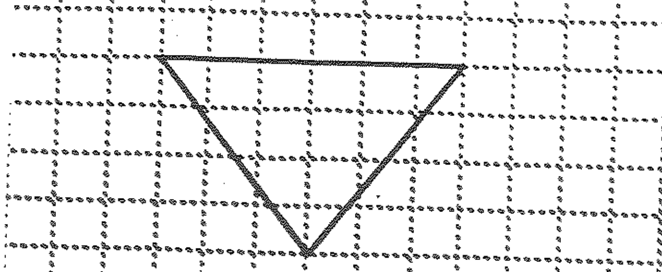


It takes 6 x 3 = 18 cans to paint the whole rectangle. So, it takes ½ x 18 = 9 cans to paint the shaded triangle.

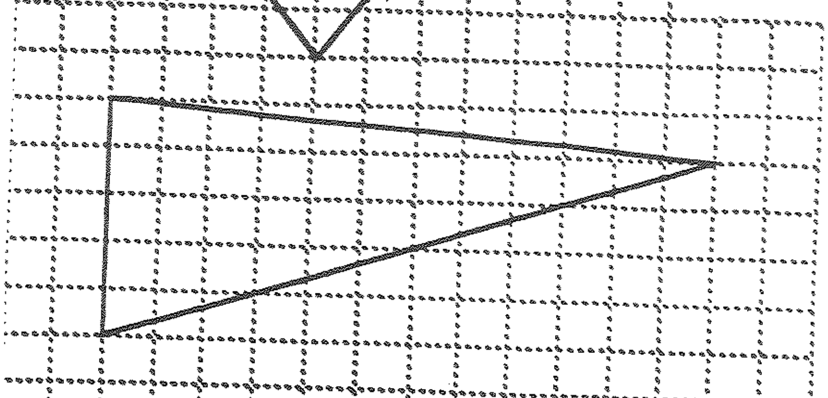
Do the following triangles the "fast way."



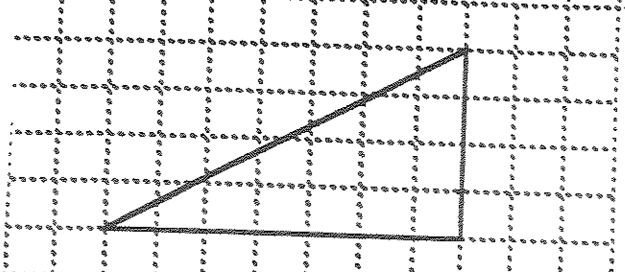
______ cans



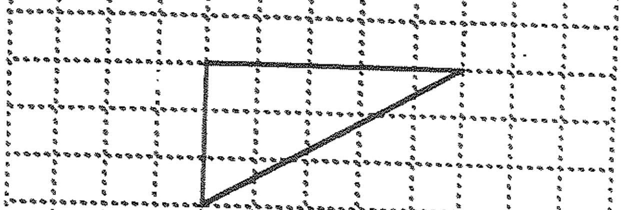
______ cans



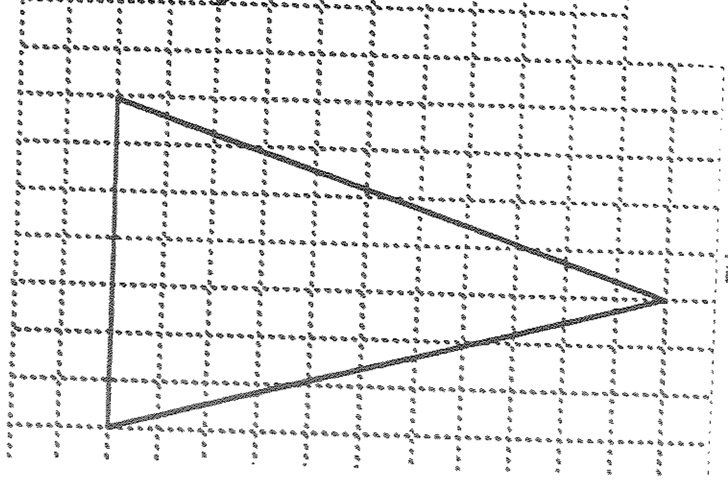
______ cans



______ cans



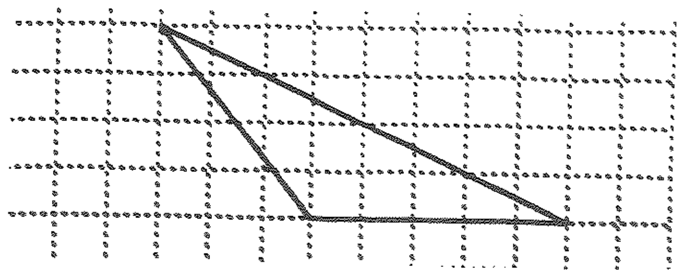
______ cans (There should be a fraction in this answer. That's okay.)



______ cans (There should be a fraction in this answer.)

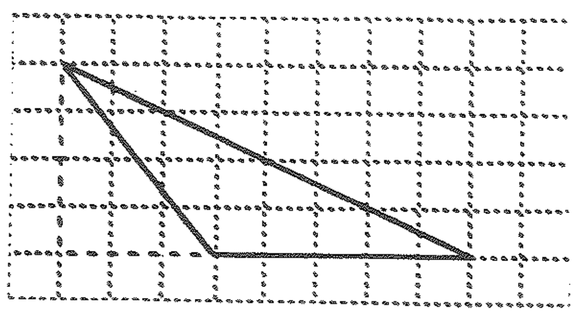
Leaning Triangles

One day, our little painter, who was getting better all the time, got a job to paint this leaning triangle.

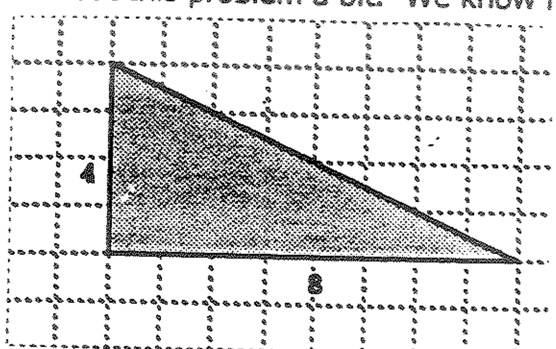


Can you figure out how much paint he will need?

Here's a hint:

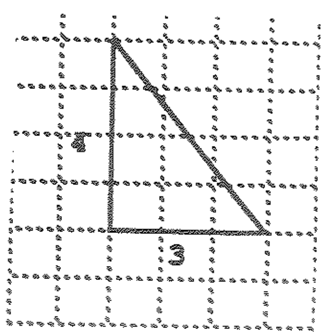


Think about this problem a bit. We know how many cans of paint to paint:



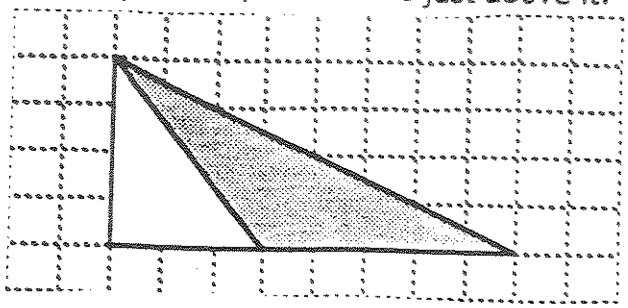
Since 4 x 8 = 32, it takes
½ x 32 = 16 cans
to paint this triangle.

We also know how many cans it will take to paint:



Since 4 x 3 = 12, it takes
½ x 12 = 6 cans
to paint this triangle.

Now we put this last triangle on top of the one just above it:

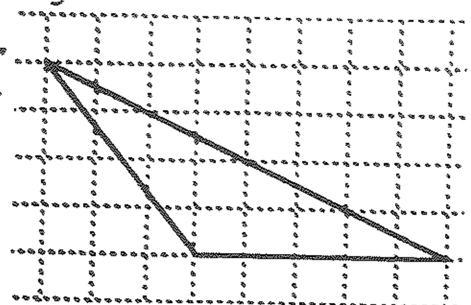


It takes 16 cans to paint the big triangle that is partly covered up by the smaller one. It takes 6 cans of paint to paint the smaller (white) triangle. So, it takes

$16 - 6 = 10$ cans

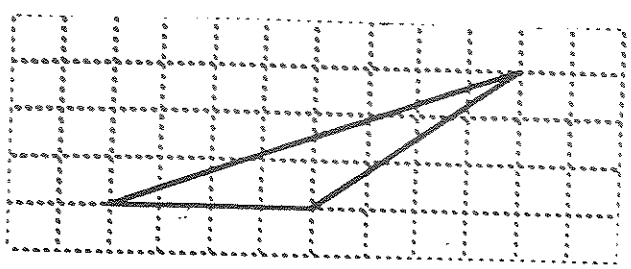
to paint the shaded part that is showing above. That is, it takes 10 cans of paint to paint the original leaning triangle.

This is the original "leaning triangle" our little painter was asked to paint.

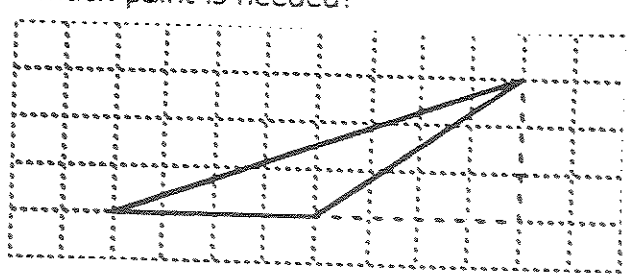


He will need 10 cans of paint to do the job. We figured it out by figuring out a big easy triangle, then a little easy triangle. We got our leaning triangle by taking the little easy one away from the big easy one.

Here's another leaning triangle.



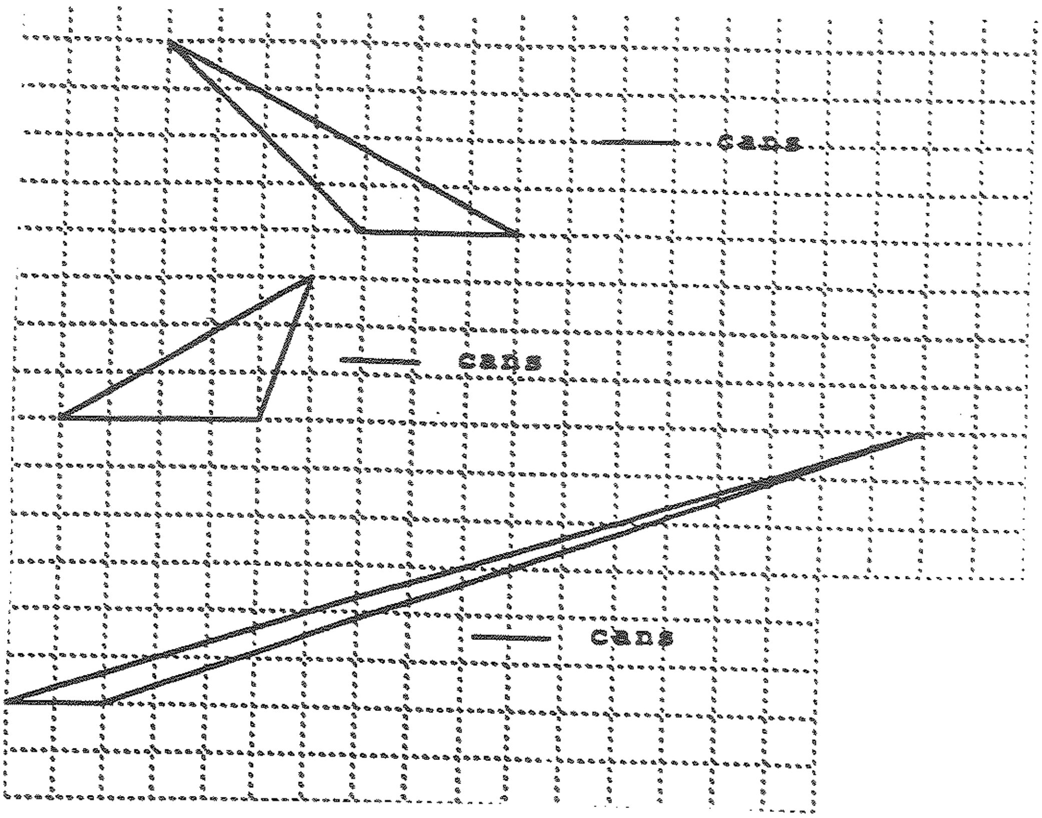
Can you figure out how much paint is needed?



$\frac{1}{2} \times 24 = 12$ cans to paint the big easy triangle.
$\frac{1}{2} \times 12 = 6$ cans to paint the piece outside the leaning triangle.

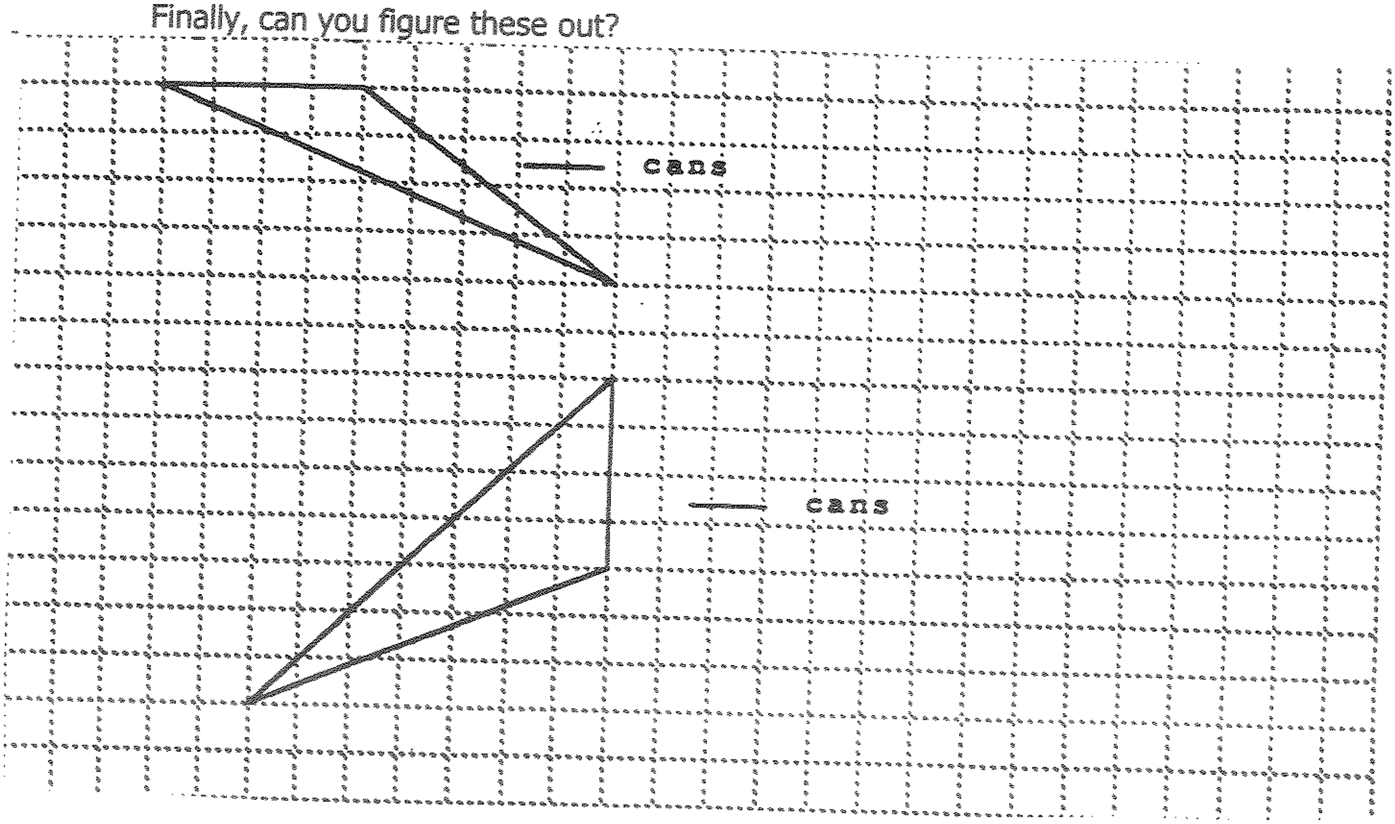
So, it takes $12 - 6 = 6$ cans to paint the leaning triangle.

Now, here are some more leaning triangles for you to figure out.



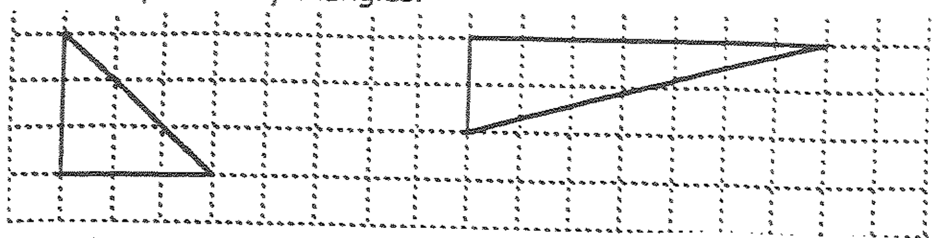
Here's some empty space. Draw a few leaning triangles of your own and figure them out.

Finally, can you figure these out?

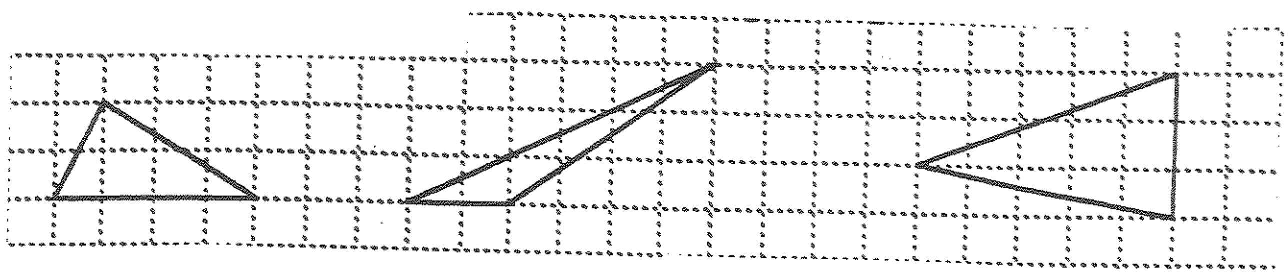


Painting Trapezoids

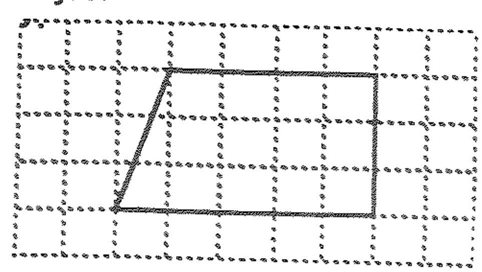
First, we learned to paint easy triangles.



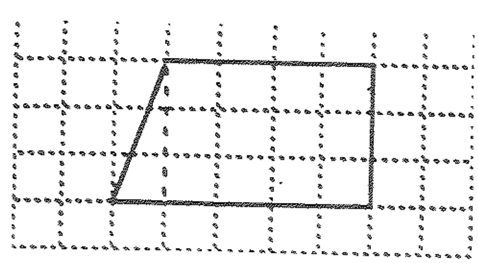
Then, we learned to paint other triangles by breaking them into easy triangles.



The "breaking-up trick" is a really good one. It works in a lot of other situations too! Suppose our little painter had this job.



How many cans of paint will he need? The question is a lot easier if we look at this hint.



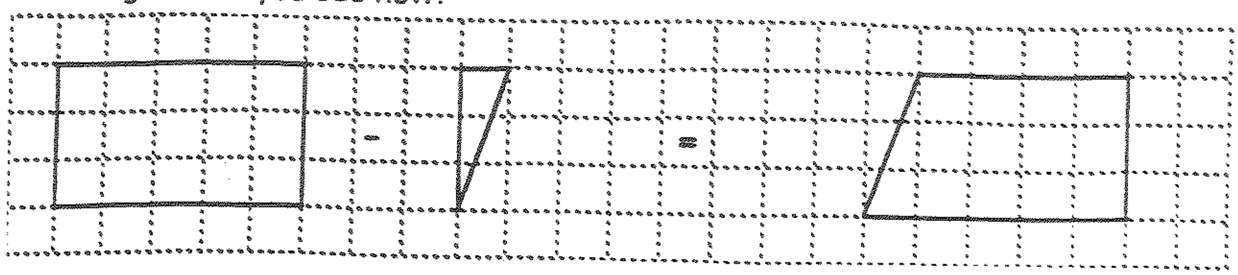
So, we can figure out the pieces.

$4 \times 3 = 12$ cans to paint the rectangle.

$\frac{1}{2} \times 3 = 1\frac{1}{2}$ cans to paint the easy triangle.

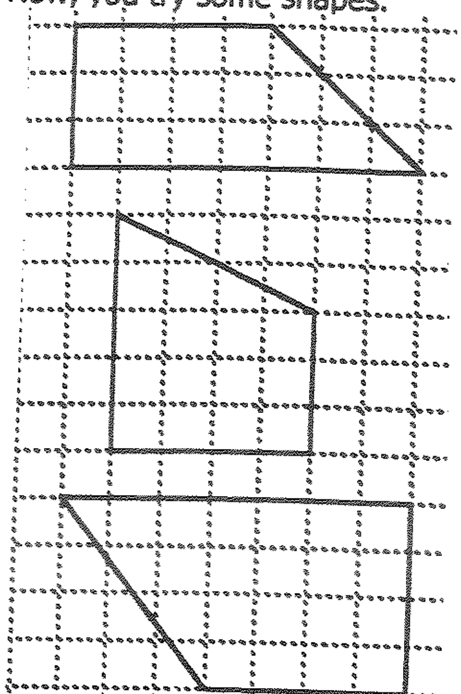
So, altogether, $12 + 1\frac{1}{2} = 13\frac{1}{2}$ cans.

Another way to figure this job out is by taking away an easy triangle from a bigger rectangle. Can you see how?



15 cans - 1 ½ cans = 13 ½ cans

Now, you try some shapes.



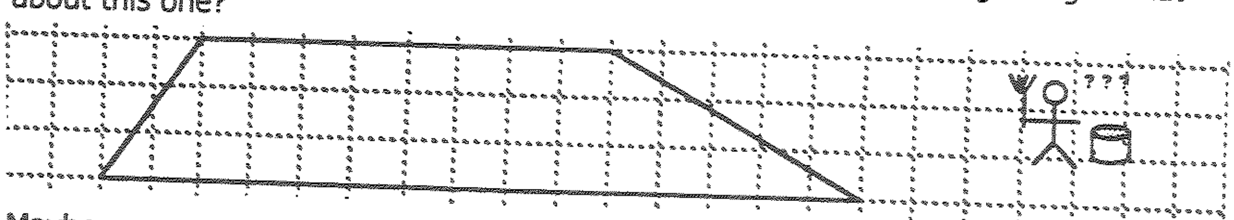
_____ cans

_____ cans

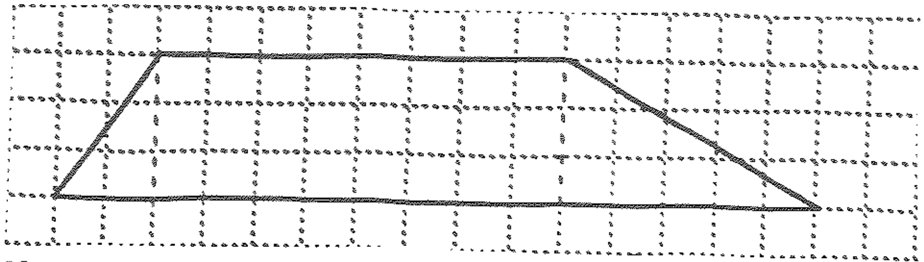
_____ cans

Find some empty spots on this page and make a few more shapes like these and figure out how much paint you will need for each of them.

We have found out how to do a new shape. But, that is only the beginning. What about this one?



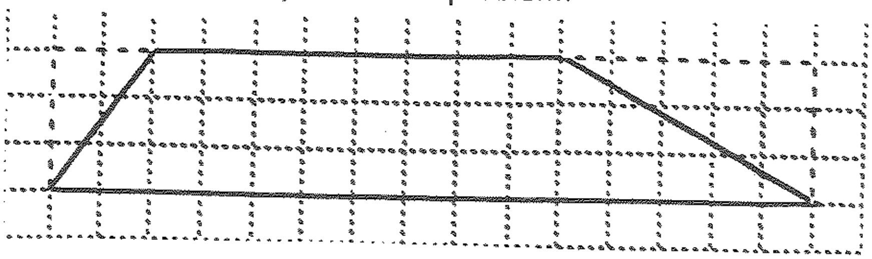
Maybe you can see right away some way to figure out how many cans of paint are needed for this job. If you want, turn to the next page where you'll find a hint.



Space 14

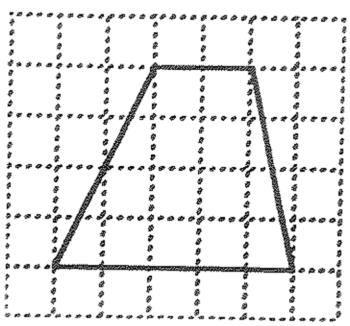
If your answer is exactly 34 ½ cans, you are right! Make sure you understand why this is the right answer.

Another way to do this problem:

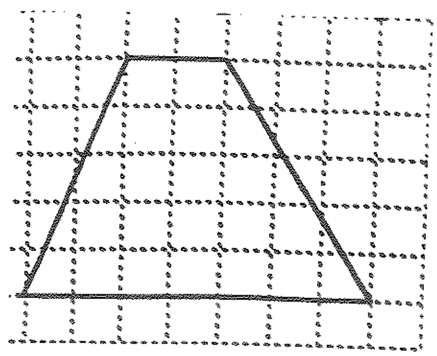


_____ - _____ = 34 ½ cans

Now, try a couple of these for yourself.

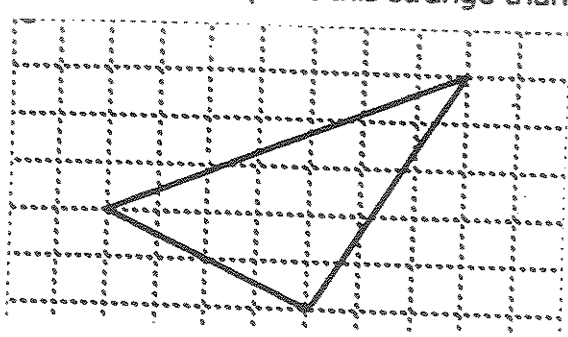


_____ cans



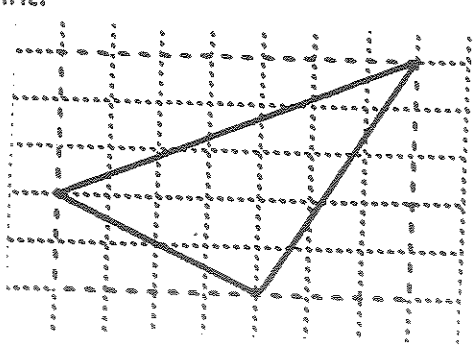
_____ cans

The "taking-away-pieces" trick is also a good one. Use it to figure out how many cans it takes to paint this strange triangle.



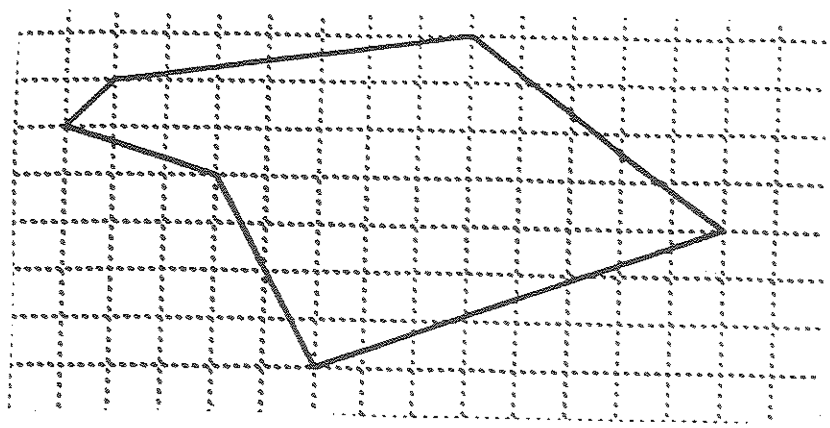
_____ cans

Hint:

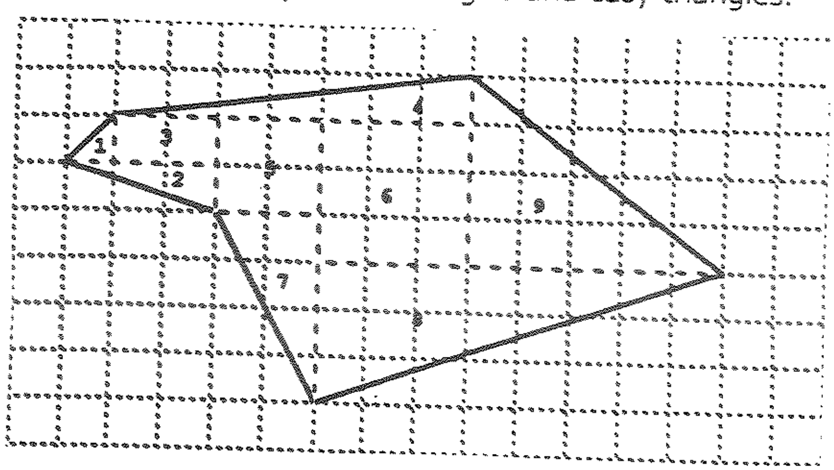


Painting Other Shapes

By now, our little painter (and we) have become so expert that we can handle almost any shape-painting job! We can even do this one:



How? We simply divide it up into rectangles and easy triangles.

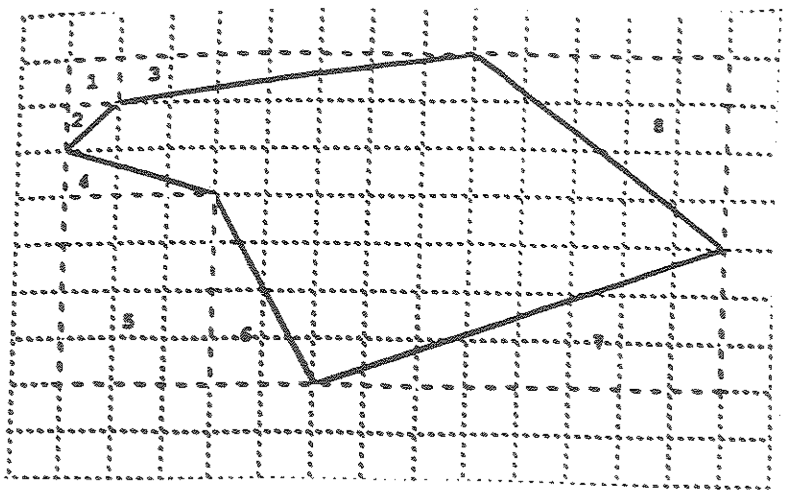


So, we can make a list.

Piece	Number of cans
1	½ x 1 = ½
2	½ x 3 = 1 ½
3	2
4	½ x 7 = 3 ½
5	4
6	9
7	½ x 8 = 4
8	½ x 24 = 12
9	½ x 20 = 10
Altogether	46 ½ cans of paint.

(Whew!)

Another way to figure this job using the take-away trick:

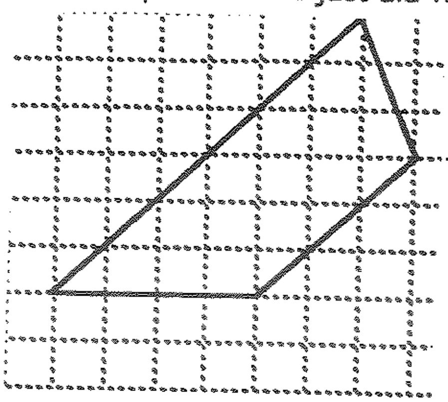


Big rectangle:
7 x 13 = 91 cans

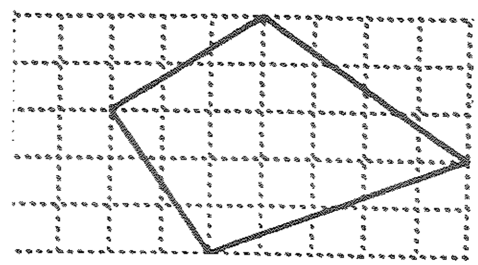
Take away: 1
½
3 ½
1 ½
12
4
12
10
44 ½ cans

So, it will take 91 – 44 ½ = 46 ½ cans of paint.

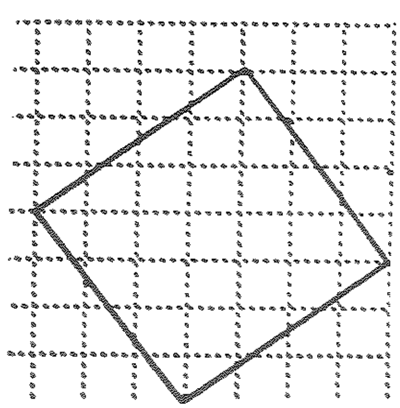
That problem we just did was huge! Try these. They're a bit easier.



_____ cans

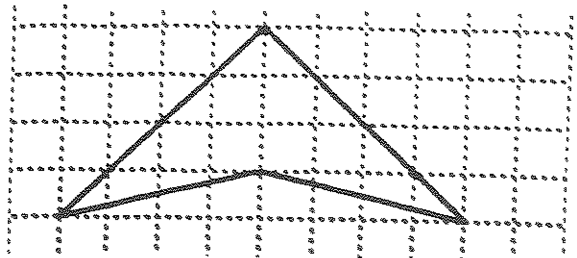


_____ cans

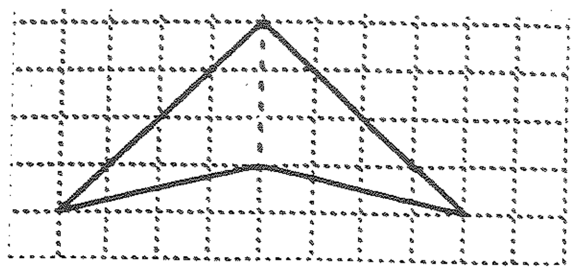


_____ cans

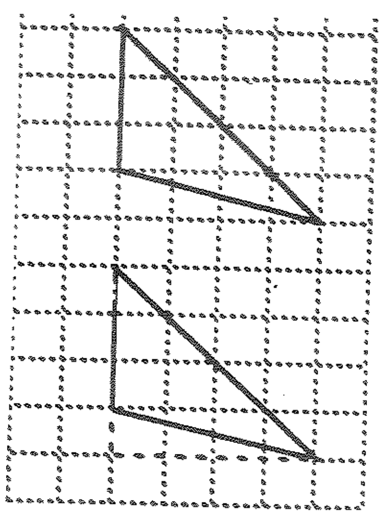
Now, maybe you are getting convinced that we can paint just about anything! Well, at least anything with straight sides and corners at places where the lines of the graph paper cross. That's true! But, to do some jobs quickly, it helps to use some more tricks. For example, if we have to paint this:



Just break it into two triangles like this.



Both triangles are the same size. (Why?) So, just figure out one of them and double your answer.



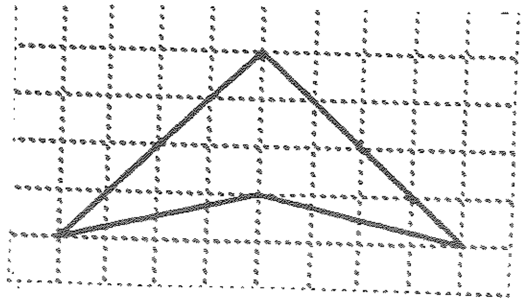
This is just one of those leaning triangles (on its side).

½ x 4 x 4 = 8

\- ½ x 4 x 1 = 2

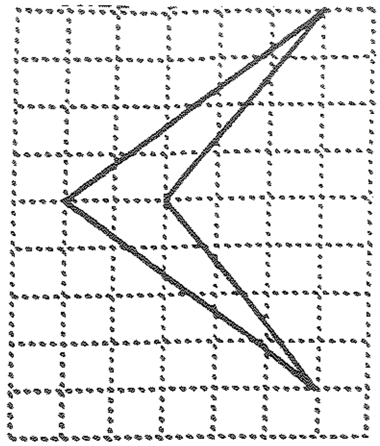
½ x 4 x 3 = 6 cans

So, it takes 2 x 6 = 12 cans of paint to paint:



Space 18

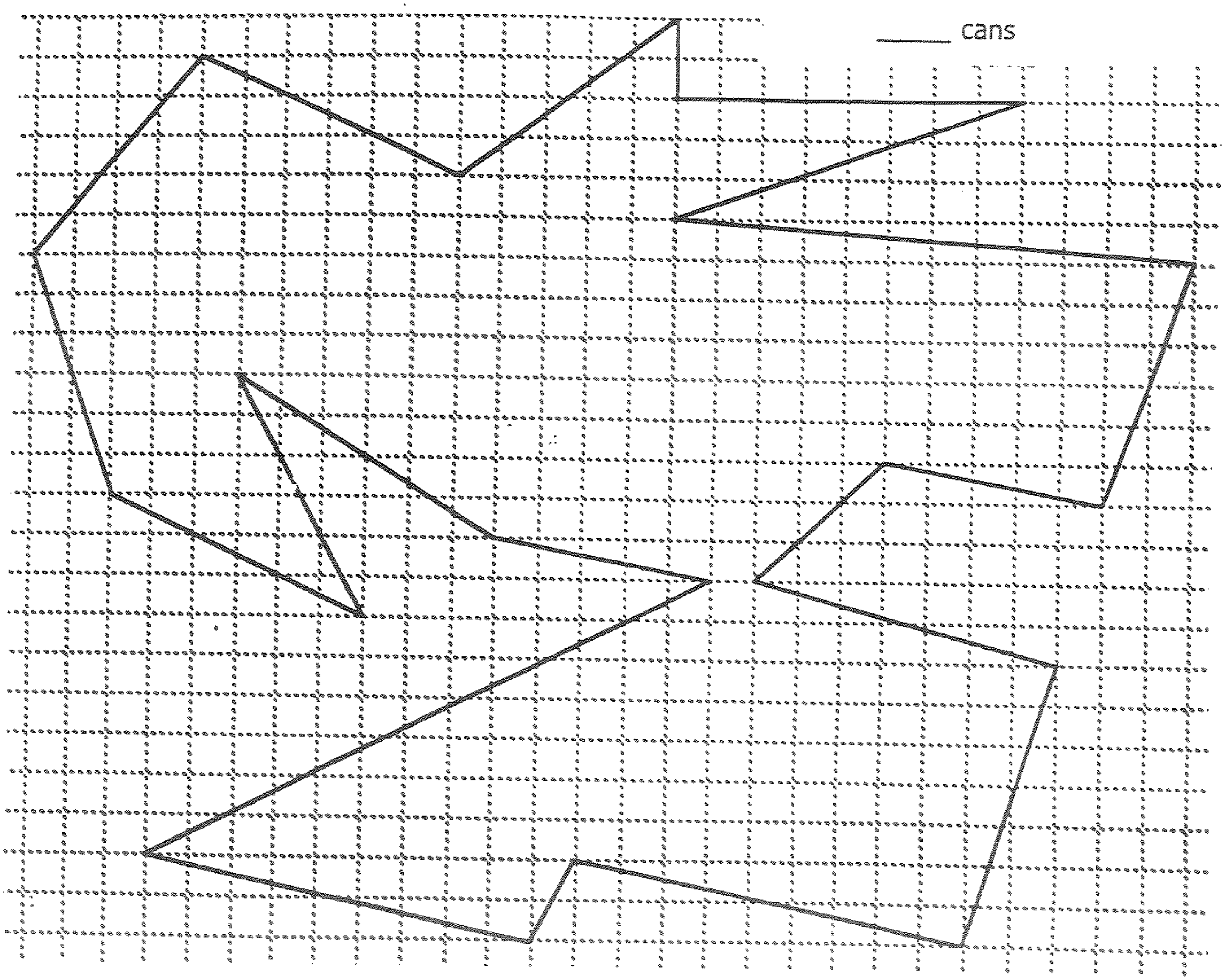
You try this one:



_______ cans

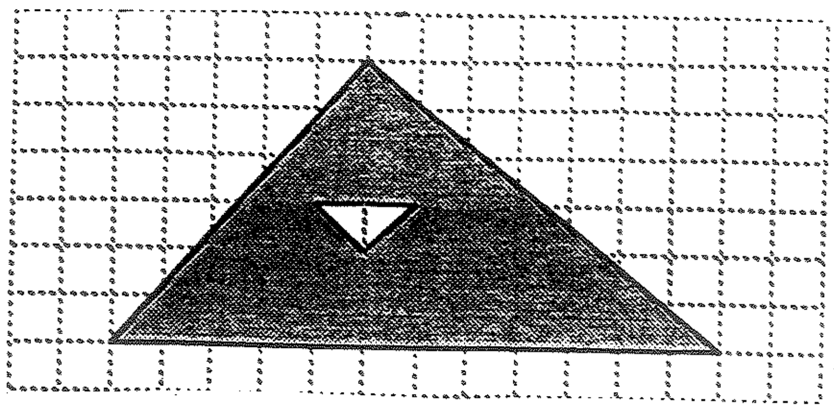
Here's a project for the rest of the day if you dare try it.

_______ cans



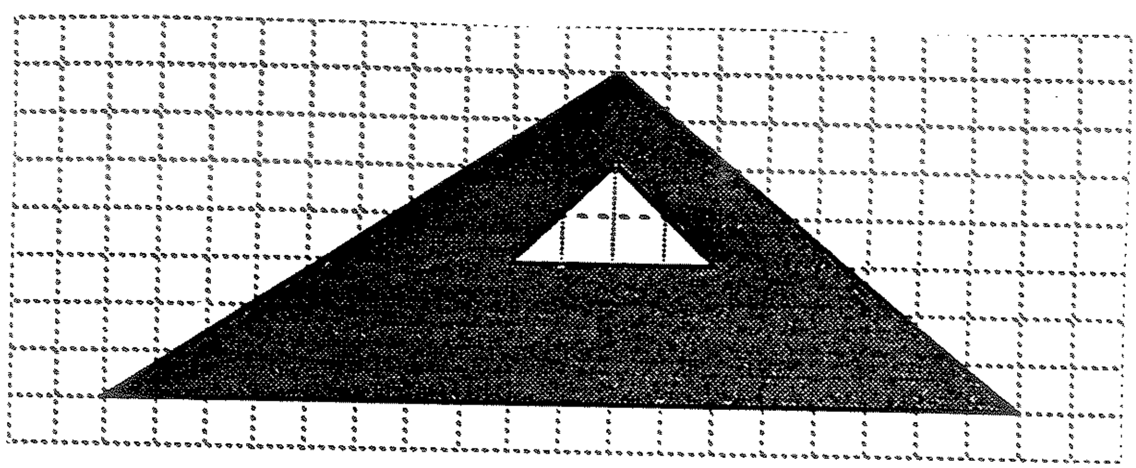
Shapes with Missing Pieces

There are other things we can figure out, too. For example, we can figure out how many cans of paint it takes to paint a triangle with a hole in it.



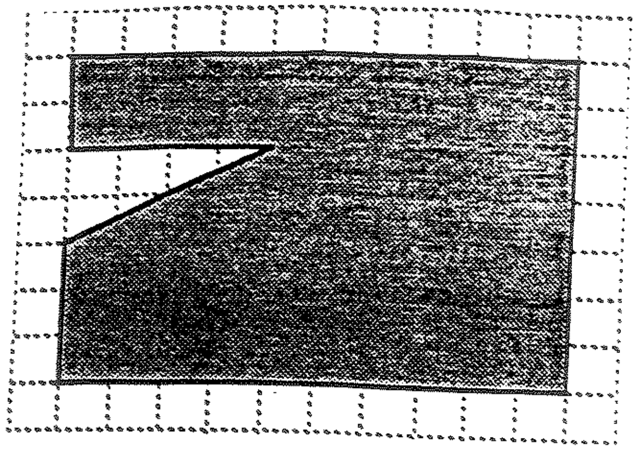
The big triangle would take $\frac{1}{2} \times 12 \times 6 = 36$ cans to paint. The little upside-down triangle would take $\frac{1}{2} \times 2 \times 1 = 1$ can to paint.
So, the shaded area would take $36 - 1 = 35$ cans to paint.

Here's one for you to try.



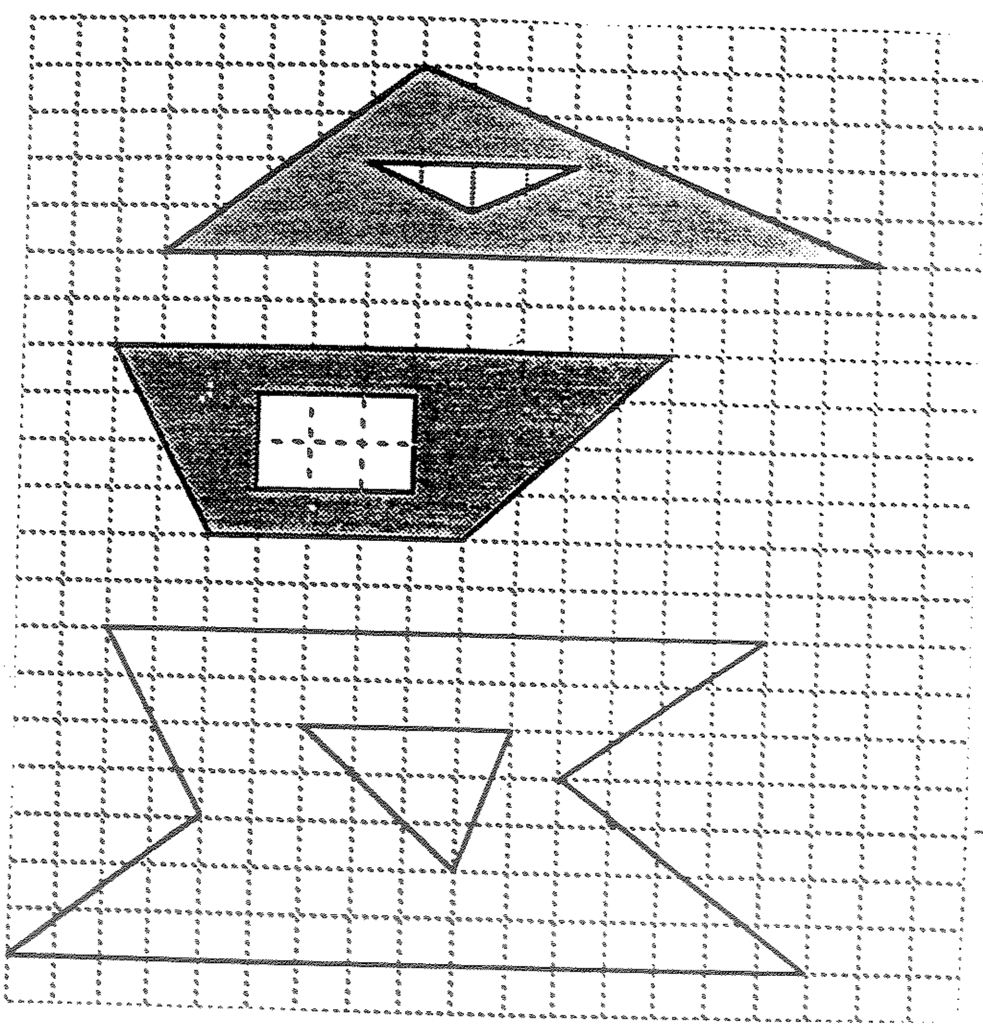
The big triangle would take _____ cans.
The inside triangle (the hole) would take _____ cans.
So, the shaded area would take _____ cans.

You can also figure things out this way when the taken-out piece isn't completely inside the big triangle. For example:



The big rectangle would take ______ cans.
The "inside" triangle (the hole) would take ______ cans.
So, the shaded area would take ______ cans.

Here are some more for you to try. (Remember, these are subtraction problems.)



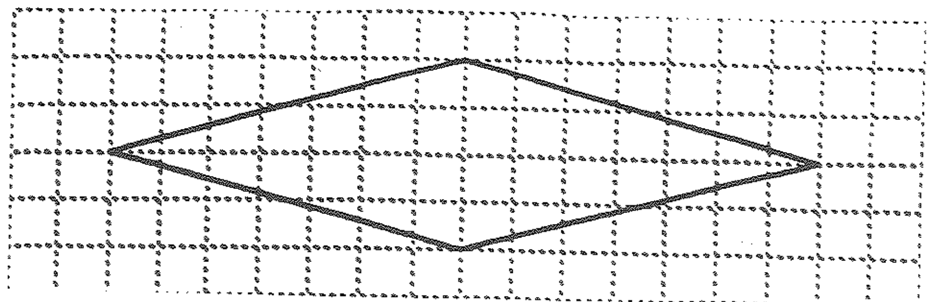
______ cans

______ cans

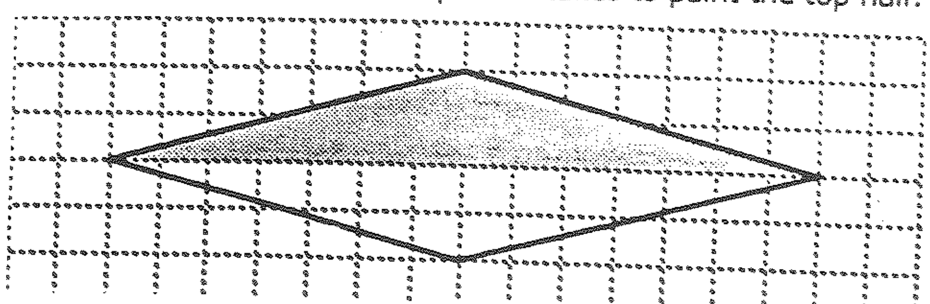
______ cans

Mirrors

Let's talk a little more about tricks and shortcuts. Sometimes, as we have seen, you only have to figure out how much paint it takes to paint part of a figure in order to get the answer for the whole figure. For example, suppose we wanted to paint this diamond.

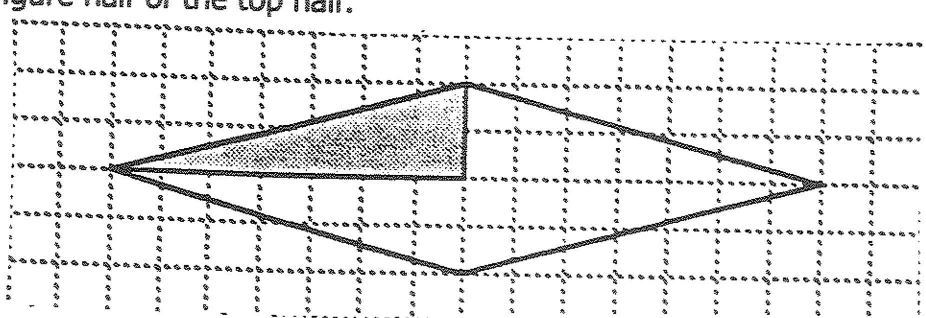


It's enough to figure out how much paint it takes to paint the top half.



We find out that answer, then double it to find out how many cans it takes to paint the whole diamond.

But wait a minute—to find out how many cans it takes to paint the top half, we can just figure half of the top half.

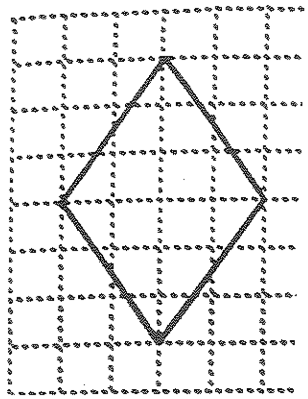


To paint the shaded triangle just above, it takes 7 cans, right? (We should be pretty expert at figuring these easy triangles by now!)

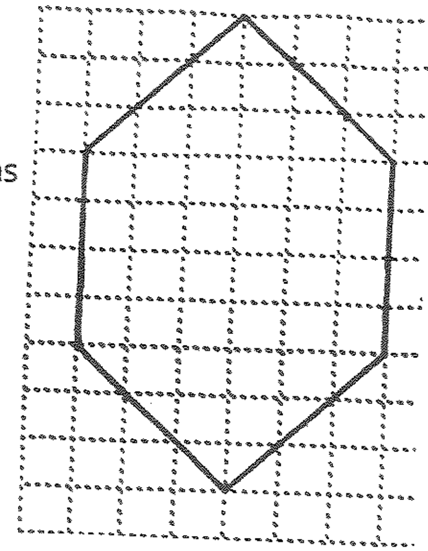
So, to paint the top half of the diamond takes 2 x 7 = 14 cans of paint. To paint the whole diamond, it takes

2 x (2 x 7) = 2 x 14 = 28 cans of paint.

Now you try some.

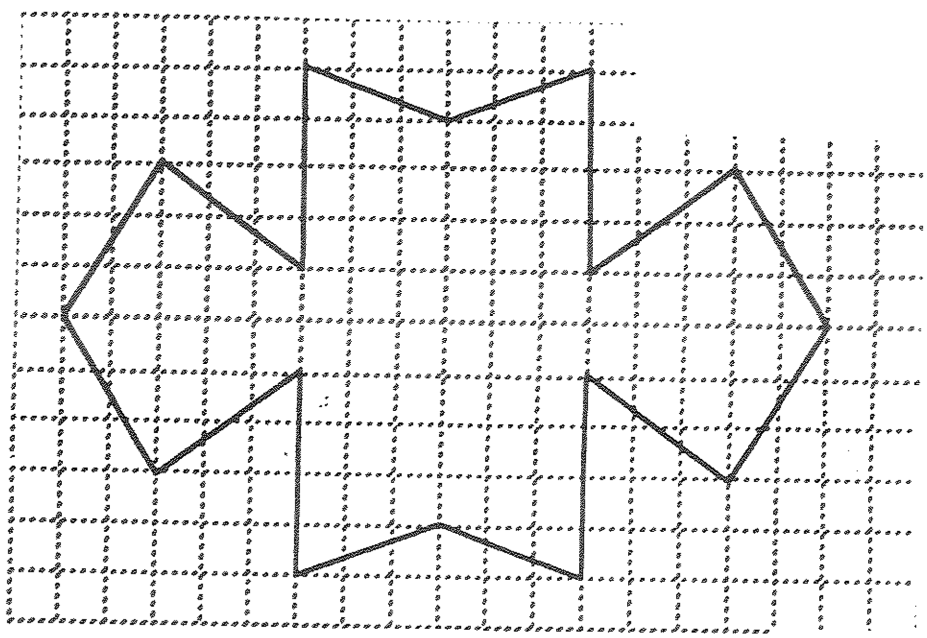


______ cans



______ cans

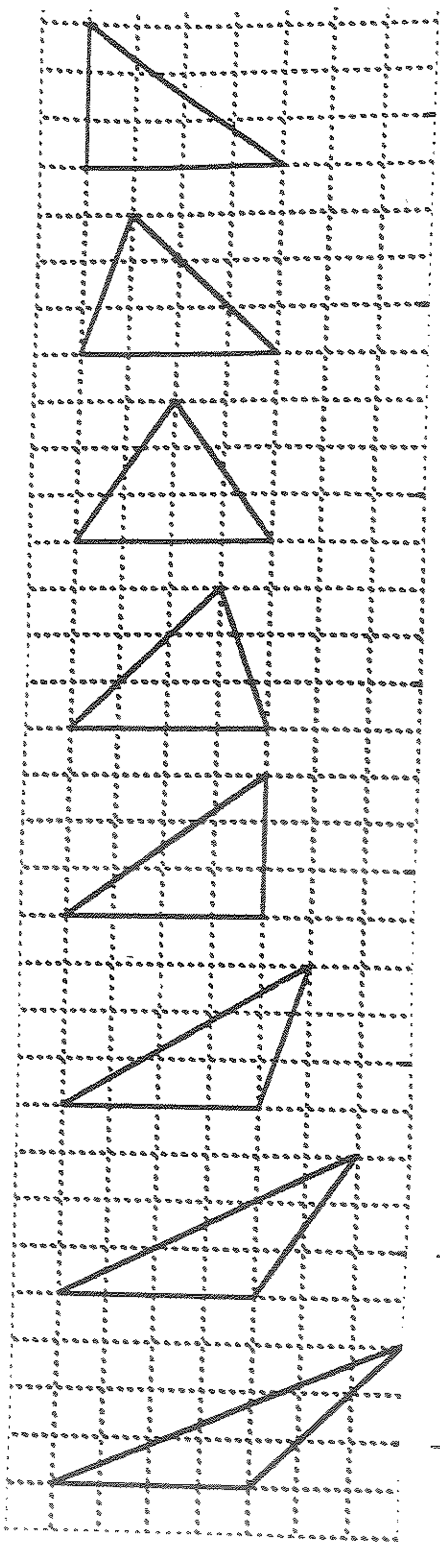
______ cans



Find any empty space and do one of your own!

Moving Triangles

Here's a whole collection of triangles. You have to paint them all! Figure out how many cans of paint you need for each one.



_____ cans

_____ cans

_____ cans

_____ cans

_____ cans

_____ cans

_____ cans

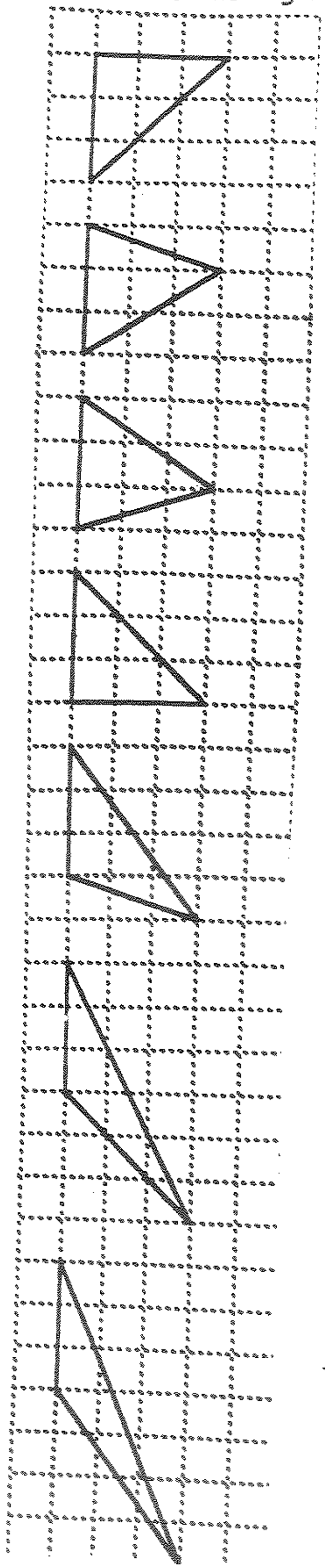
_____ cans

Look at all these triangles again. If you figured things out correctly, each triangle will need exactly the same amount of paint.

What is the same about all the triangles?

What is different in the different triangles?

Here's another set of moving triangles.



_____ cans

_____ cans

_____ cans

_____ cans

_____ cans

_____ cans

_____ cans

Do you see a pattern?

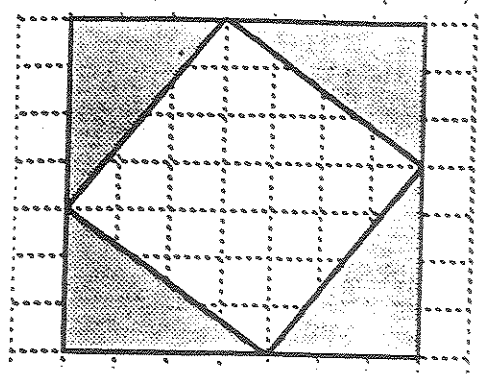
Try to make up an easy rule which tells you exactly how many cans of paint are needed to paint any of the moving triangles without having to do each one separately.

Make your own set of moving triangles and try out your rule.

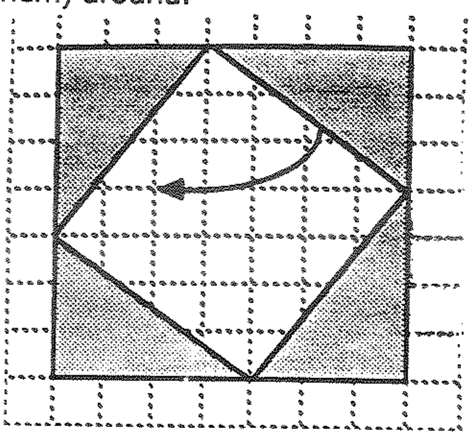
Squares in Squares

We're going to end this book with a couple of the best painting problems of all! In this section, we'll do one that we can figure out exactly, and in the next (and last) section, we'll try one that nobody can figure out exactly.

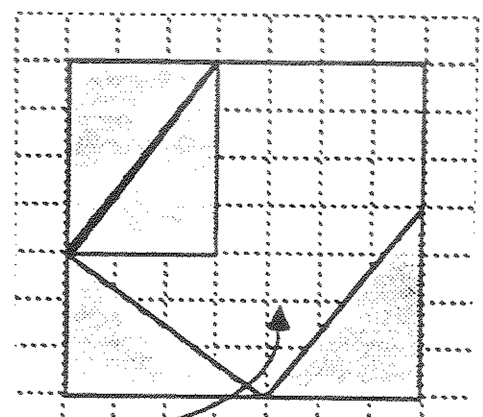
How much paint does it take to paint the inside (tilted) square in this picture?



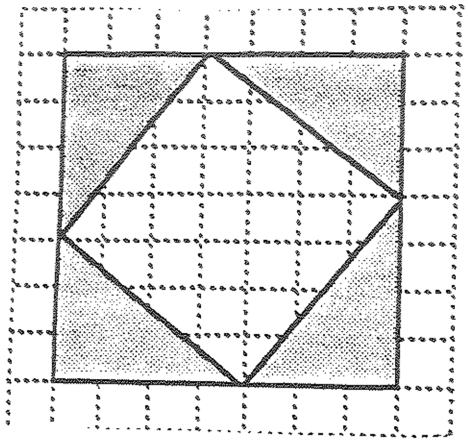
The nice thing is that there is a trick that you can do which makes it very easy to figure out the answer. The trick is to move the shaded triangles (or at least two of them) around.



First move

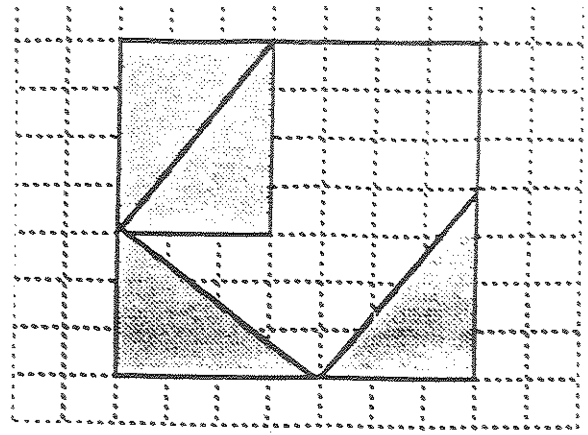


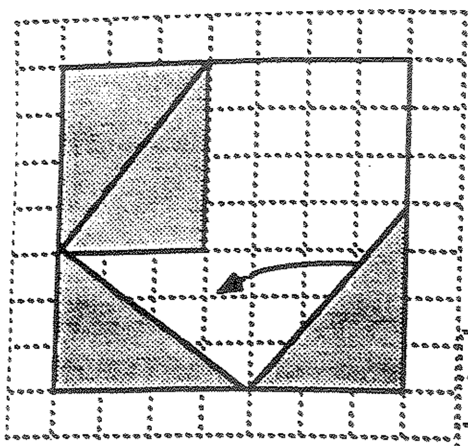
It takes exactly the same number of cans of paint to paint the unshaded part of this picture as it does to paint the tilted square in the first picture. It's like the shaded triangles were pieces of furniture in a room and we were lazy painters who only paint the part of the floor that shows, that is, the part not covered by furniture. It doesn't matter how we move the furniture around in the room, it won't make the lazy painter's job easier or harder!



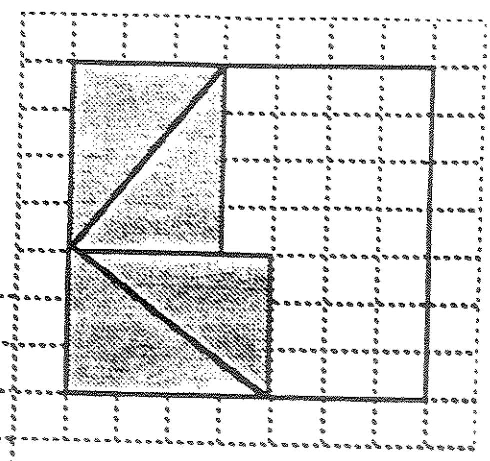
Space 26

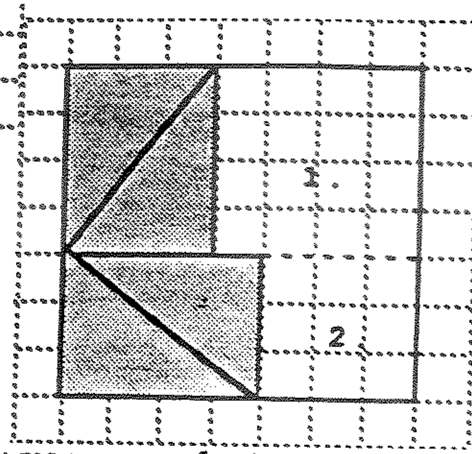
First move





Second move

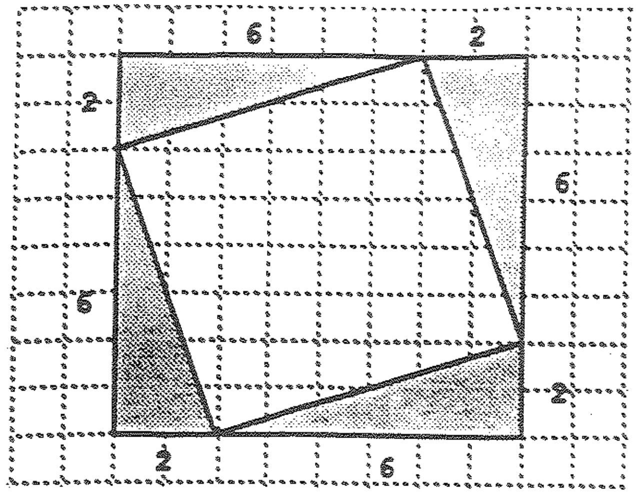




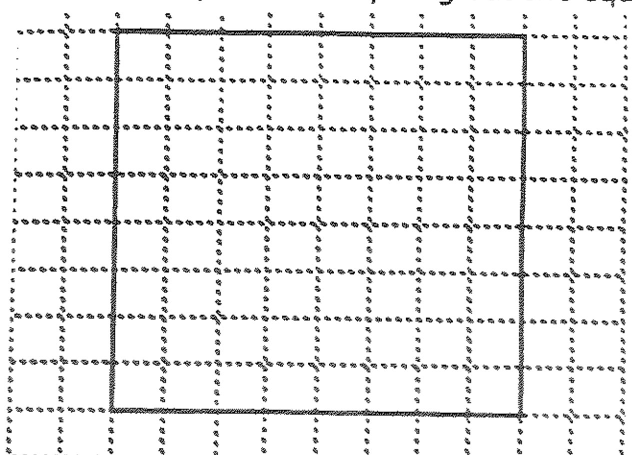
So, to figure out how many cans of paint it takes to paint the tilted square in the first picture at the top of the page, we can just figure out how many cans of paint it takes to paint the square-shaped pieces 1 and 2 in this bottom picture. We just "moved the furniture around." The amount of floor that shows is the same!

So, it takes $(4x4) + (3x3) = 16 + 9 = 25$ cans of paint to paint the tilted square!

Here's another one.



Move the triangles so that they cover everything but two square-shaped pieces.

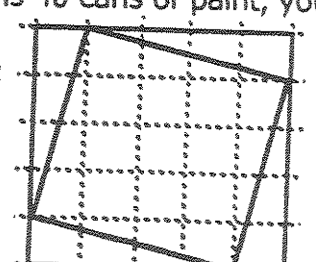


How many cans of paint does it take to paint each square?

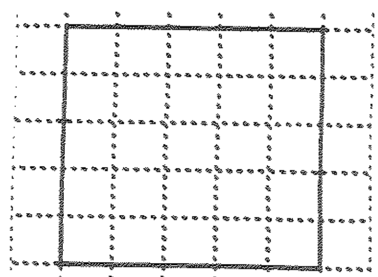
So, how many cans of paint does it take to paint the tilted square in the picture at the top of the page?

If your answer is 40 cans of paint, you're correct!

Here's another:

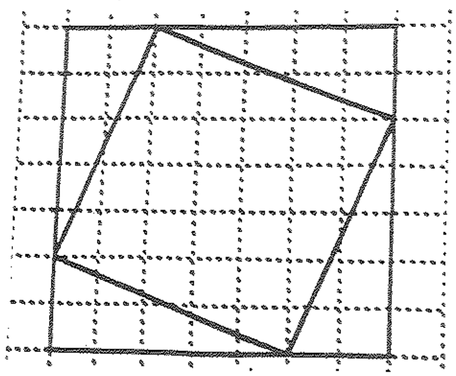


becomes

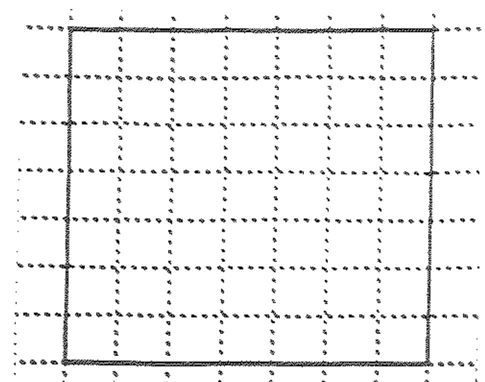


(___ x ___) + (___ x ___) = _____ cans of paint to paint the tilted square.

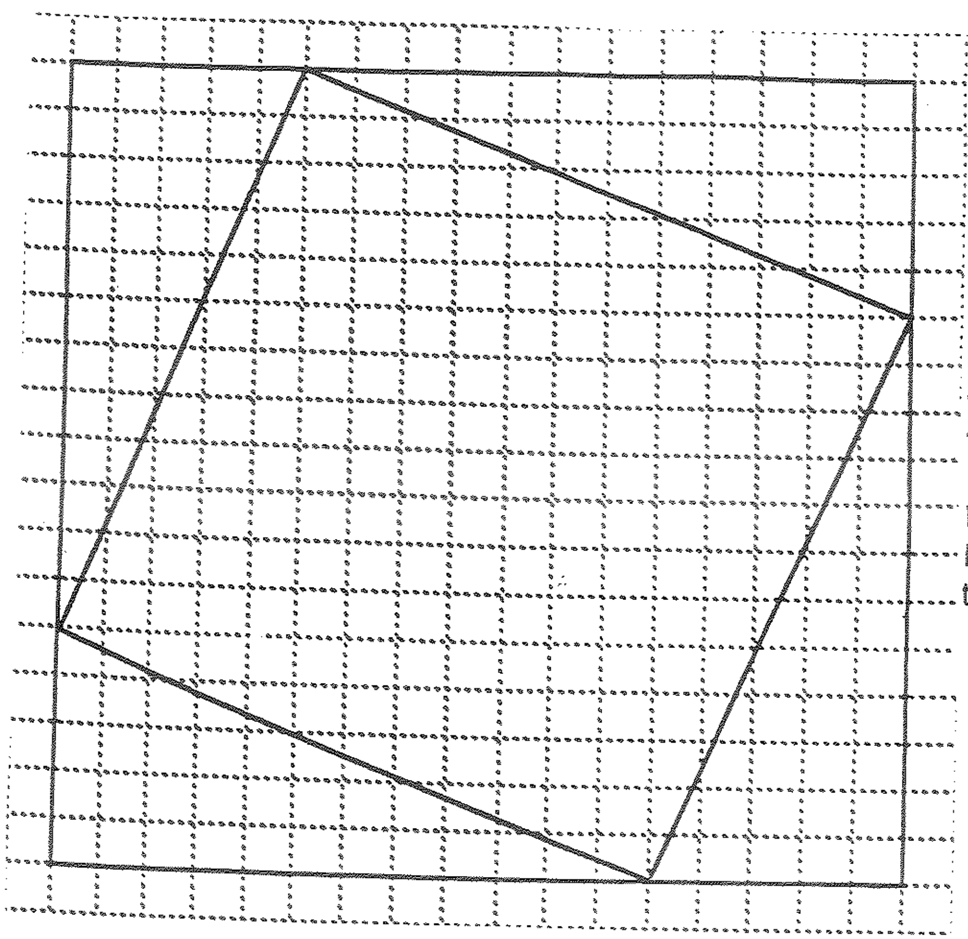
Now try these:



Becomes



So, it takes (___ x ___) + (___ x ___) = _____ cans of paint to paint the tilted square.



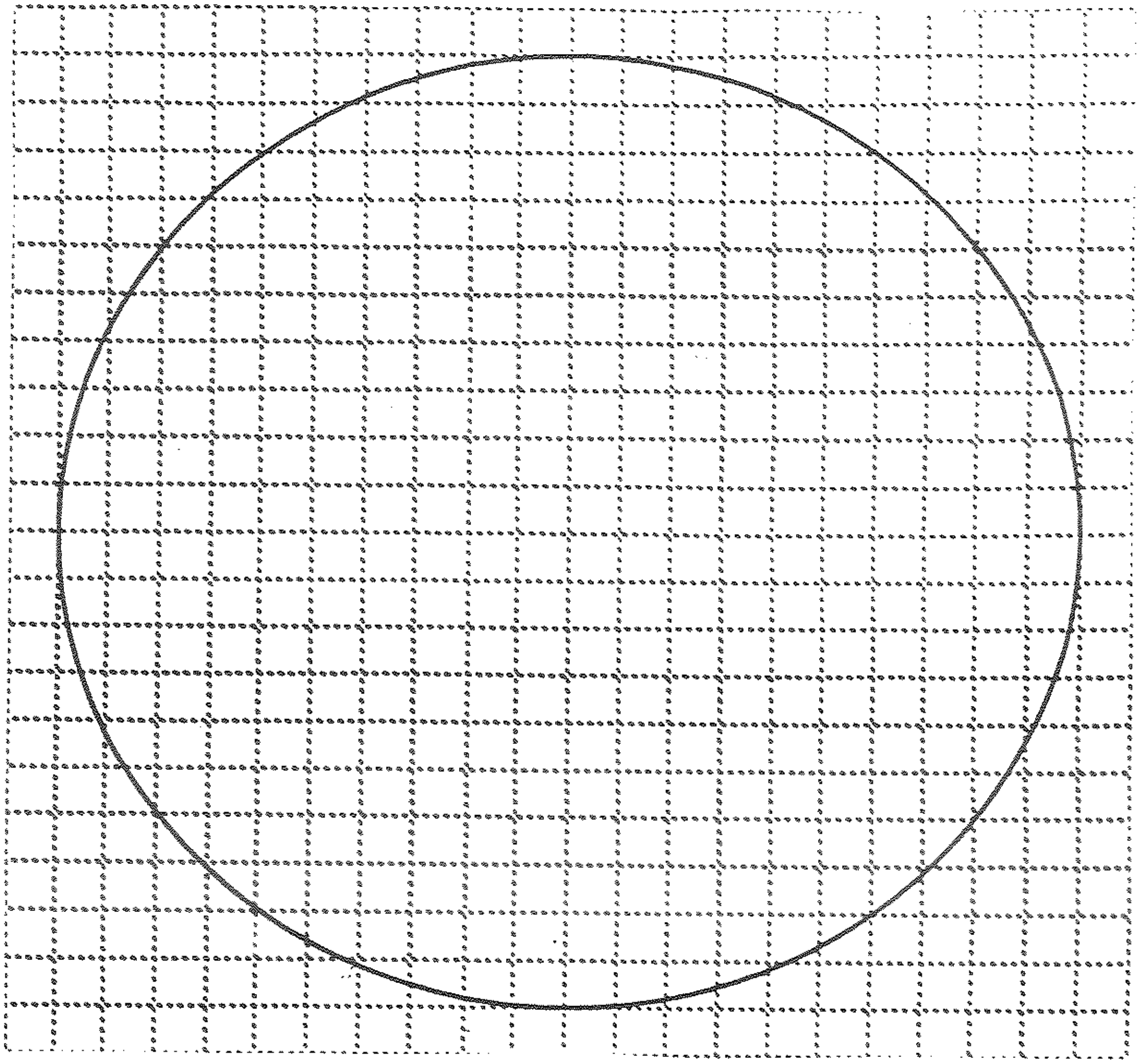
Try to figure out how many cans of paint it takes to paint the inside of the tilted square.

Extra-Credit question: How long is a side of the tilted square?

Trying to Paint the Inside of a Circle

Our little painter's final job was a wild one!

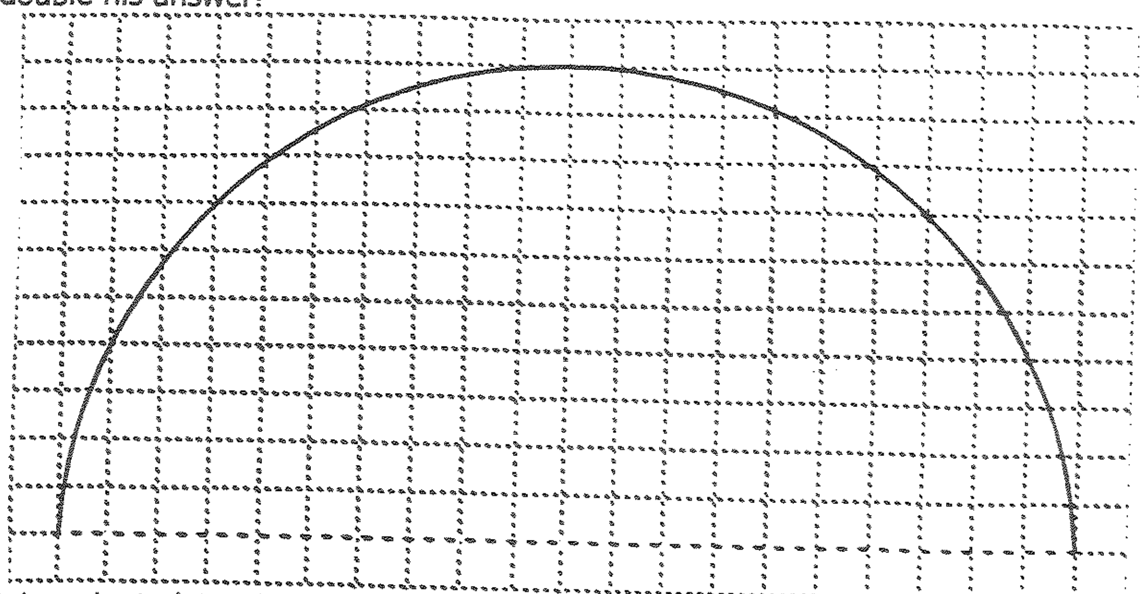
Paint this:



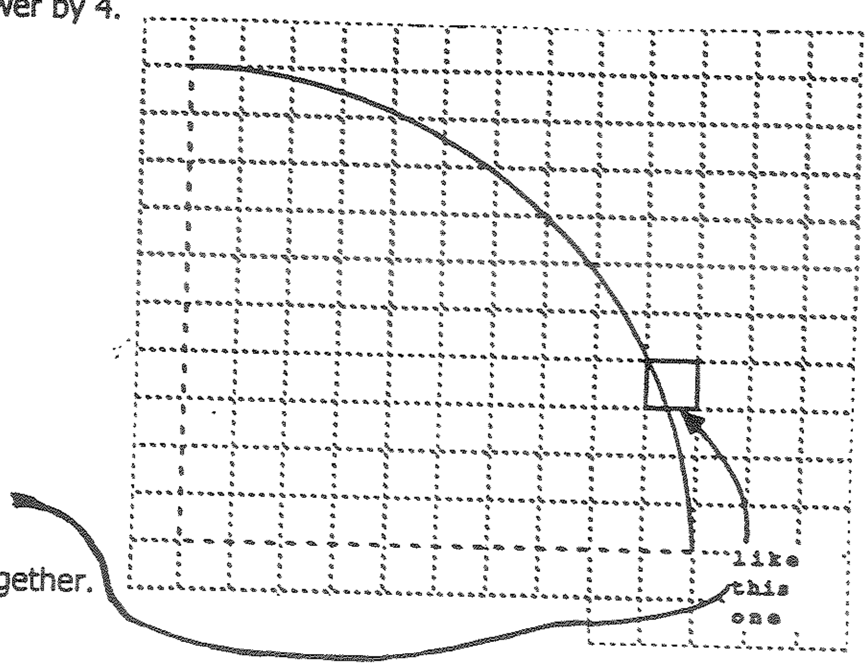
The painter got a headache trying to figure out exactly how many cans of paint it would take to paint the inside of the circle. He decided it was just not working out and that he could save time and energy by buying a large enough number of cans of paint to be sure he could do the job (and not a can more).

How can he figure out how many cans is enough? He started counting little squares which lie inside the circle. He would certainly have to buy that many cans of paint. After he finished that, he thought and realized that he was going to have to buy a little more paint to cover the little squares that are partly inside the circle. So, he counted the little squares that are partly inside and partly outside the circle.

But our painter was thinking as he was counting, and suddenly he remembered about "mirrors." Maybe he could just figure out the top half of the circle and double his answer!



But a minute later, he had an even better idea! He can just do half of that half, and then multiply his answer by 4.



Count the number of little squares that fit entirely inside this fourth of a circle.

______

Count the number of little squares that have a part inside and a part outside this fourth of a circle.

______

Add these two numbers together.

______

Multiply this last answer by 4.

______

That's how much paint the little painter bought!