5.3 Future Value of Annuities

Vocab annuity ⇒ financial plan characterized by regular payments

ordinary annuity vs. annuity due

ordinary annuity	annuity due
payments made at end of each equal payment interval	payments made at beginning of each equal payment interval

Ex1 Suppose you invest \$1000 at the end of each year for 5 years in an account that pays 10% interest compounded annually;

What is the value after 5 yrs?

end of yr 1: $P = 1000$, compounds for $t = 4$ yrs

$\Rightarrow S_1 = 1000(1.1)^4$

end of yr 2:

end of yr 3:

end of yr 4:

end of yr 5:

5.3 (cont)

Generally, then, for an ordinary annuity, the future value is

$$S = \frac{R(1-(1+r_c)^N)}{1-(1+r_c)}$$

$r_c = \frac{r}{n}$

R = amt deposited each period

(sum of geometric sequence)

$N = nt$

$$\Leftrightarrow \quad S = \frac{R(1-(1+r_c)^N)}{-r_c}$$

$$\boxed{S = \frac{R((1+r_c)^N - 1)}{r_c}}$$

Future value of ordinary annuity

Ex 2 (Twins story)

Twin 1: At the end of college, she invests \$2000 at the end of each year for 8 years in an account that earns 10% compounded annually. After 8 years, she contributes nothing, but it continues to earn 10% compounded annually for 36 more years. How much does she have then?

5.3 (cont)

Twin 2: At the end of college, he invests nothing for 8 years. Then he puts \$2000 into an account at the end of each year for 36 years, earning 10% interest compounded annually. How much does he have then?

Ex 3 How much should be invested quarterly (at end of each qtr) at 12% interest compounded quarterly to pay off a debt of \$30,000 in 6 yrs?

Sinking Fund

$$R = S\left(\frac{r_c}{(1+r_c)^N - 1}\right)$$

The payment that needs to be invested every period to pay of debt of S at the end.

5.3 (cont)

Ex 4 Find the future value of an account w/ $100 deposited at the beginning of each month for 5 yrs into an account that pays 8% compounded monthly.

Future value of Annuity Due

$$S = R\left[\frac{(1+r_c)^{N+1}-1}{r_c}\right] - R$$

$$= R\left[\frac{(1+r_c)^{N+1}-1-r_c}{r_c}\right]$$

$$= R\left[\frac{(1+r_c)^{N+1}-(1+r_c)}{r_c}\right]$$

$$= R(1+r_c)\left[\frac{(1+r_c)^{N}-1}{r_c}\right]$$

5.4 Present Value of Annuity

PV of an annuity $\Rightarrow$ we calculate this when we leave a lump sum of \$ in an account and make regular withdrawals (like what happens after a person retires)

ordinary annuity	vs. annuity due
withdrawals occur at end of each period	withdrawals occur at beginning of each period

Ex1 You want to withdraw \$1000 at the end of each year from an account that earns 10% interest compounded annually for 4 years. How much needs to be in the account from the start?

compounding:

$$S = P\left(1+\frac{r}{n}\right)^{nt} \qquad r_c = \frac{r}{n}, \; N = nt$$

$$\Rightarrow S = P(1+r_c)^N \quad \Rightarrow \quad P = S(1+r_c)^{-N}$$

after 1st year: we need $P_1 = 1000(1+0.1)^{-1} = 1000(1.1)^{-1}$

after 2nd year: " " $P_2 = 1000(1.1)^{-2}$

after 3rd year:

after 4th year:

$$P_1 + P_2 + P_3 + P_4 =$$

(sum of geometric sequence again!)

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5.4 (cont)

Present value (PV) of ordinary annuity	PV of annuity due
$S = \frac{R(1-(1+r_c)^{-N})}{r_c}$	$S_{due} = \frac{R(1+r_c)(1-(1+r_c)^{-N})}{r_c}$

Ex 2 Find PV of annuity that pays $4000 at the end of each month from an account that earns 8% interest compounded monthly for 25 years.

Ex 3 An inheritance of $500,000 will provide how much at the end of each year for 20 years, if money is worth 7.2% compounded annually?

Ex 4 A lottery prize worth \$1,800,000 is awarded in payments of \$10,000 at the beginning of each month for 15 years. Suppose money is worth 6.6% monthly. What is the real value of the prize?

Deferred Annuity ⇒ where 1st payment is deferred until a later date at which pt regular payments are made.

P = PV of deferred annuity ; m = # periods of deferment

N = # of regular withdrawals . R = payment each (withdrawal) period

$$P = \frac{R\left(1-(1+r_c)^{-n}\right)}{r_c(1+r_c)^m}$$

Ex 5 Carol received a trust fund inheritance of \$10,000 on her 30^{th} birthday. She plans to use it to supplement her income w/ 20 quarterly payments beginning on her 60^{th} birthday. If money is worth 8.1% compounded quarterly, how much will each payment be?

5.5 Loans and Amortization

amortization ⇒ "installment loan"; a loan is repaid by making all payments equal.

Bank is basically investing a lump sum of \$ and getting a periodic return which is exactly like PV of ordinary annuity.

$$\Rightarrow \boxed{R = S\left(\frac{r_c}{1-(1+r_c)^{-N}}\right) \text{ Amortization Formula}}$$

S = loan amt   R = payment amt

Ex1 When you graduate college, you buy a new car and can afford a monthly payment of \$250/mo. If you get a special rate of 3.6% interest, compounded monthly, for 6 years, how much can you afford to borrow?

5.5 (cont)

Ex2 Angie buys a house for \$200,000. She puts \$15,000 down and she gets a loan for the rest at 5.47% interest compounded monthly for 20 years. What will her payments be?

Amortization Schedule

A loan of \$10,000 w/ interest rate of 10% could be repaid in 5 equal annual payments.

$$R = 10000\left[\frac{0.1}{1-(1.1)^{-5}}\right] \approx \$2637.97$$

	payment	interest	principal	unpaid balance
				8362.03
1	2637.97	0.1(10000) = 1000	1637.97	6560.26
2	2637.97	0.1 (8362.03) = 836.20	1801.77	4578.32
3	2637.97	0.1(6560.26) = 656.03	1981.94	2398.18
4	2637.97	0.1(4578.32) = 457.83	2180.14	0
5	2637.97	0.1(2398.18) = 239.80	2398.18	

Ex3 A company that buys a piece of equipment by borrowing \$250,000 for 10 years at 6%, compounded monthly, has monthly payments of \$2,775.51.

(a) Find the unpaid balance (loan payoff) after 1 year?

(b) During that first year, how much interest does the company pay?

Loan Payoff Amt

$$S_{N-k} = R\left(\frac{1-(1+r_c)^{-(N-k)}}{r_c}\right)$$

k = # payments that have been paid

N = nt = total # payments that were originally due.

N−k = # payments "missing" from the loan.