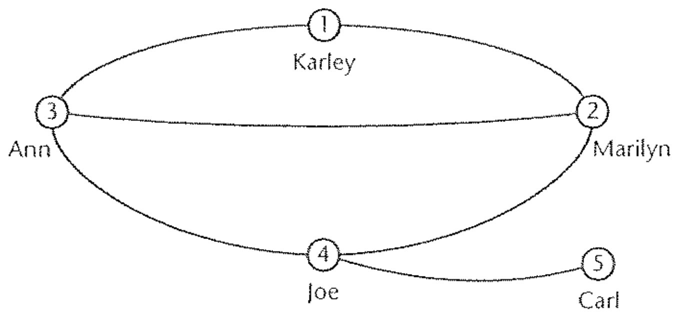
EXAMPLE 4

Routes/Graphs: According to the popular party game "Six Degrees of Kevin Bacon," every actor can link to Kevin Bacon within six actor connections. We can represent this phenomenon with a mathematical graph, connecting people. Let's use this hypothetical graph for someone named Karley and explore the connections she has with the people depicted here.



With this graph, we can use a matrix to represent the number of non-stop connections between each pair of people. Entry a_{ij} in matrix A represents the number of non-stop connections between person i and person j. For example, a_{13} is the number of non-stop connections between person 1 (Karley) and person 3 (Ann), which is only 1. Notice, then, that a_{ij} will be the same number as a_{ji} because we have no direction on the connections. So, $a_{13} = a_{31} = 1$. Also, $a_{ii} = 0$ because there is no connection from a person to him or herself. If we fill in the entire matrix A following this pattern, we'll get

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

If we square A, then each entry will be the number of one-stop connections between one person to another.

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 2 & 0 \\ 1 & 3 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

For example, the number of one-stop connections between person 1 (Karley) and person 5 (Carl) would still be zero since the shortest path between Karley and Carl is three connections. The number of one-stop connections between Karley (person 1) and Marilyn (person 2) will be represented by $a_{12} = 1$ because we can go from Karley to Ann and then to Marilyn, thereby making one stop from Karley to Marilyn. That's the only way to get from Karley to Marilyn with one stop. To get from Joe (person 4) to himself (person 4) with exactly one stop, it will be $a_{44} = 3$ ways because we can go (a) from Joe to Marilyn to Joe, (b) from Joe to Ann to Joe and (c) from Joe to Carl to Joe.

If we cube A, that is multiply A^2 by A again, then each entry a_{ij} in A^3 will represent the number of ways to get from person i to person j with exactly two stops.

$$A^3 = \begin{bmatrix} 2 & 1 & 1 & 2 & 0 \\ 1 & 3 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 5 & 2 & 2 \\ 5 & 4 & 5 & 6 & 1 \\ 5 & 5 & 4 & 6 & 1 \\ 2 & 6 & 6 & 2 & 3 \\ 2 & 1 & 1 & 3 & 0 \end{bmatrix}$$

For example, the number of two-stop connections between person 1 (Karley) and person 3 (Ann) is represented in the first-row and third-column entry, namely 5. Here are the five ways to get from Karley to Ann with two stops in between: (a) Karley to Marilyn to Joe to Ann, (b) Karley to Ann to Karley to Ann, (c) Karley to Ann to Marilyn to Ann, (d) Karley to Ann to Joe to Ann and (e) Karley to Marilyn to Karley to Ann. Notice that we still have one zero entry in this matrix, in the fifth-row and fifth-column entry. This means that person 5 (Carl) has no way to get back to himself with exactly two stops. Since the rest of the matrix has nonzero entries, it means that within two stops every person connects to every other person in this scenario.