9.1 (cont)

$$\boxed{\text{Thm If } \lim_{n\to\infty} |a_n| = 0, \text{ then } \lim_{n\to\infty} a_n = 0}$$

Ex 4 Show that if $r \in (-1,1)$, then $\lim_{n\to\infty} r^n = 0$.

pf Assume $r \neq 0$ (otherwise this result is trivial)

If $r \in (-1,1)$, then $|r| < 1 \iff \frac{1}{|r|} > 1$

Let $\frac{1}{|r|} = 1 + p$ for some $p > 0$. (since $\frac{1}{|r|} > 1$, it has to be 1 + something positive)

$$\Rightarrow \frac{1}{|r|^n} = (1+p)^n = 1 + np \underbrace{+ \cdots +}_{\text{all positive terms}} p^n$$

$$\geqslant np$$

$$\iff \frac{1}{|r|^n} \geqslant np$$

$$\iff \frac{1}{np} \geqslant |r|^n$$

$$\Rightarrow 0 \leq \lim_{n\to\infty} |r|^n \leq \lim_{n\to\infty} \frac{1}{np} = \frac{1}{p} \lim_{n\to\infty} \frac{1}{n} = 0$$

and by the squeeze theorem then

$$\lim_{n\to\infty} |r|^n = 0 \quad \forall\ |r| < 1$$

and by the Thm above $\Rightarrow \lim_{n\to\infty} r^n = 0 \quad \forall\ r \in (-1,1)$ //